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Oscillation and Nonoscillation Theorems for a Class of Fourth Order Quasilinear Difference Equations

LIU LanChu[a],* and GAO Youwu[a]

[a] College of Science, Hunan Institute of Engineering, China.

* Corresponding author.
Address: College of Science, Hunan Institute of Engineering, 88 East Fuxing Road, Xiangtan 411104, China; E-Mail: llc0202@163.com

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Abstract: In this paper,we consider certain quasilinear difference equations

$$(A) \qquad \Delta^2(\mid \Delta^2 y_n \mid^{\alpha-1} \Delta^2 y_n) + q_n \mid y_{\tau(n)} \mid^{\beta-1} y_{\tau(n)} = 0$$

where
(a) α, β are positive constants;
(b) $\{q_n\}_{n_0}^{\infty}$ are positive real sequences. $n_0 \in N_0 = \{1, 2, \cdots\}$. Oscillation and nonoscillation theorems of the above equation is obtained.

Key words: Quasilinear difference equations; Oscillation and nonoscillation theorems; Four order

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1. INTRODUCTION

In this paper, we consider certain quasilinear difference equations

$$(A) \qquad \Delta^2(\mid \Delta^2 y_n \mid^{\alpha-1} \Delta^2 y_n) + q_n \mid y_{\tau(n)} \mid^{\beta-1} y_{\tau(n)} = 0,$$

where

(a) α, β are positive constants;

(b) $\{q_n\}_{n_0}^{\infty}$ are positive real sequences. $n_0 \in N_0 = \{1, 2, \cdots\}$.

(c) $\tau(n) \leq n$, and $\lim_n \to \infty \tau(n) = \infty$

The Equation (A) can also be expressed as

$$\Delta^2((\Delta^2 y_n)^{\alpha_*}) + q_n(y_{\tau(n)})^{\beta_*} = 0, \tag{1.1}$$

in terms of the asterisk notation

$$\xi^{\gamma_*} = \mid \xi \mid^{\gamma} sgn\xi = \mid \xi \mid^{\gamma-1} \xi, \qquad \xi \in R, \quad \gamma > 0.$$

It is clear that if $\{y_n\}$ is a eventually positive solution of (1.1), then $-\{y_n\}$ is a eventually negative solution of (1.1).

Lemma 1.1. Assume that $\{y_n\}$ is a eventually positive solution of (1.1). then one of the following two cases holds for all sufficiently large n:

$$\text{I}: \qquad \Delta y_n > 0, \qquad \Delta^2 y_n > 0, \qquad \Delta(\Delta^2 y_n)^{\alpha_*} > 0$$

$$\text{II}: \qquad \Delta y_n > 0, \qquad \Delta^2 y_n < 0, \qquad \Delta(\Delta^2 y_n)^{\alpha_*} > 0$$

Proof. From (1.1), we have $\Delta^2((\Delta^2 y_n)^{\alpha_*}) < 0$ for all large n. It follows that Δy_n, $\Delta^2 y_n$, $\Delta(\Delta^2 y_n)^{\alpha_*}$ are eventually monotonic and one-signed.

(A) if $\Delta(\Delta^2 y_n)^{\alpha_*} < 0$ eventually. Then combining this with $\Delta^2((\Delta^2 y_n)^{\alpha_*}) < 0$, we see that $\lim_{n\to\infty}(\Delta^2 y_n)^{\alpha_*} = -\infty$. That is $\Delta^2 y_n \to -\infty$ for all large n. It follows that $\Delta y_n \to -\infty$, $y_n \to -\infty$, which contradicts the positivity of $\{y_n\}$.

(B) if $\Delta(\Delta^2 y_n)^{\alpha_*} > 0$ eventually. Then combining this with $\Delta^2((\Delta^2 y_n)^{\alpha_*}) < 0$, we see that $\Delta(\Delta^2 y_n)^{\alpha_*} \to 0$ or $\to a > 0$ so

$$(\Delta^2 y_n)^{\alpha_*} = (\Delta^2 y_N)^{\alpha_*} + \sum_{N}^{n-1} (\Delta^2 y_n)^{\alpha_*}.$$

