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Maximization of Wealth in a Jump-Diffusion Model

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Abstract: This paper study the problem of wealth optimization when jump-diffusion asset price model being driven by a count process that more general than Poisson process. It is found unique equivalent martingale measure, we employ the conventional stochastic analysis methods. It is proved that the existence of an optimal portfolio and consumption process. The optimal wealth process, the value function, the optimal portfolio and consumption process are given.

Key words: Optimization problem; Equivalent martingale measure; Count process

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1. INTRODUCTION

The wealth optimization problem is always one of the kernel problems on financial mathematics. The domestic and foreign scholars have done a great deal of researches on the wealth optimization problem and obtained many results which is instructive to financial practice. We consider the problem of maximization of expected utility form consumption only, and the antithetical problem of maximization of expected utility form terminal wealth only. When markets are complete, the existence of optimal strategies can be found Merton [1], Jeanblanc and Pontier [2]. Follmer and Leukert [3] discussed semi-martingale model, Pham [4] discussed continuous markets model, Nakano [5]discussed jump-diffusion mode. In this paper, we consider an economic agent whose behavior facing the risk is determined by a utility function $U_1(x)$, $U_2(x)$. He invests his wealth in the two assets and wants to maximize the expected utility of wealth at time T. We define the wealth optimization problem:

$$V(t,x) = \sup_{\pi \in \mathcal{A}(t,x)} E\left[\int_t^T U_1(C(s))ds + U_2(X^{x,\pi}(T))\,|X^{x,\pi}(t) = x\right]$$

where $X^{x,\pi}(t)$ is the wealth process and $\mathcal{A}$ is the set of admissible portfolios when the wealth equals to x at the time t. $c(t)$ is consumption process.

Our work extends those studies and analyses the wealth optimization problem when markets is incomplete and driven by discontinuous prices. We consider that price of underlying asset price obeys jump-diffusion process, because the generalized jump process conforms to the actual situation of stock price movement. This paper discusses jump-diffusion asset price model being driven by a count process [6] that more general than Poisson process.

2. ASSUMPTIONS AND MODELS

Let $\left(\Omega, F, P, \mathbb{F}(_t)_{0\le t\le T}\right)$ be a probability space. The market is built with a bond $B(t)$ and a risky asset $S(t)$. We suppose that $B(t)$ is the solution of the equation

$$dB(t) = B(t)r(t)dt \quad B(0) = 1 \tag{1}$$

and $S(t)$ satisfy the stochastic differential equation.

$$dS(t) = S(t-)\left(b(t)dt + \sigma(t)dW(t) + \sum_{i=1}^{k}\varphi_i(t)dM_i(t)\right) \tag{2}$$

where risk-free interest rate $r(t)$ and volatility $\sigma(t)$, $\{W(t), 0 \le t \le T\}$ be a standard Wiener process given on a probability space $(\Omega, \mathbb{F}, P)$. $M_i(t) = N_i(t) - \int_0^t \lambda_i(s)ds, T \ge t \ge 0$ is the compensated martingale of nonexplosive counting process $\{N_i(t), 0 \le t \le T\}$ with intensity parameter $\lambda_i(t)$. We assume that the filtration $(\mathbb{F}_t, 0 \le t \le T)$ is generated by the $\{W(t), 0 \le t \le T\}$ and martingale $\{M_i(t), 0 \le t \le T\}$.

We assume that the pair $(C(t), \pi(t))$ is self financing. We consider an economic agent with initial wealth x, then the investor's wealth process $X^{x,\pi}(t)$ satisfy

$$\begin{aligned} dX^{x,\pi}(t) = {} & [r(t)X^{x,\pi}(t) - C(t)]\,dt \\ & + \pi(t)\left((b(t) - r(t))\,dt + \sigma(t)dW(t) + \sum_{i=1}^{k}\varphi_i(t)dM_i(t)\right) \end{aligned}$$

Assumption 1 Function $\lambda_i(t)$, $r(t)$, $b(t)$, $\sigma(t)$, $\varphi_i(t)$ are bounded that satisfy: $\lambda_i(t) > 0$, $r(t) \ge 0$, $\varphi_i(t) > -1$, $\varphi_i(t) \ne 0$.

Assumption 2 Utility function $U_1(x)$, $U_2(x)$ is non decreasing, strictly concave.

