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A Class of Weighted Weibull Distributions and Its Properties

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Abstract: The Weibull distribution is a well known and common distribution. In this article, a skewness parameter to a Weibull distribution is introduced using an idea of Azzalini, which creates a new class of weighted Weibull distributions. This new distribution has a probability density function with skewness representing a general case of weighted probability density function of the Extreme value distribution, the Rayleigh distribution and Exponential distribution. Different properties of this new distribution are discussed and the inference of the old parameters and the skewness parameter is studied.

Key words: Hazard function; Moment generating function; Truncated distribution; Weighted gamma distribution; Selection model; Constrained normal mean vector; Probabilistic representation

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1. INTRODUCTION

The usefulness and applications of parametric distributions including the Weibull distribution, the Raleigh distribution and the extreme value distribution in various areas including reliability, renewal theory, and branching processes can be seen in recent papers by several authors including Oluyede [19] and in references therein.

Weighted distributions are used to adjust the probabilities of the events as observed and recorded. Patil and Rao [20] discussed how, for example, truncated distributions and damaged observations can give rise to weighted distributions.

Azzalini [4] was the first to introduce the skew-normal distribution to incorporate a shape/skewness parameter to a normal distribution depending on a weighted function denoted by $F(\alpha X)$ where α is a skewness parameter. Since then extensive work has been done to introduce a skewness parameter to some symmetric distributions, for instance skew-t, skew-Cauchy, skew-Laplace, skew-logistic. In general, skew-symmetric distributions have been defined and several of their properties and inference procedures have been discussed, see for example, Arnold and Beaver [3], Gupta and Kundu [14] and the recent monograph by Genton [11]. Arnold and Beaver [2] provided a nice interpretation of Azzalini's skew-normal distribution as a hidden truncation model, although the same interpretation may not be true for other skewed distributions.

Actually, Azzalini's method has been used extensively for several symmetric distributions and non-symmetric distributions. In this article, it will be observed that if we apply Azzalini's method to the weibull distribution, then it produces a new class of weighted weibull $WW(\lambda, \beta, \alpha)$ distributions with an additional parameter called "sensitive skewness parameter". From now we denote a member of this new class of weighted distributions as $WW(\lambda, \beta, \alpha)$ distribution. The sensitive skewness parameter governs essentially the shape of the probability density function of the $WW(\lambda, \beta, \alpha)$ distribution.

The main aim of this article is to introduce this distribution and study its properties. It will be observed that although the distribution has been obtained as a $WW(\lambda, \beta, \alpha)$ distribution, it has several other special cases and can be considered as a hidden truncation model.

It can be seen that the $WW(\lambda, \beta, \alpha)$ distribution function has a compact form and all its moments can be computed explicitly, for instance mean, variance, skewness, kurtosis, coefficient of variation, hazard function (HF), and mean residual lifetime.

The rest of the article is organized as follows. In Section 2, we provide the definition of $WW(\lambda, \beta, \alpha)$ distribution. Different properties are discussed in Section 3, also we introduce special cases by defining a weighted the Extreme value distribution, and a weighted Rayleigh distribution and a weighted Exponential distribution in Section 4. The inference of old parameters and sensitive skewness parameter is studied in Section 5.

2. DEFINITION OF WEIGHTED WEIBULL DISTRIBUTION

Let X be a non-negative random variable with an absolutely continuous distribution function F_X and probability density function f_X.

Let

$$\begin{aligned} l_X &= \inf\{x \in \Re : F_X(x) > 0\}, \\ u_X &= \sup\{x \in \Re : F_X(x) < 1\}, \\ S_X &= \{x : l_X \leq x \leq u_X\}. \end{aligned}$$

Let w be a non-negative function defined on the real line. Suppose that the realization x of X will be recorded with probability proportional to $w(t(x))$. Then

the recorded $t(x)$ is not an observation on X but it is an observation on the so-called weighted random variable $X|\{w\}$ with density function given by:

$$f_{X|\{w\}}(x) = \frac{w(t(X))f_X(x)}{E[w(t(X))]} \quad \text{for} \quad -\infty < x < \infty,$$

where $0 < E\left[w(t(X))\right] < \infty$.