If $(\Delta^2 y_n)^{\alpha_*} > 0$. That is $\Delta^2 y_n > 0$ is increasing and $\to C$ or ∞. It follows that $\Delta y_n > 0$; If $(\Delta^2 y_n)^{\alpha_*} < 0$. That is $\Delta^2 y_n < 0$ is increasing and $\to d$ or 0. If $\Delta y_n < 0$, then $y_n \to \infty$, it is impossible, so $\Delta y_n > 0$. This complete the proof of the lemma. □

From Lemma (1.1), we know $y_n, \Delta y_n, \Delta^2 y_n, \Delta(\Delta^2 y_n)^{\alpha_*}$ tend to finite or infinite limits as $n \to \infty$. Let

$$\lim_{n\to\infty} \Delta^i y_n = \omega_i, \quad i = 0, 1, 2, \quad \text{and} \quad \lim_{n\to\infty} \Delta(\Delta^2 y_n)^{\alpha_*} = \omega_3.$$

It is that ω_3 is a finite nonnegative number. One can easily show that:

If y_n satisfies I, then the set of its asymptotic values ω_i falls into one of the following three cases:

$\text{I}_1 : \omega_0 = \omega_1 = \omega_2 = \infty, \omega_3 \in (0, \infty)$;

$\text{I}_2 : \omega_0 = \omega_1 = \omega_2 = \infty, \omega_3 = 0$;

$\text{I}_3 : \omega_0 = \omega_1 = \infty, \omega_2 \in (0, \infty), \omega_3 = 0$.

If y_n satisfies II, then the set of its asymptotic values ω_i falls into one of the following three cases:

$\text{II}_1 : \omega_0 = \infty, \omega_1 \in (0, \infty), \omega_2 = \omega_3 = 0$

$\mathrm{II}_2 : \omega_0 = \infty, \omega_1 = \omega_2 = \omega_3 = 0$

$\mathrm{II}_3 : \omega_0 \in (0, \infty), \omega_1 = \omega_2 = \omega_3 = 0.$

Equivalent expressions for these six classes of positive solutions of (1.1) are as follows:

$\mathrm{I}_1 : \lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = const > 0;$

$\mathrm{I}_2 : \lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = 0, \lim_{n\to\infty} \frac{y_n}{n^2} = \infty;$

$\mathrm{I}_3 : \lim_{n\to\infty} \frac{y_n}{n^2} = const > 0;$

$\mathrm{II}_1 : \lim_{n\to\infty} \frac{y_n}{n} = const > 0;$

$\mathrm{II}_2 : \lim_{n\to\infty} \frac{y_n}{n} = 0, \lim_{n\to\infty} y_n = \infty;$

$\mathrm{II}_3 : \lim_{n\to\infty} y_n = const.$

Let y_n be a positive solution of (1.1) such that $y_n > 0$, $y_{\tau(n)} > 0$ for $n \geq N > n_0$. summing (1.1) from n to ∞ gives

$$\Delta(\Delta^2 y_n)^{\alpha_*} = \omega_3 + \sum_{s=n}^{\infty} q_s (y_{\tau(n)})^{\beta}, \quad n \geq N. \tag{1.2}$$

If y_n is a solution of type $I_i (i = 1, 2, 3)$, then sum (1.2) three times over $[N, n-1]$ to obtain

$$y_n = k_0 + k_1(n-N) + \sum_{s=N}^{n-1} (n-s) \left[{k_2}^{\alpha} + \sum_{r=n}^{s-1} \left(\omega_3 + \sum_{\sigma=r}^{\infty} q_\sigma (y_{\tau(\sigma)})^{\beta} \right) \right]^{\frac{1}{\alpha}}, \tag{1.3}$$

for $n \geq N$ where $k_0 = y_N,\ k_1 = \Delta y_N,\ k_2 = \Delta^2 y_N$ are nonnegative constants. The equality (1.3) gives a representation for a solution y_n of $type - I_1$. A $type - I_2$ solution y_n of (1.1) is expressed by (1.3) with $\omega_3 = 0$.