Assumption 2 imply that $U_1'(\cdot)$, $U_2'(\cdot)$ admits an inverse $I_1(\cdot)$, $I_2(\cdot)$, $V_x'(t,\cdot)$ admits an inverse $\mathcal{X}(t,\cdot)$.

Lemma 1 For all P^* in P, there exist predictable processes $\theta_1(t)$, $\theta_{2i}(t)$ $(i = 1,...,k)$ satisfy

$$b(t) - r(t) + \sigma(t)\theta_1(t) + \sum_{i=1}^{k} \lambda_i(t)\varphi_i(t)\theta_{2i}(t) = 0 \tag{3}$$

Proof. Since P^* in P, then $\frac{dP^*}{dP}|\mathbb{F}_t = L(t)$ is the P martingale, applying martingale representation theorem, there exists $\theta_1(t)$, $\theta_{2i}(t)$ $(i = 1,...,k)$ such that

$$dL(t) = L(t-)\left(\theta_1(s)dW(s) + \sum_{i=1}^{k}[\theta_{2i}(s)dM_i(s)]\right)$$

thus

$$\begin{aligned} L(t) = &\exp\left\{\int_0^t \theta_1(s)dW(s) - \frac{1}{2}\int_0^t \theta_1^2(s)ds\right\} \\ &\times \prod_{i=1}^{k}\left[\exp\left\{\int_0^t \ln(1+\theta_{2i}(s))dN_i(s) - \int_0^t \lambda_i(s)\theta_{2i}(s)ds\right\}\right] \end{aligned}$$

and under the martingale measure P^*, $W^*(t) = W(s) - \int_0^t \sigma_1(s)ds$ is a a standard Wiener process,

$$M_i^*(t) = N_i(t) - \int_0^t \lambda_i(s)(1+\theta_{2i}(s))ds$$

are P^* martingale. Let $\tilde{S}(t) = \frac{S(t)}{B(t)}$, then

$$\begin{aligned} d\tilde{S}(t) =& \tilde{S}(t-)\big((b(t)-r(t))dt + \sigma(t)dW^*(t) + \sigma(t)\theta_1(t)dt \\ &+ \sum_{i=1}^{k}[\varphi_i(t)dM_i^*(t) + \varphi_i(t)\lambda_i(t)\theta_{2i}(t)dt]\big) \\ =& \tilde{S}(t-)\big((b(t)-r(t)+\sigma(t)\theta_1(t) + \sum_{i=1}^{i}\lambda_i(t)\varphi_i(t)\theta_{2i}(t))dt \\ &+ \sigma(t)dW^*(t) + \sum_{i=1}^{k}\varphi_i(t)dM_i^*(t)\big) \end{aligned}$$

Since $\tilde{S}(t)$ is a P^* martingale, then

$$b(t) - r(t) + \sigma(t)\theta_1(t) + \sum_{i=1}^{k} \lambda_i(t)\varphi_i(t)\theta_{2i}(t) = 0$$

□

3. MAIN RESULTS

Proposition 1 There exists unique $\theta_{2i}(i=1,2,\cdots,k)$ such that

$$\mathcal{X}\left(t,(1+\theta_{2i})V'_x\right)-x=\frac{\varphi_i}{\sigma^2}\left(r-b-\sum_{i=1}^{k}\left(\lambda_i\varphi_i\theta_{2i}\right)\right)\frac{V'_x}{V''_{xx}}\quad(i=1,2,\cdots,k)\tag{4}$$

Proof. Let

$$F(z)=z+\sum_{i=1}^{k}\lambda_i\varphi_i-\sum_{i=1}^{k}\frac{\lambda_i\varphi_i}{V'_x}V'_x\left(t,x+\frac{\varphi_i}{\sigma^2}(r-b-z)\frac{V'_x}{V''_{xx}}\right)$$

assumption 2 implies that $V'_x(t,x)>0, V''_{xx}(t,x)<0$, then

$$F'(z)=1+\sum_{i=1}^{k}\frac{\lambda_i{\varphi_i}^2}{\sigma^2V''_{xx}}V''_{xx}\left(t,x+\frac{\varphi_i}{\sigma^2}(r-b-z)\frac{V'_x}{V''_{xx}}\right)>0$$

