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Generalized Second-Order Duality for a Continuous Programming Problem with Support Functions

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Abstract: A generalized second-order dual is formulated for a continuous programming problem in which support functions appear in both objective and constraint functions, hence it is nondifferentiable. Under second-order pseudoinvexity and second-order quasi-invexity, various duality theorems are proved for this pair of dual continuous programming problems. Special cases are deduced and a pair of dual continuous programming problems with natural boundary values is constructed and it is pointed out that the duality results for the pair can be validated analogously to those of the dual models with fixed end points. Finally, a close relationship between duality results of our problems and those of the corresponding (static) nonlinear programming problem with support functions is briefly mentioned.

Key words: Generalized second-order dual; Continuous programming; Second-order pseudoinvexity; Second-order quasi-invexity

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1. INTRODUCTION

Chen [1] was the first to identify second-order dual for a constrained variational problem and established various duality results under an involved invexity-like assumptions. Husain *et al.* [2] presented Mond–Weir type second-order duality for the problem of [1] and by introducing continuous-time version of second-order invexity and generalized second-order invexity, validated various duality results. Subsequently, for a class of nondifferentiable continuous programming problems, Husain and Masoodi [3] studied Wolfe type second-order duality while Husain and Srivastava [4] investigated Mond-Weir type second-order duality. Later, Husain and Srivastav [5] presented mixed type second-order duality for the problems considered in [3,4] and pointed out the relationship between their results and those of Zhang and Mond [6]. Recently, Husain and Masoodi [7] presented Wolfe type second-order duality and Husain and Srivastav [8] presented Mond-Weir type second-order duality for a class of continuous programming problem containing support functions which are somewhat more general than the square root of certain positive semidefinite quadratic form.

In this paper, a generalized second-order dual to the problem of [7,8] is formulated and duality results are established under generalized second-order invexity conditions. Problems with natural boundary conditions are also constructed. Finally, it is pointed out that our duality results are dynamic generalization of those of nonlinear programming problems with support functions treated by Husain *et al.* [9].

2. PRELIMINARIES AND STATEMENT OF THE PROBLEM

Let $I = [a, b]$ be a real interval, $\phi : I \times R^n \times R^n \to R$ and $\psi : I \times R^n \times R^n \to R^m$ be twice continuously differentiable functions. In order to consider $\phi\left(t, x\left(t\right), \dot{x}\left(t\right)\right)$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, denoted by ϕ_x and $\phi_{\dot{x}}$ the first order derivative of ϕ with respect to $x(t)$ and $\dot{x}(t)$ respectively, that is,

$$\phi_x = \left(\frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^2}, \cdots, \frac{\partial\phi}{\partial x^n}\right)^T,$$

$$\phi_{\dot{x}} = \left(\frac{\partial\phi}{\partial \dot{x}^1}, \frac{\partial\phi}{\partial \dot{x}^2}, \cdots, \frac{\partial\phi}{\partial \dot{x}^n}\right)^T$$

Denote by ϕ_{xx}, the $n \times n$ Hessian matrix of ϕ, and ψ_x the $m \times n$ Jacobian matrix respectively, that is, $\phi_{xx} = \left(\dfrac{\partial^2\phi}{\partial x^i \partial x^j}\right)$, $i, j = 1, 2, ...n$, ψ_x the $m \times n$ Jacobian matrix.

$$\psi_x = \begin{pmatrix} \frac{\partial\psi^1}{\partial x^1} & \frac{\partial\psi^1}{\partial x^2} & \cdots & \frac{\partial\psi^1}{\partial x^n} \\ \frac{\partial\psi^2}{\partial x^1} & \frac{\partial\psi^2}{\partial x^2} & \cdots & \frac{\partial\psi^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\psi^m}{\partial x^1} & \frac{\partial\psi^m}{\partial x^2} & \cdots & \frac{\partial\psi^m}{\partial x^n} \end{pmatrix}_{m\times n}.$$

The symbols $\phi_{\dot{x}}$, $\phi_{\dot{x}x}$, $\phi_{x\dot{x}}$ and $\psi_{\dot{x}}$ have analogous representations.

Designate by X the space of piecewise smooth functions $x : I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = D\,x \Leftrightarrow x(t) = \int_a^t u(s)\mathrm{d}s.$$

Thus $\dfrac{\mathrm{d}}{\mathrm{dt}} = D$ except at discontinuities.

We incorporate the following definitions which needed in the subsequent analysis.

Definition 1. (Second-Order Invex): If there exists a vector function $\eta = \eta(t, x, \bar{x}) \in R^n$ where $\eta : I \times R^n \times R^n \to R^n$ and with $\eta = 0$ at $t = a$ and $t = b$, such that for a scalar function $\phi(t, x, \dot{x})$, the functional $\int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t$ where $\phi : I \times R^n \times R^n \to R$ satisfies

$$\int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t - \int_I \left\{\phi\,(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}p^T\,(t)\,Gp\,(t)\right\}\mathrm{d}t$$
$$\geq \int_I \left\{\eta^T \phi_x\,(t, \bar{x}, \dot{\bar{x}}) + (D\eta)^T \phi_{\dot{x}}\,(t, \bar{x}, \dot{\bar{x}}) + \eta^T G\,p\,(t)\right\}\mathrm{d}t,$$

then $\int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t$ is second-order invex with respect to η where $G = \phi_{xx} - 2D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}} - D^3\phi_{\dot{x}\ddot{x}}$, $p \in C(I,\ R^n)$, the space of n-dimensional continuous vector functions.

Definition 2. (Second-Order Pseudoinvex): If the functional $\int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t$ satisfies

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\,p\,(t)\right\}\mathrm{d}t \geq 0$$
$$\Rightarrow \int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t \geq \int_I \left\{\phi\,(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}p(t)^T G\,p\,(t)\right\}\mathrm{d}t\,,$$

then $\int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t$ is said to be second-order pseudoinvex with respect to η.

Definition 3. (Second-order Strictly Pseudoinvex): If the functional $\int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t$ satisfies

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\,p\,(t)\right\}\mathrm{d}t \geq 0$$
$$\Rightarrow \int_I \phi\,(t, x, \dot{x})\,dt > \int_I \left\{\phi\,(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}p(t)^T Gp\,(t)\right\}\mathrm{d}t$$

then $\int_I \phi\,(t, x, \dot{x})\,\mathrm{d}t$ is said to be second-order strictly pseudoinvex with respect to η.

Definition 4. (Second-Order Quasi-invex): If the functional $\int_I \phi\left(t,x,\dot{x}\right)\mathrm{d}t$ satisfies

$$\int_I \phi\left(t,x,\dot{x}\right)dt \le \int_I \left\{\phi\left(t,\bar{x},\dot{\bar{x}}\right) - \frac{1}{2}p(t)^T G p\left(t\right)\right\}\mathrm{d}t$$

$$\Rightarrow \int_I \left\{\eta^T \phi_x + \left(D\eta\right)^T \phi_{\dot{x}} + \eta^T G\left(t\right)p\left(t\right)\right\}\mathrm{d}t \le 0,$$

then $\int_I \phi\left(t,x,\dot{x}\right)\mathrm{d}t$ is said to be second-order quasi-invex with respect to η.

Consider the following nondifferentiable continuous programming problem with support functions treated by Husain and Jabeen [10]:

(CP): Minimize

$$\int_I \left\{f\left(t,x,\dot{x}\right) + S\left(x\left(t\right)|K\right)\right\}\mathrm{d}t$$

subject to

$$x\left(a\right) = 0 = x\left(b\right), \tag{1}$$

$$g^j\left(t,x,\dot{x}\right) + S\left(x\left(t\right)|C^j\right) \le 0, \quad j = 1,2...m, \quad t \in I, \tag{2}$$

where f and g are continuously differentiable and each C^j, $(j = 1,2,...,m)$ is a compact convex set in R^n.

