CSCanada
Studies in Mathematical Sciences
Vol. 6, No. 1, 2013, pp. [1–7]
DOI: 10.3968/j.sms.1923845220130601.2290

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

General Numerical Radius Inequalities

Mohammed Al-Dolat[a] and Mohammed Ali[a],*

[a] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan.

* Corresponding author.
Address: Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan; E-Mail: myali@just.edu.jo

Received: November 6, 2012/ Accepted: January 18, 2013/ Published: February 28, 2013

Abstract: In this article, we establish new numerical radius inequalities for bounded linear operators on a complex Hilbert space. Also, we generalize some known inequalities.

Key words: Numerical radius; Commuting operators; Hilbert space

Al-Dolat, M., & Ali, M. (2013). General Numerical Radius Inequalities. *Studies in Mathematical Sciences, 6*(1), 1–7. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220130601.2290 DOI: 10.3968/j.sms.1923845220130601.2290

1. INTRODUCTION

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$, and let $B(H)$ be the space of C^*-algebra of all bounded linear operators on H. For $T \in B(H)$, let T^* denote the adjoint operator of T. Also, let $w(T)$ denote the numerical radius of T given by

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in H, ||x|| = 1\}.$$

It is well known that $w(.)$ is a norm on $B(H)$, which is equivalent to the usual operator norm $\| . \|$ defined, for $T \in B(H)$, by

$$\| T \| = \sup \{\|Tx\| : x \in H, \|x\| = 1\},$$

where $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. More particular, for $T \in B(H)$, it is known [1] that

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|. \tag{1.1}$$

It is clear that when $T^2 = 0$, we get $w(T) = \frac{1}{2}\|T\|$ (see [2]); however, $w(T) = \|T\|$ is satisfied when A is normal bounded operator.

Several numerical radius inequalities that provide alternative lower and upper bounds for $w(.)$ have received much attention from many authors. We refer the readers to [3], [4], [1], and [5] for the history and significance, and [6], [7], [2], [8], and [9] for recent developments in this area. For example, Kittaneh [2] proved that for $T \in H$,

$$w(T) \leq \frac{1}{2}\||T| + |T^*|\| \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{\frac{1}{2}}\right), \tag{1.2}$$

where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T. Kittaneh [9] improved inequality (1.1), and in [9], he determined that

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{1.3}$$

Although some open problems related to the numerical radius inequalities for bounded linear operator still remain open, the investigation to establish numerical radius inequalities for several bounded linear operators has been started, (see for instance [10] and [1]). For example, if $T_1,\, T_2 \in B(H)$, it is known [1] that

$$w(T_1T_2) \leq 4w(T_1)w(T_2).$$

Moreover, in the case $T_1T_2 = T_2T_1$, it is verified in [1] that

$$w(T_1T_2) \leq 2w(T_1)w(T_2).$$

However, the sharp inequality

$$w(T_1T_2) \leq w(T_1)w(T_2)$$

still has not been reached. A useful result in this direction, which can be found in [11], says that for any $T_1,\, T_2 \in B(H)$,

$$w(T_1T_2 \pm T_2T_1^*) \leq 2\|T_1\|\, w(T_2).$$

If $T_1,\, T_2 \in B(H)$, and T_1 is positive operator, Kittaneh in [12] showed that

$$w(T_1T_2 - T_2T_1) \leq \frac{1}{2}\|T_1\|\left(\|T_2\| + \left\|T_2^2\right\|^{\frac{1}{2}}\right).$$

Recently, the authors of [13] applied a different approach to obtain a new numerical radius inequality for commutators of Hilbert space operators. They showed that for $T_1, T_2, T_3, T_4 \in B(H)$,

$$w(T_1T_3T_2^* \pm T_2T_4T_1^*) \leq 2\|T_1\|\|T_2\|\, w\left(\begin{bmatrix} 0 & T_3 \\ T_4 & 0 \end{bmatrix}\right). \tag{1.4}$$

The following numerical radius inequality for certain 2×2 operator matrices is obtained in [14],

$$\sqrt[2n]{\max\left\{w((XY)^n), w((YX)^n)\right\}} \leq w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{\|X\| + \|Y\|}{2},\ \ (X, Y \in B(H)). \tag{1.5}$$

The purpose of this work is to establish various numerical radius inequalities for bounded linear operators on a complex Hilbert space. In particular, we use a tranquil approach to generalize inequalities (1.4) and (1.5).

2. THE MAIN RESULTS

The aim of this section is to establish new numerical radius inequalities and to generalize inequalities (1.4)-(1.5). To achieve our target, we need the following lemma.

Lemma 2.1 ([3]). *Let $X_1, X_2, \ldots, X_m \in B(H)$. Then*

$$w\left(\begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_m \end{bmatrix}\right) = \max\left\{w(X_1), w(X_2), \ldots, w(X_m)\right\}.$$

Using a straightforward technique and some known inequalities, we derive our first result.