The random variable $X|\{w\}$ is called the weighted version of X, and its distribution relative to X is called the weighted distribution of X with weight function w. Let $l_X \geq 0$ and $w(t(x)) = x$, $x \in S_X$, for some positive integer. Then the corresponding weighted distribution is called a size biased distribution. If this distribution is of order one, then it is simply called a length-biased distribution (see Blumenthal [6], Gupta and Kirmani [12], Mahfoud and Patil [16]).

Now, the new class of weighted distributions is defined by the corresponding density function:

$$f_{X|\{\alpha\}}(x) = \frac{F_X(\alpha x)f_X(x)}{E\left[F(\alpha X)\right]}, \quad \text{for} \quad x > 0, \tag{1}$$

with

$$w(t(x)) = F(\alpha x).$$

According to (1) the new class of weighted Weibull distributions can be derived as follows. Let X be distributed according to a Weibull distribution with parameters λ and β, and density function as follow

$$f_{X|\{\lambda,\beta\}} = \lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}, \tag{2}$$

where λ is a scale parameter, β is a shape parameter. The distribution function of $X|\{\beta,\lambda\}$ is

$$F_{X|\{\lambda,\beta\}}(x) = 1 - e^{-\lambda x^{\beta}}.$$

Hence, we get

$$F_{X|\{\lambda,\beta\}}(\alpha x) = 1 - e^{-\lambda(\alpha x)^{\beta}},$$

and

$$E\left[F_{X|\{\lambda,\beta\}}(\alpha X)\right] = \int_x F_{X|\{\lambda,\beta\}}(\alpha x)f_{X|\{\lambda,\beta\}}(x)dx = \frac{\alpha^{\beta}}{1+\alpha^{\beta}},$$

Inserting into (1) yields

$$f_{X|\{\lambda,\beta,\alpha\}}(x) = \frac{\lambda\beta(1+\alpha^{\beta})x^{\beta-1}e^{-\lambda x^{\beta}}\left(1 - e^{-\lambda(x\alpha)^{\beta}}\right)}{\alpha^{\beta}}, \quad \text{for} \quad x > 0, \tag{3}$$

and $f_{X|\{\lambda,\beta,\alpha\}}(x) = 0$ otherwise. The density function (3) is referred to as $WW(\lambda,\beta,\alpha)$.

Suppose X_1 and X_2 are two i.i.d. random variables with probability density function f_Y and distribution function F_Y. Then for any $\alpha > 0$, and $\beta = 2$, consider the new random variable X with $X = X_1$ given that $\alpha X_1 > X_2$. Azzalini [4] obtained the weighted skew-normal distribution from two i.i.d. normal distributions and Mahdy [15] obtained the weighted gamma distribution from two i.i.d. gamma distributions as follows

$$f_{X|\{\lambda,2,\alpha\}}(x) = \frac{1}{P_{X_1,X_2}\left(\{(x_1,x_2) : \alpha x_1 > x_2\}\right)}f_Y(x)F_Y(\alpha x). \tag{4}$$

Now, (1) can be obtained explicitly from (4) by inserting

$$f_{X|\{\lambda,2\}}(x) = 2\lambda x e^{-\lambda x^2},$$
$$F_{X|\{\lambda,2\}}(x) = 1 - e^{-\lambda x^2},$$

and

$$P_{X_1,X_2}\left(\{(x_1,x_2) : \alpha x_1 > x_2\}\right) = \int_0^{\infty}\int_0^{\alpha x_1} f_{X_1,X_2}(x_1,x_2)dx_2dx_1 = \frac{\alpha^2}{1+\alpha^2}.$$

3. SOME PROPERTIES OF THE $WW(\lambda, \beta, \alpha)$ MODEL

In this section we study the $WW(\lambda, \beta, \alpha)$ distribution. Without loss of generality, the density function of the $WW(\lambda, \beta, \alpha)$ distribution is provided by (3). The graphs of the density function of WW for different values of α are displayed in Figure 1.