Husain and Jabeen [10] derived the following optimality condition for (CP):

Lemma 1. (Fritz-John Necessary Optimality Conditions): If the problem (CP) attains a minimum at $x = \bar{x} \in X$, then there exist $r \in R$ and piecewise smooth function $\bar{y} : I \to R^m$ with $\bar{y}\left(t\right) = \left(\bar{y}^1\left(t\right), \bar{y}^2\left(t\right), ...\bar{y}^m\left(t\right)\right)$, $\bar{z} : I \to R^n$ and $w^j : I \to R^n$, $j = 1,2,...,m$, such that

$$r\left[f_x\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{z}\left(t\right)\right] + \sum_{j=1}^{m} \bar{y}^j\left(t\right)\left[g_x^j\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{w}^j\left(t\right)\right]$$

$$= D\left[r f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{y}(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right], \quad t \in I$$

$$\sum_{j=1}^{m} \bar{y}^j\left(t\right)\left[g^j\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{x}(t)^T \bar{w}^j\left(t\right)\right] = 0, \quad t \in I$$

$$\bar{x}(t)^T \bar{z}\left(t\right) = S\left(\bar{x}\left(t\right)|K\right), \quad t \in I$$

$$\bar{x}(t)^T \bar{w}^j\left(t\right) = S\left(\bar{x}\left(t\right)|C^j\right), \quad j = 1,2...m, \quad t \in I$$

$$\bar{z}\left(t\right) \in K, w^j\left(t\right) \in C^j, \quad j = 1,2...m, \quad t \in I$$

$$\left(r, \bar{y}\left(t\right)\right) \ge 0, \quad t \in I$$

$$\left(r, \bar{y}\left(t\right)\right) \ne 0, \quad t \in I$$

The minimum $\bar{x}(t)$ of (CP) may be described as normal, if $\bar{r} = 1$ so that the Fritz John optimality conditions reduce to Karush–Kuhn–Tucker optimality conditions. It suffices for $\bar{r} = 1$ that Slater's [10] condition holds at $\bar{x}(t)$.

Now we review some well known facts about a support function for easy reference.

Let K be a compact set in R^n, then the support function of K is defined by

$$S\left(x\left(t\right)\left|K\right.\right)=\max\left\{x(t)^T v\left(t\right): v\left(t\right)\in K, t\in I\right\}$$

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis, i.e., there exist $z\left(t\right)\in R^n$, $t\in I$ such that

$$S\left(\left.y(t)\right|K\right)-S\left(\left.x(t)\right|K\right)\geq\left(y(t)-x(t)\right)^T z\left(t\right)$$

From [11], the subdifferential of $S\left(x\left(t\right)\left|K\right.\right)$ is given by

$$\partial S\left(x\left(t\right)\left|K\right.\right)=\left\{z\left(t\right)\in K,\quad t\in I\;\left|x(t)^T z\left(t\right)=S\left(x\left(t\right)\left|K\right.\right)\right.\right\}.$$

For any set $\Gamma\subset R^n$, the normal cone to Γ at a point $x(t)\in\Gamma$ is defined by

$$N_\Gamma\left(x(t)\right)=\left\{\left.y(t)\in R^n\right|y(t)\left(z\left(t\right)-x\left(t\right)\right)\leq 0,\quad z\left(t\right)\in\Gamma\right\}$$

It can be verified that for a compact convex set C, $y(t)\in N_C(x(t))$ if and only if

$$S\left(\left.y(t)\right|C\right)=x(t)^T y(t),\quad t\in I.$$

3. GENERALIZED SECOND-ORDER DUALITY

In this section, we present the formulation of a generalized differentiable second-order Mond–Weir type dual to (CP) which jointly represents Wolfe and Mond-Weir type duals to (CP). The Wolfe and Mond–Weir type second-order duals to (CP) were treated in [7] and [8] as the following differentiable continuous programming problems:

(CD): Maximize

$$\int\limits_I\left\{f\left(t,u,\dot{u}\right)+u(t)^T z\left(t\right)+\sum_{i=1}^m y^i(t)^T\left(g^i\left(t,u,\dot{u}\right)+u(t)^T w^i\left(t\right)\right)-\frac{1}{2}p(t)^T H\left(t\right)p\left(t\right)\right\}\mathrm{d}t$$

subject to

$$u\left(a\right)=0=u\left(b\right)$$

$$\begin{aligned}&f_u\left(t,u,\dot{u}\right)+z\left(t\right)+\sum_{i=1}^m y^i(t)^T\left(g_u^i\left(t,u,\dot{u}\right)+w^i\left(t\right)\right)\\&-D\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+y(t)^T g_{\dot{u}}\left(t,u,\dot{u}\right)\right)+H\left(t\right)p\left(t\right)=0\quad t\in I\\&z\left(t\right)\in K,\;w^i\left(t\right)\in C^i,\;t\in I,\quad i=1,2...m.\end{aligned}$$

$$y\left(t\right)\geq 0,\quad t\in I.$$

$$\begin{aligned}H\left(t\right)=&f_{uu}\left(t,u,\dot{u}\right)+\left(y(t)^T g_u\left(t,u,\dot{u}\right)\right)_u\\&-2D\left[f_{u\dot{u}}\left(t,u,\dot{u}\right)+\left(y(t)^T g_u\left(t,u,\dot{u}\right)\right)_{\dot{u}}\right]\\&+D^2\left[f_{\dot{u}\dot{u}}\left(t,u,\dot{u}\right)+\left(y(t)^T g_{\dot{u}}\left(t,u,\dot{u}\right)\right)_{\dot{u}}\right]\end{aligned}$$

(M-WCD): Maximize

$$\int_I \left(f(t,u,\dot{u}) + u(t)^T z(t) - \frac{1}{2}p(t)^T F p(t)\right)\mathrm{d}t$$

subject to

$$u(a) = 0 = u(b)$$

$$f_u + z(t) + \sum_{i=1}^{m} y^i(t)\left(g_u^i + w^i(t)\right) - D\left(f_{\dot{u}} + y(t)^T g_{\dot{u}}\right) + (F+G)\,p(t) = 0$$

$$\int_I \left(\sum_{i=1}^{m} y^i(t)\left(g^i + u(t)^T \omega^i(t)\right) - \frac{1}{2}p(t)^T G p(t)\right)\mathrm{d}t \geq 0$$

$$z(t) \in K, \quad w^i(t) \in C^i, \quad t \in I, \quad i = 1, 2, .., m$$

$$y(t) \geq 0, \quad t \in I$$

where,

(i) $p(t) \in R^n, \quad t \in I$

(ii) $F = f_{uu} - 2Df_{u\dot{u}} + D^2 f_{\dot{u}\dot{u}} - D^3 f_{\dot{u}\ddot{u}}, \quad t \in I$

(iii) $G = y(t)^T g_{uu} - 2D\left(y(t)^T g_{\dot{u}}\right)_u + D^2\left(y(t)^T g_{\dot{u}\dot{u}}\right) - D^3\left(y(t)^T g_{\dot{u}\ddot{u}}\right)$.