Theorem 2.2. *Let $A_1, A_2, ..., A_n, X_1, X_2, ..., X_n \in B\left(H\right)$, and let*

$$T = \begin{bmatrix} 0 & 0 & ... & 0 & X_1 \\ \vdots & \vdots & \iddots & X_2 & 0 \\ \vdots & \iddots & X_3 & \iddots & \vdots \\ 0 & \iddots & \iddots & \vdots & \vdots \\ X_n & 0 & \cdots & \ldots & 0 \end{bmatrix}$$

Then if n is even,

$$w\left(\sum_{k=1}^{n} A_k X_k {A^*}_{n-k+1}\right) \leq 2w(T) \sum_{k=1}^{\frac{n}{2}} \left\|A_k\right\| \left\|A_{n-k+1}\right\|,$$

and if n is odd,

$$w\left(\sum_{k=1}^{n} A_k X_k {A^*}_{n-k+1}\right) \leq \left(\left\|A_{\frac{n+1}{2}}\right\|^2 + \sum_{k=1}^{\frac{n-1}{2}} 2\left\|A_k\right\| \left\|A_{n-k+1}\right\|\right) w(T).$$

Proof. Let $x, z_1, z_2, \ldots, z_n \in H$ with $\left(\sum\limits_{i=1}^{n} \left\|z_i\right\|^2\right) \neq 0$. Define $z = \frac{1}{\sqrt{\sum\limits_{i=1}^{n} \left\|z_i\right\|^2}} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$.

Since z is unit vector in $\overset{n}{\underset{i=1}{\oplus}} H_i$, we obtain

$$w(T) \sum_{k=1}^{n} \left\|z_k\right\|^2 \geq \left|\sum_{k=1}^{n} \left\langle X_k z_{n-k+1}, z_k\right\rangle\right|.$$

Replace z_k by $A_k x$ and z_{n-k+1} by $A_{n-k+1}x$, we reach to

$$\left|\sum_{k=1}^{n} \left\langle A_k^* X_k A_{n-k+1} x, x\right\rangle\right| = \left|\sum_{k=1}^{n} \left\langle X_k A_{n-k+1} x, A_k x\right\rangle\right|$$

$$\begin{aligned} &\leq w(T)\sum_{k=1}^{n}\|A_k x\|^2 \\ &\leq w(T)\|x\|^2\sum_{k=1}^{n}\|A_k\|^2. \end{aligned}$$

Thus,

$$w\left(\sum_{k=1}^{n} A_k^* A X_{n-k+1}\right) \leq w(T)\sum_{k=1}^{n}\|A_k\|^2.$$

On one hand, if n is even, then

$$w\left(\sum_{k=1}^{n} A_k^* A X_{n-k+1}\right) \leq w(T)\sum_{k=1}^{\frac{n}{2}}\left(\|A_k\|^2 + \|A_{n-k+1}\|^2\right).$$

For $t > 0$, replace A_k by tA_k and A_{n-k+1} by $\frac{1}{t}A_{n-k+1}$, we derive

$$w\left(\sum_{k=1}^{n} A_k X_k A_{n-k+1}^*\right) \leq w(T)\sum_{k=1}^{\frac{n}{2}}\left(\frac{t^4\|A_k\|^2 + \|A_{n-k+1}\|^2}{t^2}\right).$$

Therefore,

$$\begin{aligned} w\left(\sum_{k=1}^{n} A_k X_k A_{n-k+1}^*\right) &\leq w(T)\min_{t>0}\left(\sum_{k=1}^{\frac{n}{2}}\frac{t^4\|A_k\|^2 + \|A_{n-k+1}\|^2}{t^2}\right) \\ &\leq w(T)\sum_{k=1}^{\frac{n}{2}}\min_{t>0}\left(\frac{t^4\|A_k\|^2 + \|A_{n-k+1}\|^2}{t^2}\right) \\ &\leq 2w(T)\sum_{k=1}^{\frac{n}{2}}\|A_k\|\,\|A_{n-k+1}\|. \end{aligned}$$

On the other hand, if n is odd, following the same manner used above, we achieve that

$$\begin{aligned} w\left(\sum_{k=1}^{n} A_k^* X_k A_{n-k+1}\right) &\leq w(T)\sum_{k=1}^{n}\|A_k\|^2 \\ &= w(T)\left(\left\|A_{\frac{n+1}{2}}\right\|^2 + \sum_{k=1}^{\frac{n-1}{2}}\left(\|A_k\|^2 + \|A_{n-k+1}\|^2\right)\right). \end{aligned}$$

We end the proof by replacing A_k by tA_k and A_{n-k+1} by $\frac{1}{t}A_{n-k+1}$ for $t > 0$. $\square$

Applying Theorem 2.2 with $n = 2$, $A_1 = A$, $A_2 = I$, $X_1 = B$, and $X_2 = \pm B$; and using inequality (1.1), we achieve

$$w(AB \pm BA^*) \leq 2\,\|A\|\,w(B).$$

Following the same manner; take $n = 2$, $A_1 = T_1$, $X_1 = T_3$, $A_2 = T_2$, as well as $X_2 = e^{\mp i\pi}T_4 = \pm T_4$, and use [14], Lemma 2.1(b) to reach inequality (1.4).