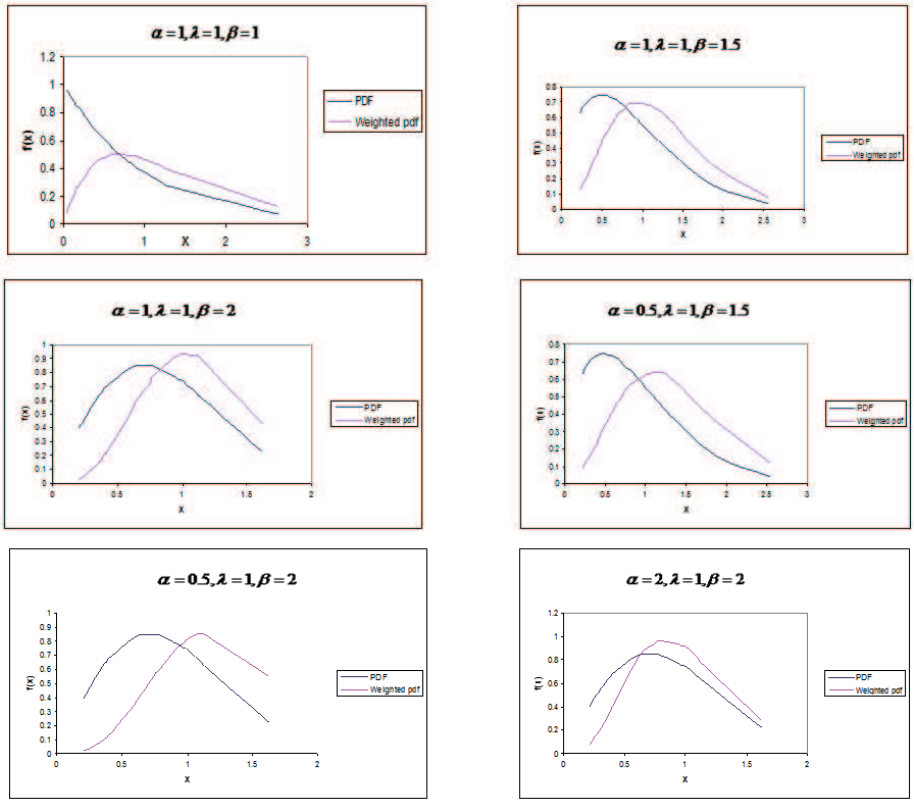

Figure 1
Weighted Weibull Densities

It is easy to see that as α increases, the skewness of the distribution increases. If X is a random variable with probability density function $WW(\lambda, \beta, \alpha)$, then

the kth moment of $X|\{\lambda,\beta,\alpha\}$ is given by

$$E_{WW(\lambda,\beta,\alpha)}(X^k)=\frac{\lambda^{-k/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{k+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(k+\beta)/\beta}\right). \tag{5}$$

From (5),the first moment of $WW(\lambda,\beta,\alpha)$ is obtained:

$$E_{WW(\lambda,\beta,\alpha)}(X)=\frac{\lambda^{-1/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{1+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(1+\beta)/\beta}\right).$$

Similarly, the variance is obtained as

$$V_{WW(\lambda,\beta,\alpha)}(x)=a_1-a_2,$$

where,

$$(i).a_1=E_{WW(\lambda,\beta,\alpha)}(X^2)=\frac{\lambda^{-2/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{2+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(2+\beta)/\beta}\right), \text{ and}$$

$$(ii).a_2=E^2_{WW(\lambda,\beta,\alpha)}(X)=\left[\frac{\lambda^{-1/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{1+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(1+\beta)/\beta}\right)\right]^2.$$