Using the second-order invexity conditions of

(i) $\int_I \left\{f(t,.,.) + (\cdot)^T z(t)\right\}\mathrm{d}t$ and $\sum_{i=1}^{m}\int_I \left\{y^i(t)\ \left(g^i(t,.,.) + (.)\,w^i(t)\right)\right\}\mathrm{d}t$ or

Second-order pseudoinvexity of

(ii) $\int_I \left\{f(t,.,.) + (.)^T z(t) + \sum_{j=1}^{m} y^j(t)^T\left(g^j(t,.,.) + (.)\,w^i(t)\right)\right\}\mathrm{d}t,$

Husain and Masoodi [7] established various duality results between (CP) and (CD), and Husain and Srivastava [8] validated duality theorems between (CP) and (M-WCD) under the assumptions that with respect to the same η,

(iii) $\int_I \left(f(t,.,.) + (.)^T z(t)\right)\mathrm{d}t$ is second-order pseudoinvex, and

(iv) $\int_I \sum_{i=1}^{n}\left(y^i(t)\,g^i(t,.,.)^T + (.)^T w^i(t)\right)\mathrm{d}t$ is second-order quasi-invex.

We now construct the following generalized second-order dual to (CP):

(GCD): Maximize

$$\int_I \left[f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{i\in I_0} y^i(t)\left(g^i(t,u,\dot{u}) + u(t)^T w^i(t)\right) - \frac{1}{2}p(t)^T H^0 p(t)\right]\mathrm{d}t$$

subject to

$$u(a) = 0, \quad u(b) = 0 \tag{3}$$

$$\begin{aligned} &f_u(t,u,\dot{u}) + z(t) + \sum_{i=1}^{m} y^i(t)^T\left({g^i}_u(t,u,\dot{u}) + w^i(t)\right) \\ &- D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u})\right) + Hp(t) = 0, \quad t \in I \end{aligned} \tag{4}$$

$$\int_I \left(\sum_{i \in I_\alpha} y^i(t) \left(g^i(t,u,\dot{u}) + u(t)^T w^i(t) \right) - \frac{1}{2} p(t)^T G^\alpha p(t) \right) \mathrm{d}t \geq 0, \quad \alpha = 1,2,3....r \tag{5}$$

$$z(t) \in K, \quad w^i(t) \in C^i, \quad t \in I, \quad i = 1,2,..,m \tag{6}$$

$$y(t) \geq 0, \quad t \in I \tag{7}$$

where

(i) $I_\alpha \subseteq M = \{1,2,....m\}$, $\alpha = 0,1,2,...r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$,

$$\begin{aligned}
\text{(ii) } H^0 =& f_{uu}(t,u,\dot{u}) + \sum_{i \in I_0} \left(y^i(t) g^i(t,u,\dot{u}) \right)_{uu} \\
&- 2D \left(f_{u\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} \left(y^i(t) g^i(t,u,\dot{u}) \right)_{u\dot{u}} \right) \\
&+ D^2 \left(f_{\dot{u}\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} \left(y^i(t) {g^i}_{\dot{u}}(t,u,\dot{u}) \right)_{\dot{u}} \right) \\
&- D^3 \left(f_{\dot{u}\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} \left(y^i(t) {g^i}_{\dot{u}}(t,u,\dot{u}) \right)_{\ddot{u}} \right),
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } H =& f_{uu}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u}) \right)_u \\
&- 2D \left(f_{u\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u}) \right)_{\dot{u}} \right) \\
&+ D^2 \left(f_{\dot{u}\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u}) \right)_{\dot{u}} \right) \\
&- D^3 \left(f_{\dot{u}\ddot{u}}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u}) \right)_{\ddot{u}} \right),
\end{aligned}$$

$$\begin{aligned}
\text{(iv) } G^\alpha =& \sum_{i \in I_\alpha} \left(y^i(t) {g_u}^i(t,u,\dot{u}) \right)_u - 2D \left(\sum_{i \in I_\alpha} \left(y^i(t) {g_u}^i(t,u,\dot{u}) \right)_{\dot{u}} \right) \\
&+ D^2 \left(\sum_{i \in I_\alpha} \left(y^i(t) {g_{\dot{u}}}^i(t,u,\dot{u}) \right)_{\dot{u}} \right) - D^3 \left(\sum_{i \in I_\alpha} \left(y^i(t) g^i(t,u,\dot{u}) \right)_{\dot{u}} \right)_{\ddot{u}}, \quad \alpha = 1,2,...,r,
\end{aligned}$$

and

$$\begin{aligned}
\text{(v) } G =& \sum_{i \in M - I_0} \left(y^i(t) {g_u}^i(t,u,\dot{u}) \right)_u - 2D \left(\sum_{i \in M - I_0} \left(y^i(t) {g_u}^i(t,u,\dot{u}) \right)_{\dot{u}} \right) \\
&+ D^2 \left(\sum_{i \in M - I_0} \left(y^i(t) {g_{\dot{u}}}^i(t,u,\dot{u}) \right)_{\dot{u}} \right) - D^3 \left(\sum_{i \in M - I_0} \left(y^i(t) g^i(t,u,\dot{u}) \right)_{\dot{u}} \right)_{\ddot{u}}.
\end{aligned}$$

Theorem 1 (Weak Duality): Let $x(t)$ be feasible for (CP) and

$$\left(u, y, z, w^1, w^2, ...w^m, p\left(t\right)\right)$$

feasible for (GCD). If for all feasible $\left(x, u, y, z, w^1, w^2, ...w^m, p\left(t\right)\right)$,

$$\int\limits_I \left(f\left(t, ., .\right) + (.)\ z\left(t\right) + \sum_{i \in I_0} \left(g^i\left(t, ., .\right) + (.)^T w^i\left(t\right)\right)\right) \mathrm{d}t$$

is second-order pseudoinvex and

$$\sum_{i \in I_\alpha} \int\limits_I y^i\left(t\right) \left(g^i\left(t, ., .\right) + (.)^T w^i\left(t\right)\right) \mathrm{d}t, \quad \alpha = 1, 2, ...r$$

is second-order quasi-invex with respect to the same η, then

$$\text{infimum (CP)} \ \geq \ \text{supremum (GCD)}.$$

Proof. By the feasibility of $x(t)$ and $\left(u, y, z, w^1, w^2, ...w^m, p\left(t\right)\right)$ for (CP) and (GCD) respectively, we have

$$\begin{aligned} &\int\limits_I \left(\sum_{i \in I_\alpha} y^i\left(t\right) \left(g^i\left(t, x, \dot{x}\right) + x(t)^T w^i\left(t\right)\right)\right) \mathrm{d}t \\ \leq \ &\int\limits_I \left(\sum_{i \in I_\alpha} y^i\left(t\right) \left(g^i\left(t, u, \dot{u}\right) + u(t)^T w^i\left(t\right)\right) - \frac{1}{2} p(t)^T G^\alpha\ p\left(t\right)\right) \mathrm{d}t, \quad \alpha = 1, 2, ...r. \end{aligned}$$

By second-order quasi-invexity of $\sum\limits_{i \in I_\alpha} \int\limits_I y^i(t) \left(g^i(t, ., .) + (.)^T w^i(t)\right) \mathrm{d}t$, $\alpha = 1, 2, ...r$ this inequality yields

$$\begin{aligned} &\int\limits_I \left[\left(\eta^T \sum_{i \in I_\alpha} y^i\left(t\right) \left(g^i{}_u\left(t, u, \dot{u}\right) + w^i\left(t\right)\right)\right)\right. \\ &\quad \left. + (D\eta)^T \left(\sum_{i \in I_\alpha} y^i\left(t\right) g^i{}_{\dot{u}}\left(t, u, \dot{u}\right)\right) + \eta^T\ G^\alpha p\left(t\right)\right] \mathrm{d}t \leq 0, \qquad \alpha = 1, 2..., r. \end{aligned}$$