So, we can assert that $F(z)$ is strictly non decreasing. By differential mean value theorem, there exists ξ_i such that

$$V'_x\left(t,x+\frac{\varphi_i}{\sigma^2}(r-b)\frac{V'_x}{V''_{xx}}\right)-V'_x(t,x)=\frac{\varphi_i}{\sigma^2}(r-b)\frac{V'_x}{V''_{xx}}V''_{xx}(t,\xi_i)$$

We reduce

$$\begin{aligned}F(0)&=\sum_{i=1}^{k}\lambda_i\varphi_i-\sum_{i=1}^{k}\frac{\lambda_i\varphi_i}{V'_x}V'_x\left(t,x+\frac{\varphi_i}{\sigma^2}(r-b)\frac{V'_x}{V''_{xx}}\right)\\&=-\sum_{i=1}^{k}\lambda_i{\varphi_i}^2\frac{1}{\sigma^2}(r-b)\frac{1}{V''_{xx}}V''_{xx}(t,\xi_i)\end{aligned}$$

$$F(0)F(r-b)=-\sum_{i=1}^{k}\lambda_i{\varphi_i}^2\frac{1}{\sigma^2}(r-b)^2\frac{1}{V''_{xx}}V''_{xx}(t,\xi_i)<0$$

By using $F(z)$ is strictly non decreasing, we have $F(z)$ exists unique zero point $z_0\in R$.

Let $\theta_{2i}+1=\frac{1}{V'_x}V'_x\left(t,x+\frac{\varphi_i}{\sigma^2}(r-b-z)\frac{V'_x}{V''_{xx}}\right)$, so $F(z)=z-\sum\limits_{i=1}^{k}\lambda_i\varphi_i\theta_{2i}$ has unique zero point $z_0=\sum\limits_{i=1}^{k}\lambda_i\varphi_i\theta_{2i}$, there exists unique $\theta_{2i}(i=1,2,...,k)$ such that

$$\theta_{2i}+1=\frac{1}{V'_x}V'_x\left(t,x+\frac{\varphi_i}{\sigma^2}(r-b-\sum_{i=1}^{k}\left(\lambda_i\varphi_i\theta_{2i}\right))\frac{V'_x}{V''_{xx}}\right)$$

then $\mathcal{X}(t,(1+\theta_{2i})V'_x)-x=\frac{\varphi_i}{\sigma^2}\left(r-b-\sum\limits_{i=1}^{k}\left(\lambda_i\varphi_i\theta_{2i}\right)\right)\frac{V'_x}{V''_{xx}}$ $(i=1,2,\cdots,k)$ has unique solution.

Let $\theta_1(t)=\frac{1}{\sigma(t)}\left[r(t)-b(t)-\sum\limits_{i=1}^{k}\lambda_i(t)\varphi_i(t)\theta_{2i}(t)\right]$, $\frac{dP^*}{dP}=L(T)$ define a unique equivalent martingale measure P^*.

□

Proposition 2 We assume that utility function $U_1(x)$, $U_2(x)$ satisfies a polynomial growth condition ($\exists C > 0, p \in N$, $\forall x \in R$, $|U_i(x)| \leq C(1+|x|^p)$), then

$$\pi^*(s) = \frac{r - b - \sum\limits_{i=1}^{k} \lambda_i \varphi_i \theta_{2i}}{\sigma^2} \frac{\mathcal{Y}^{-1}(s, X^{x,\pi^*}(s))}{{\mathcal{Y}^{-1}}_x{}'(s, X^{x,\pi^*}(s))} \tag{5}$$

$$c^*(u) = I_1(\mathcal{Y}^{-1}(t,x) Y^{(t,1)}(u))$$

Proof. H-J-B equation is

$$\begin{aligned} V_t' + \sup_{\pi} \Bigg\{ &[xr - c + \pi(b - \sum_{i=1}^{k} \lambda_i \varphi_i - r)] V_x' + \frac{1}{2}\pi^2 \sigma^2 V_{xx}'' \\ &+ \sum_{i=1}^{k} \lambda_i [V(t, x + \pi\varphi_i) - V] \Bigg\} + U_1(c) = 0 \end{aligned}$$

the boundary condition is $V(T,x) = U_2(x)$, then $V(T,x)$ satisfies a polynomial growth condition, we have $c(t) = I_1(V_x'(t,x))$ and $(b - \sum\limits_{i=1}^{k} \lambda_i \varphi_i - r) V_x' + \pi \sigma^2 V_{xx}'' + \sum\limits_{i=1}^{k} \lambda_i \varphi_i V_x'(t, x + \pi\varphi_i) = 0$, so

$$\begin{aligned} &\left(r - b - \pi\sigma^2 \frac{V_{xx}''}{V_x'} \right) + \sum_{i=1}^{k} \lambda_i \varphi_i \\ &- \sum_{i=1}^{k} \frac{\lambda_i \varphi_i}{V_x'} V_x' \left(t, x + \frac{\varphi_i}{\sigma^2} (r - b - (r - b - \pi\sigma^2 \frac{V_{xx}''}{V_x'})) \frac{V_x'}{V_{xx}''} \right) = 0 \end{aligned} \tag{6}$$

