CSCanada
Studies in Mathematical Sciences
Vol. 5, No. 2, 2012, pp. [13–25]
**DOI:** 10.3968/j.sms.1923845220120502.1452

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

# New Exact Solutions of Nonlinear Partial Differential Equations Using Tan-Cot Function Method

Anwar Ja'afar Mohamad Jawad[a],*

[a] Computer Engineering Technique Department, Al-Rafidain University College, Baghdad, Iraq.

* Corresponding author.
Address: Computer Engineering Technique Department, Al-Rafidain University College, Baghdad, Iraq; E-Mail: anwar_jawad2001@yahoo.com

Received: August 2, 2012/ Accepted: September 17, 2012/ Published: November 30, 2012

**Abstract:** In this paper, we established a traveling wave solution by using the proposed Tan-Cot function algorithm for nonlinear partial differential equations. The method is used to obtain new solitary wave solutions for various type of nonlinear partial differential equations such as, the (2+1) - dimensional nonlinear Schrödinger equation, Gardner equation, the modified KdV equation, perturbed Burgers equation, general Burger's-Fisher equation, and Benjamin-Bona-Mahony equation, which are the important Soliton equations. Proposed method has been successfully implemented to establish new solitary wave solutions for the nonlinear PDEs.

**Key words:** Nonlinear PDEs; Exact solutions; Tan-cot function method; Schrödinger equation; Gardner equation; The modified KdV equation; Perturbed Burgers equation; General Burger's-Fisher equation; Benjamin-Bona-Mahony equation

Anwar Ja'afar Mohamad Jawad (2012). New Exact Solutions of Nonlinear Partial Differential Equations Using Tan-Cot Function Method. *Studies in Mathematical Sciences, 5*(2), 13–25. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220120502.1452 DOI: 10.3968/j.sms.1923845220120502.1452

## 1. INTRODUCTION

Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for exact solutions of nonlinear partial differential equations [1]. Exact solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh-sech method [2–4], extended tanh method [5–7], hyperbolic function method [8,9], Jacobi elliptic function expansion method [10], F-expansion method [11], and the First Integral method [12,13]. The sine-cosine method [3,14,15] has been used to solve different types of nonlinear systems of PDEs.

In this paper, we applied the Tan-Cot method to solve the (2+1) - dimensional nonlinear Schrödinger equation, Gardner equation, modified KdV equation, perturbed Burgers equation, general Burger's-Fisher equation, and Benjamin-Bona-Mahony equation given respectively by:

$$iq_t + aq_{xx} - bq_{yy} + c|q|^2 q = 0 \tag{1}$$

$$u_t - 6(u + \varepsilon^2 u^2)u_x + u_{xxx} = 0 \tag{2}$$

$$u_t - \delta u^2 u_x + u_{xxx} = 0 \tag{3}$$

$$u_t + auu_x + bu_{xx} = cu^2 u_x + \beta uu_{xx} + \gamma(u_x)^2 + \delta u_{xxx} \tag{4}$$

$$u_t - au^n u_x + bu_{xx} + cu(1 - u^n) = 0 \tag{5}$$

$$u_t - u_{xxt} + u_x + uu_x = 0 \tag{6}$$

## 2. THE TAN-COT FUNCTION METHOD

Consider the nonlinear partial differential equation in the form

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \dots \dots \dots) = 0 \tag{7}$$

where $u(x, y, t)$ is a traveling wave solution of nonlinear partial differential equation Equation (7). We use the transformations,

$$u(x, y, t) = f(\xi) \tag{8}$$

where $\xi = x + y - \lambda t$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(\cdot) = -\lambda \frac{d}{d\xi}(\cdot), \ \frac{\partial}{\partial x}(\cdot) = \frac{d}{d\xi}(\cdot), \ \frac{\partial}{\partial y}(\cdot) = \frac{d}{d\xi}(\cdot) \tag{9}$$

Using Equation (9) to transfer the nonlinear partial differential equation Eq.(7) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \dots) = 0 \tag{10}$$