Moreover, we get

$$E_{WW(\lambda,\beta,\alpha)}(X^3)=\frac{\lambda^{-3/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{3+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(3+\beta)/\beta}\right).$$

Some other measures, like the coefficient of variation $CV_{WW(\lambda,\beta,\alpha)}$ of $WW(\lambda,\beta,\alpha)$ and the skewness coefficient $\tau_{WW(\lambda,\beta,\alpha)}$ can also be easily obtained in explicit forms:

$$CV_{WW(\lambda,\beta,\alpha)}(x)=\left(\sqrt{a_1-a_2}\right)/\frac{\lambda^{-1/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{1+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-1/\beta}\right),$$

and

$$\tau_{WW(\lambda,\beta,\alpha)}=\frac{s_1-s_2}{\left(a_1-a_2\right)^{3/2}},$$

where

$$\begin{aligned}s_1=&\frac{\lambda^{-3/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{3+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(3+\beta)/\beta}\right)\\&-3(a_1-a_2)\frac{\lambda^{-1/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{1+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(1+\beta)/\beta}\right);\\ s_2=&\left(\frac{\lambda^{-1/\beta}}{\alpha^{\beta}}\left(1+\alpha^{\beta}\right)\Gamma\left(\frac{1+\beta}{\beta}\right)\left(1-\left(1+\alpha^{\beta}\right)^{-(1+\beta)/\beta}\right)\right)^3.\end{aligned}$$

Next, we provide the distribution function, survival function, hazard function, reversed hazard function, and mean residual life function for the $WW(\lambda,\beta,\alpha)$ distribution:

(i). The distribution function $F_{WW(\lambda,\beta,\alpha)}$ of $WW(\lambda,\beta,\alpha)$ is given by

$$F_{WW(\lambda,\beta,\alpha)}(x) = \frac{1}{\alpha^{\beta}}\left[\left(1+\alpha^{\beta}\right)\left(1-e^{-\lambda x^{\beta}}\right)+e^{-\lambda x^{\beta}(1+\alpha^{\beta})}-1\right].$$

(ii). The survival function $\overline{F}_{WW(\lambda,\beta,\alpha)}$ of $WW(\lambda,\beta,\alpha)$ is given by

$$\overline{F}_{WW(\lambda,\beta,\alpha)}(x) = 1-\frac{1}{\alpha^{\beta}}\left[\left(1+\alpha^{\beta}\right)\left(1-e^{-\lambda x^{\beta}}\right)+e^{-\lambda x^{\beta}\left(1+\alpha^{\beta}\right)}-1\right]. \quad (6)$$

(iii). The hazard function $h_{WW(\lambda,\beta,\alpha)}$ of $WW(\lambda,\beta,\alpha)$ is given by

$$h_{WW(\lambda,\beta,\alpha)} = \frac{\left(1+\alpha^{\beta}\right)\lambda\beta x^{\beta-1}\exp\left(-\lambda x^{\beta}\right)\left[1-\exp\left(-\lambda(x\alpha)^{\beta}\right)\right]}{\left[\left(1+\alpha^{\beta}\right)e^{-\lambda x^{\beta}}-\exp\left(-\lambda x^{\beta}\left(1+\alpha^{\beta}\right)\right)-a_1+1\right]}.$$

(iv). The reversed hazard function $r_{WW(\lambda,\beta,\alpha)}$ of $WW(\lambda,\beta,\alpha)$ is given by

$$r_{WW(\lambda,\beta,\alpha)} = \frac{\lambda\beta\left(1+\alpha^{\beta}\right)x^{\beta-1}e^{-\lambda x^{\beta}}\left(1-e^{-\lambda(x\alpha)^{\beta}}\right)}{\left(1+\alpha^{\beta}\right)\left(1-e^{-\lambda x^{\beta}}\right)+e^{-\lambda x^{\beta}(1+\alpha^{\beta})}-1}.$$