By Proposition 1, Equation (6) has unique solution

$$\pi^*(t) = \frac{r - b - \sum\limits_{i=1}^{k} \lambda_i \varphi_i \theta_{2i}}{\sigma^2} \frac{V_x'(t,x)}{V_{xx}''(t,x)}$$

At time $s \geq t$, wealth is $X^{x,\pi^*}(s)$, so

$$\pi^*(s) = \frac{r - b - \sum\limits_{i=1}^{k} \lambda_i \varphi_i \theta_{2i}}{\sigma^2} \frac{V_x'(s, X^{x,\pi^*}(s))}{V_{xx}''(s, X^{x,\pi^*}(s))}$$

Let

$$\begin{aligned} Y^{(t,y)}(s) =& y \exp\left\{ - \int_t^s r(u) du + \int_t^s \theta_1(u) dW(u) - \frac{1}{2} \int_t^s \theta_1^2(u) du \right\} \\ &\times \prod_{i=1}^{k} \left[\exp\left\{ \int_t^s \ln(1 + \theta_{2i}(u)) dN_i(u) - \int_t^s \lambda_i(u) \theta_{2i}(u) du \right\} \right] \end{aligned}$$

We have $Y^{(t,y)}(s) = yY^{(t,1)}(s) = y\dfrac{B(t)L(s)}{B(s)L(t)}$ is a P martingale. Let

$$\mathcal{Y}(t,y) = E^*\left[\int_t^T e^{-\int_t^s r(u)du} I_1(Y^{(t,y)}(s))ds + e^{-\int_t^s r(u)du} I_2(Y^{(t,y)}(T))\right]$$

Therefore

$$\begin{aligned}
\mathcal{Y}(t,y) &= E^*\left\{E^*\left[\int_t^T e^{-\int_t^s r(u)du} I_1(Y^{(t,y)}(s))ds + e^{-\int_t^s r(u)du} I_2(Y^{(t,y)}(T))\,|\mathbb{F}_t\right]\right\}\\
&= E\left\{\frac{L(T)}{L(t)}E\left[\int_t^T e^{-\int_t^s r(u)du} L(s)I_1(Y^{(t,y)}(s))ds + e^{-\int_t^s r(u)du} L(T)I_2(Y^{(t,y)}(T))\,|\mathbb{F}_t\right]\right\}\\
&= \frac{1}{y}E\left\{L(T)E\left[\int_t^T Y^{(t,y)}(s)I_1(Y^{(t,y)}(s))ds + Y^{(t,y)}(T)I_2(Y^{(t,y)}(T))\,|\mathbb{F}_t\right]\right\}\\
&= \frac{1}{y}E\left[\int_t^T Y^{(t,y)}(s)I_1(Y^{(t,y)}(s))ds + Y^{(t,y)}(T)I_2(Y^{(t,y)}(T))\,|\mathbb{F}_t\right]
\end{aligned}$$

The wealth process of an agent handled with a portfolio $\pi^*(s)$ and a consumption $C^*(s)$, this process satisfies

$$\begin{aligned}
&\frac{X^{x,\pi^*}(s)B(t)}{B(s)} + \int_t^s \frac{c^*(u)B(t)}{B(u)}du\\
&= x + \int_t^s \frac{\pi^*(u)B(t)}{B(u)}\sigma(u)\mathrm{d}W^*(u) + \sum_{i=1}^k \int_t^s \frac{\pi^*(u)B(t)}{B(u)}\varphi_i(u)\mathrm{d}M_i{}^*(u)
\end{aligned} \tag{7}$$