This implies

$$\begin{aligned} 0 \geq \ &\int\limits_I \left[\left(\eta^T \sum_{i \in M - I_0} y^i\left(t\right) \left(g^i{}_u\left(t, u, \dot{u}\right) + w^i\left(t\right)\right)\right)\right. \\ &\quad \left. + (D\eta)^T \left(\sum_{i \in M - I_0} y^i\left(t\right) g^i{}_{\dot{u}}\left(t, u, \dot{u}\right)\right) + \eta^T G p\left(t\right)\right] \mathrm{d}t \\ = \ &\int\limits_I \eta^T \left[\left(\sum_{i \in M - I_0} y^i\left(t\right) \left(g^i{}_u\left(t, u, \dot{u}\right) + w^i\left(t\right)\right)\right)\right. \\ &\quad \left. - D\left(\sum_{i \in M - I_0} y^i\left(t\right) g^i{}_{\dot{u}}\left(t, u, \dot{u}\right)\right) + G p\left(t\right)\right] \mathrm{d}t \\ &\quad + \eta^T \left(\sum_{i \in M - I_0} y^i\left(t\right) \left(g^i_{\dot{u}}\left(t, u, \dot{u}\right) + w^i\left(t\right)\right)\right) \Bigg|_{t = a}^{t = b}. \end{aligned}$$

Using $\eta = 0$ at $t = a$ and $t = b$, we obtain,

$$\int_I \eta^T \left[\left(\sum_{i \in M - I_0} y^i(t) \left(g^i{}_u(t,u,\dot{u}) + w^i(t) \right) \right) \right.$$
$$\left. -D \left(\sum_{i \in M - I_0} y^i(t) g^i{}_{\dot{u}}(t,u,\dot{u}) \right) + Gp(t) \right] \mathrm{d}t \leq 0$$

Using (4), we have

$$\begin{aligned}
0 \leq\ & \int_I \eta^T \left[f_u(t,u,\dot{u}) + z(t) + \sum_{i \in I_0} y^i(t) \left(g_u^i(t,u,\dot{u}) + w^i(t) \right) \right. \\
& \left. -D \left(f_{\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g_u^i(t,u,\dot{u}) \right) + H^0 p(t) \right] \mathrm{d}t \\
=\ & \int_I \left[\eta^T \left(f_u(t,u,\dot{u}) + z(t) + \sum_{i \in I_0} y^i(t) \left(g_u^i(t,u,\dot{u}) + w^i(t) \right) \right) \right. \\
& + (D\eta)^T \left(f_{\dot{u}} + \sum_{i \in I_0} y^i(t) \left(g_u^i(t,u,\dot{u}) + w^i(t) \right) \right) \\
& \left. + \eta^T H^0 p(t) \right] \mathrm{d}t - \eta^T \left(f_{\dot{u}} + \sum_{i \in I_0} y^i(t)\ g_{\dot{u}}^i(t,u,\dot{u}) \right) \Bigg|_{t=a}^{t=b}.
\end{aligned}$$

From this, as earlier $\eta = 0$ at $t = a$ and $t = b$, we get,

$$\int_I \left[\eta^T \left(f_u\ (t,u,\dot{u}) +\ z(t) + \sum_{i \in I_0} y^i(t) \left(g_u^i(t,u,\dot{u}) + w^i(t) \right) \right) \right.$$
$$\left. +(D\eta)^T \left(f_{\dot{u}}\ (t,u,\dot{u}) + \sum_{i \in I_0} y^i(t)\ g_{\dot{u}}^i(t,u,\dot{u}) \right) + \eta^T H^0 p(t) \right] \mathrm{d}t \geq 0,$$

which by second-order pseudoinvexity of

$$\int_I \left(f(t,x,\dot{x}) + (.)^T z(t) + \sum_{i \in I_0} y^i(t) \left(g^i(t,x,\dot{x}) + (.)^T w^i(t) \right) \right) \mathrm{d}t$$

implies

$$\begin{aligned}
& \int_I \left(f(t,x,\dot{x}) + x(t)^T z(t) + \sum_{i \in I_0} y^i(t) \left(g^i(t,x,\dot{x}) + x(t)^T w^i(t) \right) \right) \mathrm{d}t \\
\geq & \int_I \left[\left(f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{i \in I_0} y^i(t) \left(g^i(t,u,\dot{u}) + u(t)^T w^i(t) \right) \right) - \frac{1}{2} p(t)^T H^0 p(t) \right] \mathrm{d}t.
\end{aligned}$$

Thus from $y(t) \geq 0$ and $g^i(t,x,\dot{x}) + S\left(x(t)\,|C^i\right) \leq 0$, $i = 1, 2...m$, $t \in I$. The above gives,

$$\int\limits_I \left(f\left(t,x,\dot{x}\right)+x(t)^T z\left(t\right)\right)\mathrm{d}t$$
$$\geq \int\limits_I \Bigg[\Bigg(f\left(t,u,\dot{u}\right)+u(t)^T z\left(t\right)+\sum_{i\in I_0} y^i\left(t\right)\left(g^i\left(t,u,\dot{u}\right)+u(t)^T w^i\left(t\right)\right)\Bigg)$$
$$-\frac{1}{2}p(t)^T H^0 p\left(t\right)\Bigg]\,\mathrm{d}t.$$

Since $x(t)^T z\left(t\right)\leq S\left(x\left(t\right)|K\right)$, $t\in I$, this inequality implies

$$\int\limits_I \left(f\left(t,x,\dot{x}\right)+S\left(x\left(t\right)|K\right)\right)\mathrm{d}t$$
$$\geq \int\limits_I \Bigg[\Bigg(f\left(t,u,\dot{u}\right)+u(t)^T z\left(t\right)+\sum_{i\in I_0} y^i\left(t\right)\left(g^i\left(t,u,\dot{u}\right)+u(t)^T\, w^i\left(t\right)\right)\Bigg)$$
$$-\frac{1}{2}p(t)^T H^0 p\left(t\right)\Bigg]\,\mathrm{d}t$$

yielding,

$$\text{infimum (CP)} \;\geq\; \text{supremum (GCD).}$$

□

Theorem 2 (Strong Duality): If $\bar{x}(t)$ is an optimal solution of (CP) and normal [3], then there exist piecewise smooth $\bar{y}:I\to R^m$, $\bar{z}:I\to R^n$ and $\bar{w}^i:I\to R^n$, $i=1,2,...,m$ such that $\left(\bar{x},\bar{y},\bar{z},\bar{w}^1,\bar{w}^2,......\bar{w}^m,\bar{p}\left(t\right)=0\right)$ is feasible for (GCD), and the corresponding values of (CP) and (GCD) are equal. If for all feasible $\left(\bar{x},\bar{y},\bar{z},\bar{w}^1,\bar{w}^2,......\bar{w}^m,\bar{p}\left(t\right)\right)$,

$$\int\limits_I \left\{f\left(t,.,.\right)+(.)^T\bar{z}\left(t\right)+\sum_{i\in I_0}\bar{y}^i\left(t\right)\left(g^i\left(t,.,.\right)+(.)^T\bar{w}^i\left(t\right)\right)\right\}\mathrm{d}t$$

is second-order pseudo-invex and

$$\sum_{i\in I_\alpha}\int\limits_I \bar{y}^i\left(t\right)\left(g^i\left(t,.,.\right)+(.)^T\bar{w}^i\left(t\right)\right)\mathrm{d}t\quad \alpha=1,2,...r$$

is second-order quasi-invex, then $\left(\bar{x},\bar{y},\bar{z},\bar{w}^1,\bar{w}^2,......\bar{w}^m,p(t)\right)$ is an optimal solution of (GCD).