(v). The mean reversed residual function $m_{WW(\lambda,\beta,\alpha)}$ of $WW(\lambda,\beta,\alpha)$ is given by

$$m_{WW(\lambda,\beta,\alpha)}(x) = \frac{\int_0^x F_{WW(\lambda,\beta,\alpha)}(u)du}{F_{WW(\lambda,\beta,\alpha)}(x)},$$

which equals to

$$m_{WW(\lambda,\beta,\alpha)}(x) = \frac{\lambda\beta x^{\beta-1}x\alpha^{\beta}+\left(1+\alpha^{\beta}\right)\left(e^{-\lambda x^{\beta}}-e^{-\lambda x^{\beta}\left(1+\alpha^{\beta}\right)}\right)}{\lambda\beta x^{\beta-1}\left(1+\alpha^{\beta}\right)\left(1-e^{-\lambda x^{\beta}}\right)-\lambda\beta x^{\beta-1}\left(1-e^{-\lambda x^{\beta}(1+\alpha^{\beta})}\right)}.$$

It can be shown that $m_{WW(\lambda,\beta,\alpha)}$ is an increasing function of x.

Table 1 contains the values of survival function (6). Looking at this table we can see that the survival probability of the distribution decreasing with increase in the value of α for a holding x and λ and β at a fixed level. Further, from the table we can see that; for fixed α, λ and β; the survival probability decreases with increase in x.

4. SPECIAL CASES

4.1. The Weighted Extreme Value Distribution

In many fields of modern science, engineering and insurance, extreme value distribution is well established (see e. g. Embrechts *et al.* [10], Reiss and Thomas [21]). Recently, more and more research has been undertaken to analyze the extreme variations that financial markets are subject to, mostly because of currency crises, stock market crashes and large credit defaults. The tail behavior of financial series has, among others, been discussed in McNeil and Frey [18], Coles [7], Beirlant *et al.* [5], Mandira [17], Cooley *et al.* [8] and Andjelic *et al.* [1]. An interesting discussion about the potential of extreme value theory in risk management is given in Diebold *et al.* [9].

Table 1
Survival Function of Weighted Weibull Distribution for $\lambda = 1$; $\beta = 1$

x	$\alpha = 1$	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$	$\alpha = 5$
0.1	0.990944	0.990103	0.989273	0.988454	0.987645	0.986847	0.976043
0.2	0.967141	0.964309	0.96155	0.958862	0.956243	0.95369	0.911138
0.3	0.932825	0.927457	0.922294	0.917326	0.912545	0.907943	0.855922
0.4	0.891311	0.883268	0.875625	0.868361	0.861454	0.854883	0.78624
0.5	0.845182	0.83458	0.824628	0.81528	0.806494	0.798231	0.717879
0.6	0.796429	0.783542	0.771586	0.760484	0.750166	0.740568	0.653109
0.7	0.746574	0.731755	0.718165	0.705685	0.694211	0.68365	0.592903
0.8	0.696761	0.680399	0.665559	0.652078	0.639813	0.628634	0.537549
1	0.600424	0.58211	0.565852	0.551383	0.538474	0.526926	0.44096
1.1	0.554939	0.536162	0.519664	0.505123	0.492267	0.480865	0.399173
1.2	0.51167	0.492722	0.476237	0.461842	0.449227	0.438129	0.361284
1.3	0.47079	0.451918	0.435657	0.421585	0.409354	0.398677	0.326956
1.4	0.432384	0.413795	0.397927	0.384312	0.372572	0.362398	0.295871
1.5	0.396473	0.378336	0.362992	0.349935	0.33876	0.329141	0.267732

Let random variable X has a Weibull distribution in (2), then the weighted probability density function of $Y = -\beta \log\left(X\lambda^{1/\beta}\right)$ is given by

$$f_{Y|\{\beta,\alpha\}}(y) = \frac{\left(1+\alpha^{\beta}\right)}{\alpha^{\beta}} e^{-y} e^{-e^{-y}} \left(1 - e^{-\alpha^{\beta} e^{-y}}\right),$$

which is weighted extreme value distribution.