We have from equation (7) that $\dfrac{X^{x,\pi^*}(s)B(t)}{B(s)} + \int_t^s \dfrac{c^*(u)B(t)}{B(u)}du$ is a P^* martingale, as well as "Bayes's rule", we get

$$\begin{aligned}
&E^*\left[\frac{X^{x,\pi^*}(T)B(t)}{B(T)} + \int_t^T \frac{c^*(u)B(t)}{B(u)}du\,|\mathbb{F}_t\right]E\left[\frac{L(T)}{L(t)}\,|\mathbb{F}_t\right]\\
&= E\left[\left[\frac{X^{x,\pi^*}(T)B(t)}{B(T)} + \int_t^T \frac{c^*(u)B(t)}{B(u)}du\right]\frac{L(T)}{L(t)}\,|_t\right]
\end{aligned}$$

That is $E\left[X^{x,\pi^*}(T)Y^{(t,1)}(T) + \int_t^T c^*(u)Y^{(t,1)}(u)du\,|\mathbb{F}_t\right] = x$

Since

$$\begin{aligned}
x &= \mathcal{Y}(t,\mathcal{Y}^{-1}(t,x))\\
&= E\left[\int_t^T Y^{(t,1)}(s)I_1(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(s))ds + Y^{(t,1)}(T)I_2(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(T))\,|\mathbb{F}_t\right]
\end{aligned}$$

We have

$$X^{x,\pi^*}(T) = I_2(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(T)),$$
$$c^*(u) = I_1(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(u)),$$

$$c^*(t) = I_1(V_x'(t,x)),$$
$$Y^{(t,1)}(t) = 1$$
$$\text{and} \ \ c^*(u) = I_1(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(u))$$

implies $V_x'(t,x) = \mathcal{Y}^{-1}(t,x)$. So

$$\pi^*(s) = \frac{r - b - \sum_{i=1}^{k} \lambda_i \varphi_i \theta_{2i}}{\sigma^2} \frac{\mathcal{Y}^{-1}(s, X^{x,\pi^*}(s))}{{\mathcal{Y}^{-1}}_x{}'(s, X^{x,\pi^*}(s))}$$

□

Proposition 3 Let $G(t,y) = E\left[\int_t^T U_1\left(I_1(yY^{(t,1)}(s))\right) ds + U_2\left(I_2(yY^{(t,1)}(T))\right)\right]$, then the optimal wealth and the value function satisfy

$$X^{x,\pi^*}(s) = \frac{1}{Y^{(t,1)}(s)} E\left[X^{x,\pi^*}(T)Y^{(t,1)}(T) + \int_s^T c^*(u)Y^{(t,1)}(u)du \,|\mathbb{F}_s\right]$$
$$V(t,x) = G(t, \mathcal{Y}^{-1}(t,x))$$

Proof. Since $\dfrac{X^{x,\pi^*}(s)B(t)}{B(s)} + \int_t^s \dfrac{c^*(u)B(t)}{B(u)}du$ is a P^* martingale, we obtain

$$\frac{X^{x,\pi^*}(s)B(t)}{B(s)} + \int_t^s \frac{c^*(u)B(t)}{B(u)}du = E^*\left[\frac{X^{x,\pi^*}(T)B(t)}{B(T)} + \int_t^T \frac{c^*(u)B(t)}{B(u)}du \,|\mathbb{F}_s\right]$$

$$\frac{X^{x,\pi^*}(s)B(t)}{B(s)} = E^*\left[\frac{X^{x,\pi^*}(T)B(t)}{B(T)} + \int_t^T \frac{c^*(u)B(t)}{B(u)}du \,|\mathbb{F}_s\right] - \int_t^s \frac{c^*(u)B(t)}{B(u)}du$$
$$= \frac{E\left[\frac{X^{x,\pi^*}(T)B(t)L(T)}{B(T)L(t)} + \int_t^T \frac{c^*(u)B(t)L(u)}{B(u)L(t)}du \,|\mathbb{F}_s\right]}{E\left[\dfrac{L(T)}{L(t)}|\mathbb{F}_s\right]} - \int_t^s \frac{c^*(u)B(t)}{B(u)}du$$
$$= \frac{L(t)}{L(s)}E\left[\frac{X^{x,\pi^*}(T)B(t)L(T)}{B(T)L(t)}|\mathbb{F}_s\right] + \frac{L(t)}{L(s)}E\left[\int_s^T \frac{c^*(u)B(t)L(u)}{B(u)L(t)}du \,|\mathbb{F}_s\right]$$