If y_n is a solution of type I_3, then, first summing (1.1) from n to ∞ and then summing the resulting equation twice times over $[N, n-1]$ to obtain

$$y_n = k_0 + k_1(n-N) + \sum_{s=N}^{n-1} (n-s) \left[{\omega_2}^{\alpha} - \sum_{r=s}^{\infty} (r-s) q_r (y_{\tau(r)})^{\beta} \right]^{\frac{1}{\alpha}}, \quad n > N \tag{1.4}$$

A representation for a solution y_n of type II_1 is derived by summing (1.2) with $\omega_3 = 0$ twice from n to ∞ and then once from N to $n-1$:

$$y_n = k_0 + \sum_{s=N}^{n-1} \left(\omega_1 + \sum_{r=s}^{\infty} \left[\sum_{\sigma=r}^{\infty} (\sigma - r) q_\sigma (y_{\tau(\sigma)})^{\beta} \right] \right)^{\frac{1}{\alpha}}, \quad n > N \tag{1.5}$$

a representation for a solution y_n of type II_2 is given by (1.5) with $\omega_1 = 0$. a representation for a solution y_n of type II_3 is derived by summing (1.2) with $\omega_3 = 0$ three times from n to ∞ yield

$$y_n = \omega_0 - \sum_{s=n}^{\infty} (s-n) \left[\sum_{r=s}^{\infty} (r-s) q_r (y_{\tau(r)})^{\beta} \right]^{\frac{1}{\alpha}}, \quad n > N \tag{1.6}$$

2. NONOSCILLATION CRITERIA

Theorem 1. The equation (1.1) has a positive of $type - I_1$ if and only if

$$\sum_{n=n_0}^{\infty} q_n(\tau(n))^{2+\frac{1}{\alpha}}\beta < \infty \tag{2.1}$$

Proof. Necessary. Suppose that (1.1) has a positive of $type - I_1$, then, it satisfies (1.3) for $n \geq N$, which implies that

$$\sum_{n=N}^{\infty} q_n(y_{\tau(n)})^{\beta} < \infty$$

This together with the asymptotic relation $\lim\limits_{n\to\infty} \dfrac{y_n}{n^{2+\frac{1}{\alpha}}} = const > 0$; shows that the condition (2.1) is satisfied.

Sufficiently. Suppose now that (2.1) holds. Let $k > 0$ be any given constant. Choose $N > n_0$ large enough so that

$$\left(\frac{\alpha^2}{(\alpha+1)(2\alpha+1)}\right)^{\beta} \sum_{n=n_0}^{\infty} q_n(\tau(n))^{2+\frac{1}{\alpha}}\beta \leq \frac{(2k)^{\alpha} - k^{\alpha}}{(2k)^{\beta}} \tag{2.2}$$

Put $N_* = min\{N, \inf\limits_{n>N} \tau(n)\}$, and define

$$G(n,N) = \sum_{s=N}^{n-1}(n-s)(s-N)^{\frac{1}{\alpha}} = \frac{\alpha^2}{(\alpha+1)(2\alpha+1)}(n-N)^{\frac{2}{1+\alpha}} \quad n \geq N$$
$$G(n,N) = 0 \qquad\qquad n < N$$

Let B_N be the Banach space of all real sequences $Y = \{y_n\}$, with the norm $\| Y \| = \sup\limits_{n>n_0} | y_n | < \infty$ we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega = \{Y = \{y_n\} \in B_N \quad kG(n,N) \leq y_n \leq 2kG(n,N),\ n \geq N_*\}$$

Define the map $T : \Omega \to B_N$ as follows:

$$\begin{cases} Ty_n = \sum\limits_{N}^{n-1}(n-s)\left[\sum\limits_{N}^{s-1}(k^{\alpha} + \sum\limits_{\sigma=r}^{\infty} q_{\sigma}(y_{\tau(\sigma)})^{\beta})\right]^{\frac{1}{\alpha}}, & n \geq N \\ Ty_n = Ty_N, & N_* \leq n \leq N \end{cases} \tag{2.3}$$

I) T maps Ω into Ω. For $y_n \in \Omega$, then for $n \geq N$

$$Ty_n \geq k\sum_{N}^{n-1}(n-s)(s-N)^{\frac{1}{\alpha}} = kG(n,N)$$

and

$$\begin{aligned} Ty_n &\le \sum_{N}^{n-1}(n-s)\left[\sum_{N}^{s-1}\left(k^{\alpha}+\sum_{\sigma=r}^{\infty}q_{\sigma}(2k\tau(\tau(\sigma),N)^{\beta})\right)\right]^{\frac{1}{\alpha}} \\ &\le \sum_{N}^{n-1}(n-s)\left[\sum_{N}^{s-1}\left(k^{\alpha}+(\frac{2k\alpha^2}{(\alpha+1)(2\alpha+1)})^{\beta}\right)\sum_{\sigma=r}^{\infty}q_{\sigma}(\tau(\sigma))^{(2+\frac{1}{\alpha})\beta}\right]^{\frac{1}{\alpha}} \\ &\le 2k\sum_{N}^{n-1}(n-s)(s-N)^{\frac{1}{\alpha}} = 2kG(n,N) \end{aligned}$$