The ordinary differential equation (10) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form:

$$f(\xi) = \alpha \tan^{\beta}(\mu\xi), \quad |\xi| \le \frac{\pi}{2\mu}$$

or in the form

$$f(\xi) = \alpha \cot^{\beta}(\mu\xi), \quad |\xi| \leq \frac{\pi}{2\mu} \tag{11}$$

where $\alpha$, $\mu$, and $\beta$ are parameters to be determined, $\mu$ and $c$ are the wave number and the wave speed, respectively. We use

$$\begin{aligned} f(\xi) &= \alpha \tan^{\beta}(\mu\xi) \\ f' &= \alpha\beta\mu\left[\tan^{\beta-1}(\mu\xi) + \tan^{\beta+1}(\mu\xi)\right] \\ f'' &= \alpha\beta\mu^2\left[(\beta-1)\tan^{\beta-2}(\mu\xi) + 2\beta\tan^{\beta}(\mu\xi) + (\beta+1)\tan^{\beta+2}(\mu\xi)\right] \\ f''' &= \beta\mu^3\alpha\left[(\beta-1)(\beta-2)\tan^{\beta-3}(\mu\xi) + (3\beta^2-3\beta+2)\tan^{\beta-1}(\mu\xi)\right. \\ &\quad \left. + (\beta+1)(\beta+2)\tan^{\beta}(\mu\xi) + 2\beta^2\tan^{\beta+1}(\mu\xi) + (\beta+1)(\beta+2)\tan^{\beta+2}(\mu\xi)\right] \end{aligned} \tag{12}$$

and their derivative. Or use

$$\begin{aligned} f(\xi) &= \alpha \cot^{\beta}(\mu\xi) \\ f' &= -\alpha\beta\mu\left[\cot^{\beta-1}(\mu\xi) + \cot^{\beta+1}(\mu\xi)\right] \\ f'' &= \alpha\beta\mu^2\left[(\beta-1)\cot^{\beta-2}(\mu\xi) + 2\beta\cot^{\beta}(\mu\xi) + (\beta+1)\cot^{\beta+2}(\mu\xi)\right] \end{aligned} \tag{13}$$

and so on. We substitute (12) or (13) into the reduced equation (10), balance the terms of the tan functions when (12) are used, or balance the terms of the cot functions when (13) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\tan^k(\mu\xi)$ or $\cot^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's $\alpha$, $\mu$ and $\beta$, and solve the subsequent system.

## 3. APPLICATIONS

### *3.1. Schrödinger Equation*

Let us first consider the (2+1) - dimensional nonlinear Schr$\ddot{o}$dinger equation (16) that reads:

$$iq_t + aq_{xx} - bq_{yy} + c|q|^2 q = 0 \tag{14}$$

where $a$, $b$ and $c$ are nonzero constants. Firstly, we introduce the transformations

$$q(x, y, t) = e^{(i\theta)} \cdot u(\xi), \quad \theta = \alpha x + \omega y + \delta t, \quad \xi = k(x + ly - \lambda t) \tag{15}$$

where $\alpha$, $\omega$, $\delta$, $k$, $l$ and $\lambda$ are real constants. Substituting Equation (15) into Equation (14), we obtain the $\lambda = 2(a\alpha - b\omega l)$ and $u(\xi)$ satisfy into the ODE:

$$-(\delta + a\alpha^2 - b\omega^2)u(\xi) + (a - bl^2)k^2 u''(\xi) + c(u(\xi))^3 = 0 \tag{16}$$

Rewrite this second-order ordinary differential equation as follows:

$$u'' + k_1 u^3 - k_2 u = 0 \tag{17}$$

Where

$$k_1 = \frac{c}{(a - bl^2)k^2}, \quad k_2 = \frac{\delta + a\alpha^2 - b\omega^2}{(a - bl^2)k^2} \tag{18}$$

Seeking solutions of the form (12) we get:

$$\begin{aligned}&\alpha\beta\mu^2\left[(\beta-1)\tan^{\beta-2}(\mu\xi)+2\beta\tan^{\beta}(\mu\xi)+(\beta+1)\tan^{\beta+2}(\mu\xi)\right]\\&+k_1\alpha^3\tan^{3\beta}(\mu\xi)-k_2\alpha\tan^{\beta}(\mu\xi)=0\end{aligned}\tag{19}$$

Equating the exponents and the coefficients of each pair of the tan functions we find the following algebraic system:

$$\begin{aligned}&\beta+2=3\beta \quad \text{then} \quad \beta=1\\&\alpha\beta\mu^2(\beta+1)+k_1\alpha^3=0\\&2\alpha\beta^2\mu^2-k_2\alpha=0\end{aligned}\tag{20}$$

By solving the algebraic system (20), we get

$$\alpha=\mp i\sqrt{\frac{\delta+a\alpha^2-b\omega^2}{c}},\quad \mu=\mp\sqrt{\frac{\delta+a\alpha^2-b\omega^2}{2(a-bl^2)k^2}}\tag{21}$$

Then by substituting Equation (21) into Equation (12) then, the exact soliton solution of equation (14) can be written in the form:

$$u(\xi)=\mp i\sqrt{\frac{\delta+a\alpha^2-b\omega^2}{c}}\tan\left(\mp\sqrt{\frac{\delta+a\alpha^2-b\omega^2}{2(a-bl^2)}}(x+ly-\lambda t)\right)\tag{22}$$