That is

$$X^{x,\pi^*}(s) = \frac{1}{Y^{(t,1)}(s)} E\left[X^{x,\pi^*}(T)Y^{(t,1)}(T) + \int_s^T c^*(u)Y^{(t,1)}(u)du \,|\mathbb{F}_s\right]$$

Let $G(t,y) = E\left[\int_t^T U_1\left(I_1(yY^{(t,1)}(s))\right)ds + U_2\left(I_2(yY^{(t,1)}(T))\right)\right]$, Proposition 2 implies that

$$X^{x,\pi^*}(T) = I_2\left(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(T)\right)$$
$$c^*(u) = I_1\left(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(u)\right)$$

We have

$$
\begin{aligned}
V(t,x) &= E\left[\int_t^T U_1(C^*(s))ds + U_2(X^{x,\pi^*}(T))\,\Big|X^{x,\pi^*}(t)=x\right] \\
&= E\left[\int_t^T U_1\left(I_1(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(s))\right)ds + U_2\left(I_2(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(T))\right)\right] \\
&= G\left(t,\mathcal{Y}^{-1}(t,x)\right)
\end{aligned}
$$

□

Proposition 4 Assume $U_1(x)=U_2(x)=\log x,\ 0<x<\infty$. The value function, the optimal wealth process and the optimal portfolio satisfy

$$
\begin{aligned}
X^{x,\pi^*}(s) &= \frac{x(T-s+1)}{T-t+1}\frac{B(s)L(t)}{B(t)L(s)} \\
V(t,x) &= (T-t+1)\log\frac{x}{T-t+1} - E\left[\int_t^T \log Y^{(t,1)}(s)ds + \log Y^{(t,1)}(T)\right] \\
\pi^*(s) &= \frac{(b-r+\sum\limits_{i=1}^{k}\lambda_i\varphi_i\theta_{2i})}{\sigma^2}X^{x,\pi^*}(s) \\
c^*(u) &= \frac{x}{T-t+1}\frac{B(u)L(t)}{B(t)L(u)}
\end{aligned}
$$

Proof. Because of $U_1(x)=U_2(x)=\log x,\ 0<x<\infty$ in this case, $I_1(y)=I_2(y)=\frac{1}{y}$, $\mathcal{Y}(t,y)=\frac{T-t+1}{y}$, and $\mathcal{Y}^{-1}(t,x)=\frac{T-t+1}{x}$, $G(t,y)=(t-T-1)\log y - E\left[\int_t^T \log Y^{(t,1)}(s)ds + \log Y^{(t,1)}(T)\right]$, then the consumption and optimal terminal wealth are given respectively by

$$
\begin{aligned}
c^*(u) &= \frac{x}{T-t+1}\frac{B(u)L(t)}{B(t)L(u)}, \\
X^{x,\pi^*}(T) &= \frac{x}{T-t+1}\frac{B(T)L(t)}{B(t)L(T)}.
\end{aligned}
$$

The investor's wealth process

$$
\begin{aligned}
X^{x,\pi^*}(s) &= \frac{1}{Y^{(t,1)}(s)}E\left[X^{x,\pi^*}(T)Y^{(t,1)}(T) + \int_s^T c^*(u)Y^{(t,1)}(u)du\,|\mathbb{F}_s\right] \\
&= \frac{x(T-s+1)}{T-t+1}\frac{B(s)L(t)}{B(t)L(s)}
\end{aligned}
$$

The value function satisfies

$$
\begin{aligned}
V(t,x) &= G(t,\mathcal{Y}^{-1}(t,x)) \\
&= (T-t+1)\log\frac{x}{T-t+1} - E[\int_t^T \log Y^{(t,1)}(s)ds + \log Y^{(t,1)}(T)] \\
\pi^*(s) &= \frac{r-b-\sum_{i=1}^{k}\lambda_i\varphi_i\theta_{2i}}{\sigma^2}\frac{\mathcal{Y}^{-1}(s,X^{x,\pi^*}(s))}{\mathcal{Y}^{-1}{}_x{}'(s,X^{x,\pi^*}(s))} \\
&= \frac{(b-r+\sum_{i=1}^{k}\lambda_i\varphi_i\theta_{2i})}{\sigma^2}X^{x,\pi^*}(s)
\end{aligned}
$$