II) T is continuous. Let $y^{(k)} \in \Omega$ such that $\lim_{k\to\infty} \| y^{(k)} - y \| = 0$

$$\begin{aligned} &\left|(Ty^{(k)})_n - (Ty)_n\right| \\ = &\sum_{N}^{n-1}(n-s)\left[\sum_{N}^{s-1}\left(k^{\alpha}+\sum_{\sigma=r}^{\infty}q_{\sigma}(y^{(k)}_{\tau(\sigma)})^{\beta}\right)\right]^{\frac{1}{\alpha}} \\ &-\sum_{N}^{n-1}(n-s)\left[\sum_{N}^{s-1}\left(k^{\alpha}+\sum_{\sigma=r}^{\infty}q_{\sigma}(y_{\tau(\sigma)})^{\beta}\right)\right]^{\frac{1}{\alpha}} \end{aligned}$$

by using Lebesgue's dominated convergence theorem, we can conclude that

$$\lim_{n\to\infty} \| Ty^{(k)} - Ty \| = 0$$

III) T is uniformly-cauchy, $\forall n_1, n_2 > N_*$

$$\begin{aligned} |Ty_{n_1} - Ty_{n_2}| &= \sum_{N}^{n_2-1}(n-s)\left[\sum_{N}^{s-1}\left(k^{\alpha}+\sum_{\sigma=r}^{\infty}q_{\sigma}(y_{\tau(\sigma)})^{\beta}\right)\right]^{\frac{1}{\alpha}} \\ &\quad -\sum_{N}^{n_1-1}(n-s)\left[\sum_{N}^{s-1}\left(k^{\alpha}+\sum_{\sigma=r}^{\infty}q_{\sigma}(y_{\tau(\sigma)})^{\beta}\right)\right]^{\frac{1}{\alpha}} \\ &= \sum_{n_1}^{n_2-1}(n-s)\left[\sum_{N}^{s-1}\left(k^{\alpha}+\sum_{\sigma=r}^{\infty}q_{\sigma}(y_{\tau(\sigma)})^{\beta}\right)\right]^{\frac{1}{\alpha}} \end{aligned}$$

Therefore, by the Schauder fixed point theorem, there exists a fixed $Ty = y$, which satisfies (1.1). This completes the proof. □

Theorem 2. The equation (1.1) has a positive of $type - I_3$ if and only if

$$\sum_{n=n_0}^{\infty} nq_n(\tau(n))^{2\beta} < \infty \tag{2.4}$$

Proof. Necessary. Suppose that (1.1) has a positive of $type - I_3$, then, it satisfies (1.4) for $n \ge N$, which implies that

$$\sum_{n=N}^{\infty}(n-N)q_n(y_{\tau(n)})^{\beta} < \infty$$

This together with the asymptotic relation $\lim_{n\to\infty}\frac{y_n}{n^2}=const>0$; shows that the condition (2.2) is satisfied.

Sufficiently. Suppose now that (2.2) holds. Let $k>0$ be any given constant. Choose $N>n_0$ large enough so that

$$\sum_{n=N}^{\infty} nq_n(\tau(n))^{2\beta}\leq\frac{(2k)^{\alpha}-k^{\alpha}}{(k)^{\beta}} \tag{2.5}$$

Put $N_*=min\{N,\inf_{n>N}\tau(n)\}$. Let B_N be the Banach space of all real sequences $Y=\{y_n\}$, with the norm $\parallel Y\parallel=\sup_{n>n_0}\mid y_n\mid<\infty$ we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega=\{Y=\{y_n\}\in B_N\quad \frac{2}{k}(n-N)_+^2\leq y_n\leq k(n-N)_+^2, n\geq N_*\}$$

where $n-N_+=n-N$ if $n\geq N$, and $n-N_+=0$ if $n\leq N$. Define the map $T:\Omega\to B_N$ as follows:

$$\begin{cases} Ty_n=\sum\limits_{N}^{n-1}(n-s)\left[2k^{\alpha}-\sum\limits_{r=s}^{\infty}(r-s)q_r(y_{\tau(r)})^{\beta}\right]^{\frac{1}{\alpha}}, & n\geq N \\ Ty_n=Ty_N & N_*\leq n\leq N \end{cases}$$