Proof. Since $\bar{x}(t)$ is an optimal solution of (CP) and normal [3], then by Lemma 1, there exist piecewise smooth $\bar{y}:I\to R^m$, $\bar{z}:I\to R^n$ and $\bar{w}^i:I\to R^n$, $i=1,2,...,m$ such that

$$\left[f_x\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{z}\left(t\right)\right]+\sum_{i=1}^m\bar{y}^i\left(t\right)\left[g_x^i\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{w}^i\left(t\right)\right]$$
$$=D\left[f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{y}(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right],\quad t\in I$$

$$\sum_{i=1}^{m} \bar{y}^i(t)\left[g^i(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{w}^i(t)\right]=0,\quad t\in I$$

$$\bar{x}(t)^T\bar{z}(t)=S(\bar{x}(t)\,|K),\quad t\in I$$

$$\bar{x}(t)^T\bar{w}^i(t)=S\left(\bar{x}(t)\,|C^i\right),\quad i=1,2...m,\quad t\in I$$

$$\bar{z}(t)\in K,\bar{w}^i(t)\in C^i,\quad i=1,2...m,\quad t\in I$$

$$\bar{y}(t)\geq 0,\quad ,t\in I$$

This implies that $\left(\bar{x},\bar{y},\bar{z},\bar{w}^1,\bar{w}^2,......\bar{w}^m,\bar{p}(t)=0\right)$ is feasible for (GCD). Evidently, in view of $\bar{x}(t)^T\bar{z}(t)=S(\bar{x}(t)\,|K)$, $t\in I$, and $\sum_{i=1}^{m}\bar{y}^i(t)\left[g^i(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{w}^i(t)\right]=0$, $t\in I$, we have

$$\int_I \{f(t,\bar{x},\dot{\bar{x}})+S(\bar{x}(t)\,|K)\}\mathrm{d}t$$
$$=\int_I\left[f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{z}(t)+\sum_{i\in I_0}\bar{y}^i(t)\left(g^i(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{w}^i(t)\right)\right.$$
$$\left.-\frac{1}{2}\bar{p}(t)^TH^0\bar{p}(t)\right]\mathrm{d}t$$

and that is, objective values of (CP) and (GCD) are equal.

If

$$\int_I\left\{f(t,\bar{x},\dot{\bar{x}})+(.)^T\bar{z}(t)+\sum_{i\in I_o}\bar{y}^i(t)\ \left(g^i(t,\bar{x},\dot{\bar{x}})+(.)^T\bar{w}^i(t)\right)\right\}\mathrm{d}t$$

is second-order pseudo-invex, and

$$\sum_{i\in I_\alpha}\int_I \bar{y}^i(t)\left(g^i(t,.,.)+(.)^T\bar{w}^i(t)\right)\mathrm{d}t,\quad \alpha=1,2,....,r$$

is second-order quasi-invex with respect to the same η, then the optimality of

$$\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{w}^1(t),.....\bar{w}^m(t),\bar{p}(t)\right)$$

for (GCD) follows from weak duality theorem (Theorem 1).

□

Theorem 3 (Converse Duality): Let $\left(x,y,z,w^1,w^2,...w^m,p(t)\right)$ be an optimal solution of (GCD) at which

(A_1): for all $\alpha=1,2,3,...,r$, either

(a) $\int_I p(t)^T\left(G_\alpha+\sum_{i\in I_\alpha}(y^ig^i{}_x)_x\right)p(t)\,\mathrm{d}t>0$, and $\int_I p(t)^T\left(\sum_{i\in I_\alpha}y^i\left(g^i{}_x+w^i(t)\right)\right)\mathrm{d}t\geq 0$, or

(b) $\int_I p(t)^T\left(G_\alpha+\sum_{i\in I_\alpha}(y^ig^i{}_x)_x\right)p(t)\,\mathrm{d}t<0$, and $\int_I p(t)^T\left(\sum_{i\in I_\alpha}y^i\left(g^i{}_x+w^i(t)\right)\right)\mathrm{d}t\leq 0$, (A$_2$): the vectors H_j^0, G_j^α, $\alpha = 1,2,...,r$ and $j = 1,2,...,n$ are linearly independent, where H_j^0 is the j$^{\text{th}}$ row of the matrix H^0 and G_j^α is the j$^{\text{th}}$ row of the matrix G^α and

(A$_3$): the vectors $\sum\limits_{i\in I_\alpha}\left(y^i(t)\left(g_x^i + w^i(t)\right) - D\left(y^i(t)g_{\dot{x}}^i\right)\right)$, $\alpha = 1,2,3...r$ are linearly independent.

If, for all $\left(x, y, z, w^1, w^2, ..., p(t)\right)$,

$$\int_I \left\{ f(t,...) + (.)^T z(t) + \sum_{i\in I_0} y^i(t)\left(g^i(t,.,.) + (.)^T w^i(t)\right)\right\} \mathrm{d}t$$

is second-order pseudoinvex and

$$\int_I \sum_{i\in I_\alpha} y^i(t)\left(g^i(t,.,.) + (.)^T w^i(t)\right)\mathrm{d}t, \quad \alpha = 1,2...r$$

are second-order quasi-invex, then $x(t)$ is an optimal solution of (CP).

Proof. Since $\left(x, y, z, w^1, w^2, ...w^m, p(t)\right)$ is an optimal solution of (GCD), by Lemma 1, there exist $\tau_0 \in R$, $\tau_\alpha \in R$, $\alpha = 1,2,...,r$ and piecewise smooth $\theta : I \to R^m$, $r : I \to R^m$ and $w^i : I \to R^n$, $i = 1,2,...,m$ such that

$$\begin{aligned}
&\tau_0 \Bigg[\left(f_x + z(t) + \sum_{i\in I_0}\left(y^i(t)\left(g_x^i + w^i(t)\right)\right)\right) - D\left(f_{\dot{x}} + \sum_{i\in I_0}\left(y^i(t)g^i\right)_{\dot{x}}\right)\\
&\quad - \frac{1}{2}\left(p(t)^T H^0 p(t)\right)_x + \frac{1}{2}D\left(p(t)^T H^0 p(t)\right)_{\dot{x}} - \frac{1}{2}D^2\left(p(t)^T H^0 p(t)\right)_{\ddot{x}}\\
&\quad + \frac{1}{2}D^3\left(p(t)^T H^0 p(t)\right)_{\dddot{x}} - \frac{1}{2}D^4\left(p(t)^T H^0 p(t)\right)_{\ddddot{x}}\Bigg]\\
&+\theta(t)^T\Big[\left(f_{xx} + y(t)^T g_{xx}\right) + D\left(f_{x\dot{x}} + y(t)^T g_{x\dot{x}}\right) - D\left(D(f_{\dot{x}\dot{x}}) + y(t)^T g_{\dot{x}\dot{x}}\right)\\
&\quad + D^2\left(D(f_{\dot{x}\ddot{x}}) + (y(t)^T g_{\dot{x}})_{\ddot{x}}\right) + (Hp(t))_x - D(Hp(t))_{\dot{x}}\\
&\quad + D^2(Hp)_{\ddot{x}} - D^3(Hp)_{\dddot{x}} + D^4(Hp)_{\ddddot{x}}\Big]\\
&+\sum_{\alpha=1}^r \tau_\alpha \Bigg\{\sum_{i\in I_\alpha}\left(y^i(t)\left(g_x^i + w^i(t)\right) - D(y^i(t)g_x)\right) - \frac{1}{2}(p(t)G^\alpha p(t))_x\\
&\quad + \frac{1}{2}D(p(t)G^\alpha p(t))_{\dot{x}} - \frac{1}{2}D^2(p(t)G^\alpha p(t))_{\ddot{x}}\\
&\quad + \frac{1}{2}D^3(p(t)G^\alpha p(t))_{\dddot{x}} - \frac{1}{2}D^4(p(t)G^\alpha p(t))_{\ddddot{x}}\Bigg\} = 0
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\tau_0\left(g^i + x(t)^T w^i(t) - \frac{1}{2}p(t)^T g_{xx}^i p(t)\right)\\
&+\theta(t)^T\left(g_x^i + w^i(t) + g_{xx}^i p(t)\right) + r^i(t) = 0, \quad i \in I_0
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\tau_\alpha\left(g^i + x(t)^T w^i(t) - \frac{1}{2}p(t)^T g_{xx}^i p(t)\right)\\
&+\theta(t)\left(g_x^i + w^i(t) + g_{xx}^i p(t)\right) + r^i(t) = 0, \quad i \in I_\alpha, \quad \alpha = 1,2,3.......r
\end{aligned} \tag{10}$$