4.2. The Weighted Rayleigh Distribution

The Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences.

Let random variable Y has the weighted Rayleigh distribution $WR(\sigma, \alpha)$, then Y has the weighted Weibull distribution $WW\left((1/\sigma\sqrt{2})^2, 2, \alpha\right)$. Therefore, if $w(t(x)) = F(\alpha x)$, the weighted Rayleigh distribution has an probability density function and distribution function are given by

$$\begin{aligned} & f_{Y\left|\left\{(1/\sigma\sqrt{2})^2,2,\alpha\right\}\right.}(y) \\ = \; & \left(\left(1+\alpha^2\right)/\left(\sigma^2\alpha^2\right)\right) y \exp\left(-0.5(y/\sigma)^2\right)\left(1 - \exp\left(-0.5(y\alpha/\sigma)^2\right)\right), \end{aligned}$$

and

$$\begin{aligned} & F_{WR\left((1/\sigma\sqrt{2})^2,2,\alpha\right)}(y) \\ = \; & \left(1+\alpha^2\right)\left(1 - \exp\left(-0.5(y/\sigma)^2\right)\right) + \exp\left(-0.5(y/\sigma)^2(1+\alpha^2)\right) - 1. \end{aligned}$$

4.3. The Weighted Exponential Distribution

The second distribution can be obtained following Gupta and Kundu [13] who introduced a new weighted exponential distribution but not fixed λ. Suppose $\beta = 1$ in (2), and then we get the density function of the weighted exponential distribution of the random variable $Y|\{\lambda, \alpha\}$ as

$$f_{Y|\{\lambda,\alpha\}}(y) = \lambda(1+\alpha)\exp(-\lambda y)(1-\exp(-\lambda y\alpha))/\alpha, \quad \text{for} \quad y > 0,$$

and $f_{Y|\{\lambda,\alpha\}}$ otherwise also, the distribution function can be written as

$$F_{WE(\lambda,\alpha)}(y) = \frac{1}{\alpha}\left[(1+\alpha)(1-\exp(-\lambda y)) + \exp(-\lambda y(1+\alpha)) - 1\right].$$

5. PARAMETER ESTIMATION

This section is devoted to the estimation of the unknown parameter α in case $\beta = 2$. For this case, the maximum likelihood and the moment estimator is derived and their asymptotic distributions are given.

5.1. Maximum Likelihood Estimates

The weighted Weibull distribution is obtained by means of (3) in case $\beta = 2$. We can write it as follows:

$$f_{X|\{\lambda,\alpha\}}(x) = 2\lambda\left(1+\alpha^2\right)x\exp\left(-\lambda x^2\right)\left(1-\exp\left(-\lambda(x\alpha)^2\right)\right)/\alpha^2, \quad \text{for} \quad x > 0.$$

The likelihood function based on the observed sample $\{x_1, x_2, ..., x_n\}$ is

$$\begin{aligned} & L(x_1, x_2, ..., x_n|\lambda, \alpha) \\ = \; & 2^n\lambda^n\left(1+\alpha^2\right)^n\Pi_{i=1}^n x_i \exp\left(-\lambda\sum_{i=1}^n {x_i}^2\right)\Pi_{i=1}^n\left(1-\exp\left(-\lambda(x_i\alpha)^2\right)\right)/\alpha^{2n}. \end{aligned} \tag{7}$$

From (7) we obtain the log-likelihood function $\ell(x_1, x_2, ..., x_n|\lambda, \alpha)$:

$$\begin{aligned} & \ell(x_1, x_2, ..., x_n|\lambda, \alpha) \\ = \; & n\ln 2 + n\ln\lambda + n\ln\left(1+\alpha^2\right) + \sum_{i=1}^n \ln x_i - \lambda\sum_{i=1}^n x_i^2 \\ & + \sum_{i=1}^n \ln\left[1 - e^{-\lambda(x_i\alpha)^2}\right] - 2n\ln\alpha. \end{aligned} \tag{8}$$