□

Proposition 5 Assume utility function is $U_1(x)=U_2(x)=\dfrac{x^p}{p},\ 0<x<\infty, 0<p<1$. The value function, the optimal wealth process and the optimal portfolio satisfy

$$
\begin{aligned}
X^{x,\pi^*}(s) &= \frac{x}{\mathcal{Y}(t,1)Y^{(t,1)}(s)}E\left[[Y^{(t,1)}(T)]^{\frac{p}{p-1}} + \int_s^T [Y^{(t,1)}(u)]^{\frac{p}{p-1}}du\,|\mathbb{F}_s\right] \\
V(t,x) &= \frac{x^p}{p}\mathcal{Y}(t,1)^{1-p} \\
\pi^*(s) &= \frac{(b-r+\sum_{i=1}^{k}\lambda_i\varphi_i\theta_{2i})}{\sigma^2}\frac{X^{x,\pi^*}(s)}{1-p} \\
c^*(u) &= \frac{x}{\mathcal{Y}(t,1)}\Big[Y^{(t,1)}(u)\Big]^{\frac{1}{p-1}}
\end{aligned}
$$

Proof. $U_1(x)=U_2(x)=\dfrac{x^p}{p},\ 0<x<\infty, 0<p<1$, we have

$$
\begin{aligned}
I_1(y) &= I_2(y) = y^{\frac{1}{p-1}}, \\
\mathcal{Y}(t,y) &= \frac{1}{y}E\left[\int_t^T Y^{(t,y)}(s)I_1\left(Y^{(t,y)}(s)\right)ds + Y^{(t,y)}(T)I_2\left(Y^{(t,y)}(T)\right)|\mathbb{F}_t\right] \\
&= y^{\frac{1}{p-1}}E\left[\int_t^T Y^{(t,1)}(s)I_1(Y^{(t,1)}(s))ds + Y^{(t,1)}(T)I_2(Y^{(t,1)}(T))\,|\mathbb{F}_t\right] \\
&= y^{\frac{1}{p-1}}\mathcal{Y}(t,1), \\
\mathcal{Y}^{-1}(t,x) &= \left(\frac{x}{\mathcal{Y}(t,1)}\right)^{p-1}, \\
G(t,y) &= E\left[\int_t^T U_1\left(I_1(Y^{(t,y)}(s))\right)ds + U_2\left(I_2(Y^{(t,y)}(T))\right)\right] = \frac{1}{p}y^{\frac{p}{p-1}}\mathcal{Y}(t,1),
\end{aligned}
$$

then the consumption and optimal terminal wealth are given respectively by

$$c^*(u) = I_1\left(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(u)\right) = \frac{x}{\mathcal{Y}(t,1)}\left[Y^{(t,1)}(u)\right]^{\frac{1}{p-1}},$$

$$X^{x,\pi^*}(T) = I_2\left(\mathcal{Y}^{-1}(t,x)Y^{(t,1)}(T)\right) = \frac{x}{\mathcal{Y}(t,1)}\left[Y^{(t,1)}(T)\right]^{\frac{1}{p-1}}.$$

The wealth process

$$\begin{aligned}X^{x,\pi^*}(s) &= \frac{1}{Y^{(t,1)}(s)}E\left[X^{x,\pi^*}(T)Y^{(t,1)}(T) + \int_s^T c^*(u)Y^{(t,1)}(u)du\,|\mathbb{F}_s\right]\\ &= \frac{x}{\mathcal{Y}(t,1)Y^{(t,1)}(s)}E\left[\left[Y^{(t,1)}(T)\right]^{\frac{p}{p-1}} + \int_s^T \left[Y^{(t,1)}(u)\right]^{\frac{p}{p-1}}du\,|\mathbb{F}_s\right]\end{aligned}$$

The value function satisfy

$$V(t,x) = G(t,\mathcal{Y}^{-1}(t,x)) = \frac{x^p}{p}\mathcal{Y}(t,1)^{1-p}$$

$$\pi^*(s) = \frac{r-b-\sum_{i=1}^{k}\lambda_i\varphi_i\theta_{2i}}{\sigma^2}\frac{\mathcal{Y}^{-1}(s,X^{x,\pi^*}(s))}{\mathcal{Y}^{-1}{}_x{}'(s,X^{x,\pi^*}(s))} = \frac{(b-r+\sum_{i=1}^{k}\lambda_i\varphi_i\theta_{2i})}{\sigma^2}\frac{X^{x,\pi^*}(s)}{1-p}$$

□

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