$$\tau_0 x(t) + \theta(t) \in N_K(\bar{z}(t)) \tag{11}$$

$$\tau_0 y^i(t)\, x(t) + \theta(t)\; y^i(t) \in N_{C^i}\left(w^i(t)\right), \quad i \in I_0 \tag{12}$$

$$\tau_\alpha y^i(t)\, x(t) + \theta(t)\, y^i(t) \in N_{C^i}\left(w^i(t)\right), \quad i \in I_\alpha, \quad \alpha = 1, 2, ...r. \tag{13}$$

$$(\tau_0 p(t) - \theta(t))\, H^0 + \sum_{\alpha=1}^{r} (\tau_\alpha p(t) - \theta(t)) G^\alpha = 0 \tag{14}$$

$$\tau_\alpha \int_I \left\{ \sum_{i \in I_\alpha} \left(y^i(t) \left(g^i + x(t)^T w^i(t) \right) \right) - \frac{1}{2} p(t)^T G^\alpha p(t) \right\} \mathrm{d}t = 0, \quad \alpha = 1, 2, 3...r \tag{15}$$

$$r(t)^T y(t) = 0, \quad t \in I \tag{16}$$

$$(\tau_0, \tau_1, \tau_2, ..., \tau_r, r(t)) \geq 0, \quad t \in I \tag{17}$$

$$(\tau_0, \tau_1, \tau_2, ..., \tau_r, r(t), \theta(t)) \neq 0, \quad t \in I \tag{18}$$

Because of assumption (A_2), (14) implies

$$\tau_\alpha p(t) - \theta(t) = 0, \quad \alpha = 0, 1, 2, ...r \tag{19}$$

Multiplying (10) by $y^i(t)$, $t \in I_\alpha$, $\alpha = 1, 2, ..., r$ and summing over i, we have,

$$\begin{aligned} &\tau_\alpha \left\{ y^i(t) \left(g^i + x(t)^T w^i(t) \right) - \frac{1}{2} p(t) \left(y^i g_x{}^i \right)_x p(t) \right\} \\ &+\theta(t) \left\{ \left(y^i g_x^i \right)_x + \left(y^i g_x^i \right)_x p(t) \right\} + y^i(t)\, r^i(t) = 0, \end{aligned}$$

$$\begin{aligned} &\tau_\alpha \left\{ \sum_{i \in I_\alpha} y^i(t) \left(g^i + x(t)^T w^i(t) \right) - \frac{1}{2} p(t)^T \sum_{i \in I_\alpha} \left(y^i g_x^i \right)_x p(t) \right\} \\ &+\theta(t) \left\{ \sum_{i \in I_\alpha} \left(y^i \left(g_x^i + w^i(t) \right) \right) + \sum_{i \in I_\alpha} \left(y^i g_x^i \right)_x p(t) \right\} = 0, \quad i \in I_\alpha, \quad \alpha = 1, 2, ..., r \end{aligned} \tag{20}$$

Using (19) in (20), we have

$$\begin{aligned} &\tau_\alpha \left\{ \sum_{i \in I_\alpha} y^i(t) \left(g^i + x(t)^T w^i(t) \right) - \frac{1}{2} p(t)^T \sum_{i \in I_\alpha} \left(y^i g_x^i \right)_x p(t) \right\} \\ &+\tau_\alpha p(t) \left\{ \sum_{i \in I_\alpha} \left(y^i \left(g_x^i + w^i(t) \right) \right) + \sum_{i \in I_\alpha} \left(y^i g_x^i \right)_x p(t) \right\} = 0, \quad i \in I_\alpha, \; \alpha = 1, 2, ..., r. \end{aligned} \tag{21}$$

$$
\begin{aligned}
&-\tau_\alpha\left\{\int\limits_I p(t)^T\left(\sum_{i\in I_\alpha} y^i\left(g_x^i+w^i(t)\right)\right)dt+\int\limits_I p(t)^T\left(\sum_{i\in I_\alpha}\left(y^i g_x^i\right)_x\right)p(t)\,\mathrm{d}t\right\}\\
&+\tau_\alpha\int\limits_I\left(\sum_{i\in I_\alpha}\left(y^i\left(g^i+x(t)^T w^i(t)\right)\right)-\frac{1}{2}p(t)^T\left(\sum_{i\in I_\alpha} y^i\left(g_x^i\right)_x\right)\right)p(t)\,\mathrm{d}t=0,\\
&\qquad\qquad\alpha=1,2,..,r.
\end{aligned}
\tag{22}
$$

This implies

$$
\begin{aligned}
&\tau_\alpha\left[\int\limits_I p(t)\left(\sum_{i\in I_\alpha}\left(y^i\left(g_x^i+w^i(t)\right)\right)\right)dt+\frac{1}{2}\int\limits_I p(t)^T\left(\sum_{i\in I_\alpha} y^i\left(g_x^i+w^i(t)\right)\right)p(t)\,\mathrm{d}t\right]\\
&+\frac{\tau_\alpha}{2}\int\limits_I p(t)\,G^\alpha p(t)\,\mathrm{d}t=0
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\tau_\alpha\int\limits_I p(t)^T\sum_{i\in I_\alpha}\left(y^i\left(g_x^i+w^i(t)\right)\right)\mathrm{d}t\\
&+\frac{\tau_\alpha}{2}\int\limits_I p(t)\left(\sum_{i\in I_\alpha}\left(y^i g_x^i\right)_x+G^\alpha\right)p(t)\,\mathrm{d}t=0
\end{aligned}
\tag{24}
$$

If for all $\alpha=0,1,2,...,r$, $\tau_\alpha=0$, then (19) implies $\theta(t)=0$, $t\in I$.
From (10), we have $r(t)=0$, $t\in I$.
Thus $(\tau_0,\tau_1,\tau_2,......\tau_r,r(t),\theta(t))=0,\ t\in I$.
This gives a contradiction. Hence there exists an $\bar{\alpha}\in\{0,1,2,.....r\}$ such that $\tau_{\bar{\alpha}}>0$.
If $p(t)\neq 0$, $t\in I$, thus (19) gives $(\tau_\alpha-\bar{\tau}_\alpha)\,p(t)=0,\ \alpha=1,2,3,...,r$.
This implies that $\tau_\alpha=\bar{\tau}_\alpha>0$, from (21), we have

$$
2\int\limits_I p(t)^T\left(\sum_{i\in I_\alpha} y^i\left(g_x^i+w^i(t)\right)\right)dt+\int\limits_I p(t)^T\left(G^\alpha+\sum_{i\in I_\alpha}\left(y^i g^i\right)_{xx}\right)p(t)dt=0
$$

This contradicts (A_1). Hence $p(t)=0$, $t\in I$.
Using (4) and $p(t)=0$, $t\in I$, the relation (8) gives

$$
\sum_{\alpha=1}^{r}(\tau_\alpha-\tau_0)\left\{\sum_{i\in I_\alpha}\left(y^i\left(g_x^i+w^i(t)\right)\right)-D\sum_{i\in I_\alpha}\left(y^i g_{\dot{x}}^i\right)\right\}=0. \tag{25}
$$

This by the linear independence of

$$
\left\{\sum_{i\in I_\alpha}\left(y^i\left(g_x^i+w^i(t)\right)\right)-D\sum_{i\in I_\alpha}\left(y^i g_{\dot{x}}^i\right)\right\},\ \alpha=1,2,3,...,r
$$

yields $\tau_\alpha = \tau_0$, for all $\alpha \in \{0, 1, 2, ..., r\}$.