For determining the maximum likelihood estimators (MLE) of λ and α, the expression (8) must be maximized with respect to λ and α. Thus, firstly, the MLE of λ is obtained as a solution of the following fixed-point type equation:

$$g\left(\hat{\lambda}\right) = \hat{\lambda}, \tag{9}$$

where

$$g\left(\hat{\lambda}\right) = C_1 + \hat{\lambda}\frac{C_2}{C_3},$$
$$C_1 = \frac{n}{\sum\limits_{i=1}^{n} x_i^2},$$
$$C_2 = \sum_{i=1}^{n} (x_i\alpha)^2 \exp\left(-\lambda(x_i\alpha)^2\right),$$

and

$$C_3 = \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} \ln\left[1 - \exp\left(-\lambda(x_i\alpha)^2\right)\right].$$

The solution of (9) can be obtained by a simple iterative procedure. Suppose we start with an initial guess $\hat{\lambda}_0$, then the next iteration $\hat{\lambda}_1$ can be obtained as $\hat{\lambda}_1 = g(\hat{\lambda}_0)$, similarly, $\hat{\lambda}_2 = g(\hat{\lambda}_1)$, and so on. Finally the iterative procedure should be stopped when $\left|\hat{\lambda}_i - \hat{\lambda}_{i+1}\right| < \epsilon$, where ϵ is a preassigned tolerance value.

Secondly, the MLE of α is obtained as a solution of the following fixed-point type equation:

$$g\left(\hat{\alpha}\right) = \hat{\alpha}, \tag{10}$$

where

$$g\left(\hat{\alpha}\right) = S_1 - S_2,$$
$$S_1 = \frac{\left(1+\hat{\alpha}^2\right)}{\hat{\alpha}},$$

and

$$S_2 = \frac{\left(1+\hat{\alpha}^2\right)\sum\limits_{i=1}^{n} 2\lambda\hat{\alpha}(x_i)^2 \exp\left(-\lambda(x_i\hat{\alpha})^2\right)}{2n\sum\limits_{i=1}^{n} \ln\left(1 - \exp\left(-\lambda(x_i\hat{\alpha})^2\right)\right)}.$$

The solution of (10) can be obtained by a simple iterative procedure. Suppose we start with an initial guess $\hat{\alpha}_0$, then the next iteration $\hat{\alpha}_1$ can be obtained as $\hat{\alpha}_1 = g(\hat{\alpha}_0)$, similarly, $\hat{\alpha}_2 = g(\hat{\alpha}_1)$, and so on. Finally the iterative procedure should be stopped when $|\hat{\alpha}_i - \hat{\alpha}_{i+1}| < \epsilon$, where ϵ is a preassigned tolerance value.

5.2. The Moment Estimators

Next, we discuss the moment estimators of λ and α when $\beta = 2$. If m^2 denotes the second non-central moment, then by equating the second moment, we obtain the moment estimator of α as

$$0 = \frac{\lambda^{-1}}{\alpha^2}\left(1+\alpha^2\right)\left(1 - \left(1+\alpha^2\right)^2\right) - m^2.$$

Thus, the moment estimators of α can be obtained as a solution of the following fixed-point type equation:

$$g\left(\hat{\alpha}\right) = \hat{\alpha},$$

where

$$g(\hat{\alpha}) = \left(\frac{\lambda^{-1}}{m^2} \left(1 + \hat{\alpha}^2\right) \left(1 - \left(1 + \hat{\alpha}^2\right)^2\right) \right)^{1/2},$$

This solution can be obtained by a simple iterative procedure, analogously as in the case of the solution of (10).

Also, we can be obtained the moment estimator of λ as

$$\hat{\lambda} = \frac{1}{\alpha^2 m^2} \left(1 + \alpha^2\right) \left(1 - \left(1 + \alpha^2\right)^2\right).$$

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