The proof is similar to that of Theorem 1 and there exists an element y such that $y=Ty$, which is a $type-I_3$ solution of (1.1) with the property that $\lim_{n\to\infty}\Delta_2 y_n=2k>0$. This completes the proof. □

Theorem 3. The equation (1.1) has a positive of $type-II_1$ if and only if

$$\sum_{n=N}^{\infty}\left[\sum_{s=n}^{\infty}(s-n)q_s(\tau(s))^{\beta}\right]^{\frac{1}{\alpha}}<\infty \tag{2.6}$$

Proof. Necessary. Suppose that (1.1) has a positive of $type-II_1$, then, it satisfies (1.4) for $n\geq N$, which implies that

$$\sum_{n=N}^{\infty}(n-N)q_n(y_{\tau(n)})^{\beta}<\infty$$

This together with the asymptotic relation $\lim_{n\to\infty}\frac{y_n}{n}=const>0$; shows that the condition (2.6) is satisfied.

Sufficiently. Suppose now that (2.6) holds. Let $k>0$ be any given constant. Choose $N>n_0$ large enough so that

$$\sum_{n=N}^{\infty}\left[\sum_{s=n}^{\infty}(s-n)q_s{y_{\tau(s)}}^{\beta}\right]^{\frac{1}{\alpha}}<2^{\frac{-\beta}{\alpha}}k^{1-\frac{\beta}{\alpha}}$$

Put $N_*=min\{N,\inf_{n>N}\tau(n)\}$. Let B_N be the Banach space of all real sequences $Y=\{y_n\}$, with the norm $\parallel Y\parallel=\sup_{n>n_0}|y_n|<\infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega=\{Y=\{y_n\}\in B_N\quad kn\leq y_n\leq 2kn, n\geq N_*\}$$

Define the map $T : \Omega \to B_N$ as follows:

$$\begin{cases} Ty_n = kn + \sum\limits_{N}^{n-1} \sum\limits_{s}^{\infty} \left[\sum\limits_{r}^{\infty} (\sigma - r) q_\sigma (y_{\tau(\sigma)})^\beta \right]^{\frac{1}{\alpha}}, & n \geq N \\ Ty_n = kn & N_* \leq n \leq N \end{cases} \tag{2.7}$$

The proof is similar to that of Theorem 1 and there exists an element y such that $y = Ty$, which is a $type - II_1$ solution of (1.1) with the property that $\lim\limits_{n\to\infty} \Delta y_n = k > 0$; This completes the proof. □

Theorem 4. The equation (1.1) has a positive of $type - II_3$ if and only if

$$\sum_{n=n_0}^{\infty} n \left[\sum_{s=n}^{\infty} (s-n) q_s \right]^{\frac{1}{\alpha}} < \infty \tag{2.8}$$

Proof. Necessary. Suppose that (1.1) has a positive of $type - II_3$, then, it satisfies (1.6) for $n \geq N$, which implies that

$$\sum_{n=N}^{\infty} n \left[\sum_{s=n}^{\infty} (s-n) q_s (y_{\tau(s)})^\beta \right]^{\frac{1}{\alpha}} < \infty \tag{2.9}$$

This together with the asymptotic relation $\lim\limits_{n\to\infty} y_n = const > 0$; shows that the condition (2.8) is satisfied.

Sufficiently. Suppose now that (2.8) holds. Let $k > 0$ be any given constant. Choose $N > n_0$ large enough so that

$$\sum_{n=N}^{\infty} n \left[\sum_{s=n}^{\infty} (s-n) q_s {y_{\tau(s)}}^\beta \right]^{\frac{1}{\alpha}} < \frac{1}{2} k^{1-\frac{\beta}{\alpha}} \tag{2.10}$$

Put $N_* = min\{N, \inf\limits_{n>N} \tau(n)\}$. Let B_N be the Banach space of all real sequences $Y = \{y_n\}$, with the norm $\| Y \| = \sup\limits_{n>n_0} |y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega = \{Y = \{y_n\} \in B_N \quad \frac{k}{2} \leq y_n \leq k, n \geq N_*\}$$