From (11), (12) and (13) along with $\theta(t) = 0$, $t \in I$, we have $x(t) \in N_K(z(t))$, $x(t) \in N_{C^i}\left(w^i(t)\right)$, $i \in I_0$ and $x(t) \in N_{C^i}\left(w^i(t)\right)$, $i \in I_\alpha$, $\alpha = 1, 2, ..., r$. These implies from the definition of a cone $x(t)^T z(t) = S(x(t)/K)$, $x(t)^T w^i(t) = S\left(x(t)/C^i\right)$, $i \in I_0$ and $x(t)^T w^i(t) = S\left(x(t)/C^i\right)$, $i \in I_\alpha$, $\alpha = 1, 2, ..., r$.

Now (9) and (10) give

$$\tau_0\left(g^i + \bar{x}(t)^T w^i(t)\right) + r^i(t) = 0,\ t \in I,\ i \in I_0$$

implies

$$g^i(.) + \bar{x}(t)^T w^i(t) \leq 0, \quad i \in I_0 \tag{26}$$

$$\tau_\alpha\left(g^i + \bar{x}(t)^T w^i(t)\right) + r^i(t) = 0,\ t \in I,\ i \in I_\alpha$$

implies

$$g^i(.) + \bar{x}(t)^T w^i(t) \leq 0, \quad i \in I_\alpha, \quad \alpha = 1, 2, 3, ..., r. \tag{27}$$

Using $\bar{x}(t)^T w^i(t) = S\left(\bar{x}(t)/C^i\right)$, $i = 1, 2, ..., m$, in the above, we have $g^i(\ ) + S\left(\bar{x}(t)/C^i\right) \leq 0$, $i = 1, 2, ...m$, yielding the feasibility of $\bar{x}(t)$ for (CP).

From the above analysis, we have

$$\begin{aligned}
&\int_I \{f(t, x, \dot{x}) + S(x(t)|K)\}\mathrm{d}t \\
&= \int_I \Big[f(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T z(t) \\
&\quad + \sum_{i \in I_0} y^i(t)\left(g^i(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T w^i(t)\right) - \frac{1}{2}p(t)^T H^0 p(t)\Big]\,\mathrm{d}t
\end{aligned}$$

That is, the values of the objective functionals are identical. Consequently by weak duality theorem (Theorem 1) the optimality of $\bar{x}(t)$ for (CP) follows.

□

Theorem 4 (Strict Converse Duality): Assume that

$$\int_I \left\{f(t, x, \dot{x}) + (.)^T z(t) + \sum_{i \in I_o} y^i(t)\left(g^i(t, ., .) + (.)^T w^i(t)\right)\right\}\mathrm{d}t$$

is second-order strictly pseudo-invex and

$$\sum_{i \in I_\alpha} \int_I y^i(t)\left(g^i(t, ..,) + (.)^T w^i(t)\right)\mathrm{d}t, \quad \alpha = 1, 2, ..., r$$

are second-order quasi-invex with respect to same η. Assume also that (CP) has an optimal solution $\bar{x}(t)$. If $\left(\bar{u}, y, z, w^1, w^2, ..., w^m, p(t)\right)$ is an optimal solution of (GCD), then $\bar{u}(t)$ is an optimal solution of (CP) with $\bar{x}(t) = \bar{u}(t)$, $t \in I$.

Proof. We assume that $\bar{x}(t) \neq \bar{u}(t)$, $t \in I$ and exhibit a contradiction. Since $\bar{x}(t)$ is an optimal solutions of (CP), it follows from the Theorem 2 that there exist $y : I \to R^m$, $z : I \to R^n$ and $w^i : I \to R^n$, $i = 1, 2, ..., m$ such that $\left(\bar{x}, y, z, w^1, w^2, ..., w^m, p(t)\right)$ is an optimal solution of (GCD). Since

$$\left(\bar{u}, y, z, w^1, w^2, ..., w^m, p(t)\right)$$

is an optimal solution of (GCD), it implies that

$$\int_I \left\{ f\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{x}\left(t\right) z\left(t\right) + \sum_{i \in I_0} y^i\left(t\right) \left(g^i\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{x}(t)^T w^i\left(t\right) \right) \right\} \mathrm{d}t$$

$$= \int_I \left\{ f\left(t, \bar{u}, \dot{\bar{u}}\right) + u(t)^T z\left(t\right) + \sum_{i \in I_0} y^i\left(t\right) \left(g^i\left(t, \bar{u}, \dot{\bar{u}}\right) + \bar{u}(t)^T w^i\left(t\right) \right) - \frac{1}{2} p(t)^T H^0 p\left(t\right) \right\} \mathrm{d}t$$

This, in view of second-order strict pseudoinvexity of

$$\int_I \left\{ f\left(t, x, \dot{x}\right) + (.)^T z\left(t\right) + \sum_{i \in I_0} y^i\left(t\right) \left(g^i\left(t, ., .,\right) + (.)^T w^i\left(t\right) \right) \right\} \mathrm{d}t$$

yields

$$\begin{aligned} \int_I \Bigg\{ & \eta^T \left(f_u\left(t, u, \dot{u}\right) + z\left(t\right) + \sum_{i \in I_0} y^i\left(t\right) \left(g_u^i + w^i\left(t\right) \right) \right) \\ & + (D\eta)^T \left(f_{\dot{u}} + \sum_{i \in I_0} y^i\left(t\right) g_{\dot{u}}^i \right) + \eta^T H^0\ p(t) \Bigg\} \mathrm{d}t < 0 \end{aligned} \tag{28}$$

From the constraints of (CP) and (GCD) with $x(t)^T w^i\left(t\right) \leq S\left(x\left(t\right)/C^i\right)$, we have

$$\int_I \left(\sum_{i \in I_\alpha} y^i\left(t\right) \left(g^i\left(t, x, \dot{x}\right) + x(t)^T w^i\left(t\right) \right) \right) \mathrm{d}t$$

$$\leq \int_I \left(\sum_{i \in I_\alpha} y^i\left(t\right) \left(g^i\left(t, u, \dot{u}\right) + u(t)^T w^i\left(t\right) \right) - \frac{1}{2} p(t)^T G^\alpha p\left(t\right) \right) \mathrm{d}t \quad \alpha = 1, 2...r$$

This in view of second-order quasi-invexity of

$$\sum_{i \in I_\alpha} \int_I y^i\left(t\right) \left(g^i\left(t, ..,\right) + (.)^T w^i\left(t\right) \right) \mathrm{d}t, \quad \alpha = 1, 2, ...r,$$

yields

$$\begin{aligned} \int_I \Bigg[& \left(\eta^T \sum_{i \in I_\alpha} y^i\left(t\right) \left(g^i{}_u\left(t, u, \dot{u}\right) + w^i\left(t\right) \right) \right) + (D\eta)^T \left(\sum_{i \in I_\alpha} y^i\left(t\right) g^i{}_{\dot{u}}\left(t, u, \dot{u}\right) \right) \\ & + \eta^T G^\alpha p\left(t\right) \Bigg] \mathrm{d}t \leq 0, \quad \alpha = 1, 2, ..., r \end{aligned} \tag{29}$$