Let us use inequalities (1.1)-(1.2) and Lemma 2.1 to prove the following theorem.

Theorem 2.3. *Let* $X_1, X_2, \ldots, X_m \in B(H)$ *and* $n \in \mathbb{N}$. *Then*

$$\sqrt[2n]{\max\left\{w((X_i X_{m-i+1})^n) : i = 1, 2, \ldots, m\right\}} \leq w(T) \leq \sum_{i=1}^{m} \frac{\|X_i\|}{2} \tag{2.1}$$

where

$$T = \begin{bmatrix} 0 & \ldots & 0 & X_1 \\ \vdots & ⋰ & X_2 & 0 \\ 0 & ⋰ & ⋰ & \vdots \\ X_m & 0 & \ldots & 0 \end{bmatrix}.$$

Proof. Let

$$L_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & X_1 \\ \vdots & \vdots & ⋰ & 0 & 0 \\ \vdots & ⋰ & 0 & 0 & \vdots \\ 0 & ⋰ & ⋰ & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{m\times m},$$

$$L_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & ⋰ & X_2 & \vdots \\ \vdots & ⋰ & 0 & 0 & \vdots \\ 0 & ⋰ & ⋰ & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{m\times m},$$

$$\cdots, \quad \text{and} \quad L_m = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & ⋰ & ⋰ & 0 \\ \vdots & ⋰ & ⋰ & 0 & \vdots \\ 0 & 0 & ⋰ & \cdots & \vdots \\ X_m & 0 & \cdots & \cdots & 0 \end{bmatrix}_{m\times m}.$$

Since $L_1^2 = L_2^2 = \cdots = L_m^2 = [0]_{m\times m}$, we conclude that

$$w(T) \leq \sum_{i=1}^{m} w(L_i) = \frac{1}{2}\sum_{i=1}^{m} \|X_i\|. \tag{2.2}$$

As

$$T^{2n}=\begin{bmatrix} (X_1X_m)^n & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & (X_iX_{m-i+1})^n & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & (X_mX_1)^n \end{bmatrix},$$

we deduce, by Lemma 2.1, that

$$\max\{w((X_iX_{m-i+1})^n) : i=1,2,\dots,m\} = w(T^{2n}) \le w^{2n}(T). \tag{2.3}$$

By this and inequality (2.2), we finish the proof. □

Use the above theorem with $m=2$, we derive

$$\sqrt[2n]{\max\{w((X_1X_2)^n), w((X_2X_1)^n)\}} \le w\left(\begin{bmatrix} 0 & X_1 \\ X2 & 0 \end{bmatrix}\right) \le \frac{\|X_1\|+\|X_2\|}{2},$$

which is exactly inequality (1.5).

REFERENCES

[1] Gustafson, K., & Rao, D. (1997). *Numerical range.* New York: Springer-Verlage.

[2] Kittaneh, F. (2003). A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. *Studia Math., 1*(158), 11-17.

[3] Bhatia, R. (1997). *Matrix analysis.* Berlin: Springer-Verlag.

[4] Dragomir, S. (2007). A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces. *Banach J. Math. Anal., 2*(1), 154-175.

[5] Halmos, P. (1982). *A Hilbert space problem book* (2nd ed.). New York: Springer-Verlage.

[6] Dragomir, S. (2007). Inequalities for the norm and the numerical radius of linear operator in Hilbert spaces. *Demonstratio Math., 2*(40), 411-417.

[7] Dragomir, S. (2007). Norm and numerical radius inequalities for sums of bounded linear operators in Hilbert spaces. *Ser. Math. Inform., 1*(22), 61-75.

[8] Omidvar, E., Moslehian, M., & Niknam, A. (2009). Some numerical radius inequalities for Hilbert space operators. *Involve, a journal of mathematics, 4*(2), 471-478.

[9] Kittaneh, F. (2005). Numerical radius inequalities for Hillbert space operators. *Studia Math., 1*(168), 73-80.

[10] Dragomir, S. (2009). Power inequalities for the numarical radius of a product of two operators in Hilbert spaces. *Seraj. J. math., 18*(5), 269-278.

[11] Fong, C., & Holbrook, J. (1983). Unitarily invariant operator norms. *Can. J. Math., 135*, 274-299.

[12] Kittaneh, F. (1988). Notes on some inequalities for Hilbert space operators. *Pub1. Res. Inst. Math. Sci., 24*, 283-293.

[13] Hirzallah, O., Kittaneh, F., & Shebrawi, K. (2011). Numerical radius inequalities for commutators of Hilbert space operators. *Num. Func. Anal. and Opti., 7*(32), 739-749.

[14] Hirzallah, O., Kittaneh, F., & Shebrawi, K. (2011). Numerical radius inequalities for certian 2×2 operator matrices. *Integr. Equ. Oper. Theory, 71*, 129-147.