Define the map $T : \Omega \to B_N$ as follows:

$$\begin{cases} Ty_n = k - \sum\limits_{n}^{\infty} (s-n) \left[\sum\limits_{r=s}^{\infty} (r-s) q_r (y_{\tau(r)})^\beta \right]^{\frac{1}{\alpha}}, & n \geq N \\ Ty_n = Ty_N & N_* \leq n \leq N \end{cases} \tag{2.11}$$

The proof is similar to that of Theorem 1 and there exists an element y such that $y = Ty$, which is a $type - II_1$ solution of (1.1) with the property that $\lim\limits_{n\to\infty} \Delta y_n = k > 0$; This completes the proof. □

Theorem 5. The equation (1.1) has a positive of $type - I_2$ if

$$\sum_{n=n_0}^{\infty} q_n(\tau(n))^{(2+\frac{1}{\alpha})\beta} \leq \infty \tag{2.12}$$

and

$$\sum_{n=n_0}^{\infty} nq_n(\tau(n))^{2\beta} = \infty \tag{2.13}$$

Proof. Suppose now that (2.12) holds. Choose $N > n_0$ large enough so that

$$\sum_{n=N}^{\infty} q_n(\tau(n))^{(2+\frac{1}{\alpha})\beta} \leq \frac{1}{2^{\alpha+1}}\left(\frac{(\alpha+1)(2\alpha+1)}{\alpha^2}\right)^{\alpha} \tag{2.14}$$

Put $N_* = min\{N, \inf_{n>N} \tau(n)\}$. Let B_N be the Banach space of all real sequences $Y = \{y_n\}$, with the norm $\| Y \| = \sup_{n>n_0} |y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega = \{Y = \{y_n\} \in B_N \quad \frac{1}{2^{1+\frac{1}{\alpha}}}(n-N)_+^2 \leq y_n \leq n^{2+\frac{1}{\alpha}} \qquad n \geq N_*\}$$

Define the map $T : \Omega \to B_N$ as follows:

$$\begin{cases} Ty_n = \sum\limits_{N}^{n-1}(n-s)\left[\frac{1}{2}\sum\limits_{N}^{s-1}\sum\limits_{\sigma=r}^{\infty}(\sigma)q_\sigma(y_{\tau(\sigma)})^{\beta}\right]^{\frac{1}{\alpha}}, & n \geq N \\ Ty_n = 0 & N_* \leq n \leq N \end{cases} \tag{2.15}$$

The proof is similar to that of Theorem 1 and there exists an element y such that $y = Ty$, which is a $type - I_2$ solution of (1.1) This completes the proof. □

Theorem 6. The equation (1.1) has a positive of $type - II_2$ if

$$\sum_{n}^{\infty} n\left[\sum_{n}^{\infty}(s-n)q_s(\tau(s))^{\beta}\right]^{\frac{1}{\alpha}} < \infty \tag{2.16}$$

and

$$\sum_{n=n_0}^{\infty}\left[\sum_{n}^{\infty}(s-n)q_s\right]^{\frac{1}{\alpha}} = \infty \tag{2.17}$$

Proof. Suppose now that (2.16) holds. Choose $N > n_0$ large enough so that

$$\sum_{N}^{\infty} n\left[\sum_{n}^{\infty}(s-n)q_s(\tau(s))^{\beta}\right]^{\frac{1}{\alpha}} \leq 2^{\frac{-\beta}{\alpha}}k^{1-\frac{\beta}{\alpha}} \tag{2.18}$$

Put $N_* = min\{N, \inf_{n>N} \tau(n)\}$. Let B_N be the Banach space of all real sequences $Y = \{y_n\}$, with the norm $\| Y \| = \sup_{n>n_0} |y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega = \{Y = \{y_n\} \in B_N \quad k \leq y_n \leq 2kn, n \geq N_*\}$$

Define the map $T : \Omega \to B_N$ as follows:

$$\begin{cases} Ty_n = k + \sum_{N}^{n-1} \sum_{s}^{\infty} \left[\sum_{\sigma=r}^{\infty} (\sigma - r) q_\sigma (y_{\tau(\sigma)})^\beta \right]^{\frac{1}{\alpha}}, & n \geq N \\ Ty_n = k & N_* \leq n \leq N \end{cases} \tag{2.19}$$

The proof is similar to that of Theorem 1 and there exists an element y such that $y = Ty$, which is a $type - II_2$ solution of (1.1). This completes the proof. □

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