Combining (28) and (29), we have

$$\int_I \eta^T \Bigg[\Bigg(f_u + z\,(t) + \sum_{i=1}^{m} y^i\,(t)\left(g_u^i\,(t,u,\dot{u}) + w^i\,(t)\right)\Bigg)$$
$$- D\left(f_{\dot{u}} + y(t)^T g_{\dot{u}}\,(t,u,\dot{u})\right) + H\ p\,(t)\Bigg]\,\mathrm{d}t < 0$$

This contradicts the feasibility of $\left(\bar{u}, y, z, w^1, w^2, ..., w^m, p(t)\right)$ for (GCD). Hence $\bar{x}\,(t) = \bar{u}\,(t)$, $t \in I$. □

4. SPECIAL CASES

Let for $t \in I$, $A(t)$, $B^i(t)$, $i = 1, 2, ..., m$ be positive semidefinite matrices and continuous on I. Then

$$\left(x(t)^T A\,(t)\,x\,(t)\right)^{1/2} = S\,(x\,(t)\,|K\,)\,, \quad t \in I,$$

where

$$K = \left\{A\,(t)\,z\,(t)\,\middle|\,z(t)^T A\,(t)\,z\,(t) \le 1,\ t \in I\right\}$$

$$\left(x(t)^T B^i\,(t)\,x\,(t)\right)^{1/2} = S\left(x\,(t)\,\middle|C^i\right),\ i = 1, 2, ...m,\ t \in I$$

Replacing $S\,(x\,(t)\,|K\,)$ by $\left(x(t)^T A\,(t)\,x\,(t)\right)^{1/2}$ and $S\left(x\,(t)\,\middle|C^i\right),\ i = 1, 2, ..., m,\ t \in I$ by $\left(x(t)^T B^i\,(t)\,x\,(t)\right)^{1/2}$, we have the following problems.

(GCD): Maximize

$$\int_I \Bigg[f\,(t,u,\dot{u}) + u(t)^T A\,(t)\,z\,(t)$$
$$+ \sum_{i\in I_0} y^i\,(t)\left(g^i\,(t,u,\dot{u}) + u(t)^T B^i\,(t)\,w^i\,(t)\right) - \frac{1}{2}p(t)^T H^0 p\,(t)\Bigg]\,\mathrm{d}t$$

subject to

$$u\,(a) = 0, \quad u\,(b) = 0$$

$$f_u\,(t,u,\dot{u}) + A\,(t)\,z\,(t) + \sum_{i=1}^{m} y^i(t)^T\left({g^i}_u\,(t,u,\dot{u}) + B^i\,(t)\,w^i\,(t)\right)$$
$$- D\left(f_{\dot{u}}\,(t,u,\dot{u}) + y(t)^T g_{\dot{u}}\,(t,u,\dot{u})\right) + Hp\,(t) = 0, \quad t \in I,$$

$$\int_I \left(\sum_{i\in I_\alpha} y^i\,(t)\left({g^i}_u\,(t,u,\dot{u}) + B^i\,(t)\,w^i\,(t)\right) - \frac{1}{2}p(t)^T G^\alpha p\,(t)\right)\mathrm{d}t \ge 0,\ \ \alpha = 1, 2, 3, ..., r,$$

$$z(t)^T A\,(t)\,z\,(t) \le 1, \quad t \in I,$$

$$w^i(t)^T B^i\,(t)\,w^i\,(t) \le 1, \quad t \in I, \quad i \in 1, 2, ..., m,$$

$$y\,(t) \ge 0, \quad t \in I.$$

If $S\left(x\,(t)\,\middle|C^i\right)$, $i = 1, 2, ..., m$ are deleted from the constraints of (CP), then (CP) and (GCD) form a pair of dual problems treated by Husain and Srivastava [5].

5. PROBLEMS WITH NATURAL BOUNDARY VALUES

In this section, we formulate a pair of nondifferentiable dual variational problems with natural boundary values rather than fixed end points:

(CP$_0$): Minimize

$$\int_I \{f(t,x,\dot{x}) + S(x(t)|K)\}\, dt$$

subject to

$$x(a) = 0 = x(b)$$

$$g^i(t,x,\dot{x}) + S\left(x(t)|C^i\right) \le 0, \quad t \in I, \quad i = 1,2,...,m.$$

(GCD$_0$): Maximize

$$\int_I \left[f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{i\in I_0} y^i(t)\left(g^i(t,u,\dot{u}) + u(t)^T w^i(t)\right) - \frac{1}{2} p(t)^T H^0 p(t) \right] \mathrm{d}t$$

subject to

$$u(a) = 0, \quad u(b) = 0$$

$$f_u(t,u,\dot{u}) + z(t) + \sum_{i=1}^m y^i(t)^T \left({g^i}_u(t,u,\dot{u}) + w^i(t)\right)$$

$$- D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u})\right) + Hp(t) = 0, \quad t \in I,$$

$$\int_I \left(\sum_{i\in I_\alpha} y^i(t) \left(g^i(t,u,\dot{u}) + u(t) w^i(t)\right) - \frac{1}{2} p(t)^T G^\alpha p(t) \right) \mathrm{d}t \ge 0$$

$$\alpha = 1,2,3,...,r,$$

$$z(t) \in K, \quad w^i(t) \in C^i, \quad t \in I, \quad i = 1,2,...,m,$$

$$y(t) \ge 0, \quad t \in I,$$

$$f_{\dot{u}}(t,u,\dot{u}) = 0, \quad \text{at} \quad t = a \quad \text{and} \quad t = b,$$

$$y^i(t)^T g^i_{\dot{u}}(t,u,\dot{u}) = 0, \quad i = 1,2,....m \quad \text{at} \quad t = a \quad \text{and} \quad t = b.$$

6. NONLINEAR PROGRAMMING PROBLEMS

If all functions in the problems (CP$_0$) and (GCD$_0$) are independent of t, then these problems will reduce to the following nonlinear programming problems studied by Husain *et al.* [9].

(CP$_1$): Minimize $f(x) + S(x|K)$ subject to

$$g^i(x) + S\left(x|C^i\right) \le 0, \quad i = 1,2...m.$$

(GCD$_1$): Maximize

$$f(u) + u^T z + \sum_{i\in I_0} y^i\left(g^i + uw^i\right) - \frac{1}{2} p^T H^0 p$$

subject to

$$f_u(u) + z + \sum_{i \in I_0} y^i \left(g_u^i(u) + w^i\right) + Hp = 0,$$

$$\sum_{i \in I_\alpha} y^i \left(g^i + uw^i\right) - \frac{1}{2} p^T G^\alpha p \geq 0, \quad \alpha = 1, 2, ..., r.$$

$$z \in K, \quad w^i \in C^i, \quad i = 1, 2, ..., m., \qquad y \geq 0.$$

$$H^0 = f_{uu} + \left(\sum_{i \in I_0} y^i g^i\right)_{uu}'$$

$$H = f_{uu} + \left(\sum_{i \in I_\alpha} y^i g^i\right)_{uu}$$

and

$$G^\alpha = \left(\sum_{i \in I_\alpha} y^i g^i\right)_{uu}, \quad \alpha = 1, 2, ..., r.$$

7. CONCLUSIONS

In this paper, a generalized second-order dual is formulated for a continuous programming problem in which support functions appear in both objective and constraint functions. Various duality theorems are derived under second-order pseudoinvexity and second-order quasi-invexity for this pair of dual continuous programming problems. Some special cases are obtained. A pair of dual continuous programming problems with natural boundary values is formulated and hence it is indicated that the duality results for the pair can be proved analogously to those of the dual models with fixed end points. A linkage between our duality results and those of the corresponding (static) nonlinear programming problem with support functions is pointed out. Our results in this research can be elegantly extended in the context of a class of nondifferentiable multiobjective continuous programming problems.

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