Therefore

$$q(x,y,t)=\mp i\sqrt{\frac{\delta+a\alpha^2-b\omega^2}{c}}\tan\left(\mp\sqrt{\frac{\delta+a\alpha^2-b\omega^2}{2(a-bl^2)}}(x+ly-\lambda t)\right)e^{i(\alpha x+\omega y+\delta t)}\tag{23}$$

for $\alpha=\omega=\delta=k=l=1$, $a\neq b$, $a=c=1$, $b=.5$ and (23) becomes:

$$q(x,y,t)=\mp i\sqrt{\frac{3}{2}}\tan\left(\mp\sqrt{\frac{3}{2}}(x+y-t)\right)e^{i(x+y+t)}\tag{24}$$

### *3.2. Gardner Equation*

Consider the Gardner equation [17,18]

$$u_t-6(u+\varepsilon^2u^2)u_x+u_{xxx}=0\tag{25}$$

This equation known as the mixed KdV-mKdV equation is very widely studied in various areas of Physics that includes Plasma Physics, Fluid Dynamics, Quantum Field Theory, Solid State Physics and others [18].

We introduce the transformation $\xi=k(x-\lambda t)$, where $k$, and $\lambda$ are real constants. Equation (25) transforms to the ODE:

$$-k\lambda u'-3k(u^2)'-2\varepsilon^2k(u^3)'+k^3u'''=0\tag{26}$$

Integrating (26) once with zero constant to get the following ordinary differential equation:

$$\lambda u + 3u^2 + 2\varepsilon^2 u^3 - k^2 u'' = 0 \tag{27}$$

Seeking the solution in (13)

$$\begin{aligned} &\lambda\alpha \cot^{\beta}(\mu\xi) + 3\alpha^2 \cot^{2\beta}(\mu\xi) + 2\varepsilon^2\alpha^3 \cot^{3\beta}(\mu\xi) \\ &-\alpha\beta\mu^2 k^2 \left[(\beta-1)\cot^{\beta-2}(\mu\xi) + 2\beta\cot^{\beta}(\mu\xi) + (\beta+1)\cot^{\beta+2}(\mu\xi)\right] = 0 \end{aligned} \tag{28}$$

Equating the exponents and the coefficients of each pair of the cot functions we find the following algebraic system:

$$3\beta = \beta - 2 \ \rightarrow \ \beta = -1 \tag{29}$$

Substituting Equation (29) into Equation (28) to get:

$$\begin{aligned} &\lambda\alpha \cot^{-1}(\mu\xi) + 3\alpha^2 \cot^{-2}(\mu\xi) + 2\varepsilon^2\alpha^3 \cot^{-3}(\mu\xi) \\ &-\alpha\beta\mu^2 k^2 \left[(\beta-1)\cot^{-3}(\mu\xi) + 2\beta\cot^{-1}(\mu\xi)\right] = 0 \end{aligned} \tag{30}$$

Equating the exponents and the coefficients of each pair of the cot function, we obtain a system of algebraic equations:

$$\begin{aligned} &\cot^{-3}(\mu\xi): \ 2\varepsilon^2\alpha^3 - 2\alpha\mu^2 k^2 = 0 \\ &\cot^{-2}(\mu\xi): \ 3\alpha^2 = 0 \\ &\cot^{-1}(\mu\xi): \ \lambda\alpha - 2\alpha\mu^2 k^2 = 0 \end{aligned} \tag{31}$$

By solving the algebraic system (31), we get,

$$\lambda = 2\mu^2 k^2, \quad \alpha = \mp\frac{k\mu}{\varepsilon} \tag{32}$$

Then by substituting Equation (32) into Equation (12), the exact soliton solution of equation (27) can be written in the form

$$u(x,t) = \mp\frac{k\mu}{\varepsilon}\tan\left(\mu k(x - 2\mu^2 k^2 t)\right), \quad 0 < \mu k(x - 2\mu^2 k^2 t) < \pi \tag{33}$$

For $\mu = k = \varepsilon = 1$, then (33) becomes:

$$u(x,t) = \tan(x - 2t) \tag{34}$$

$u(x,t)$ in (34) is represented in Figure 1 for $-10 \le x \le 10$ and $0 \le t \le 1$.

### *3.3. Dispersive Equation*

Consider the (1+1) - dimensional nonlinear dispersive equation

$$u_t - \delta u^2 u_x + u_{xxx} = 0 \tag{35}$$

where $\delta$ is a nonzero positive constant. This equation is called the modified KdV equation [19], which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits

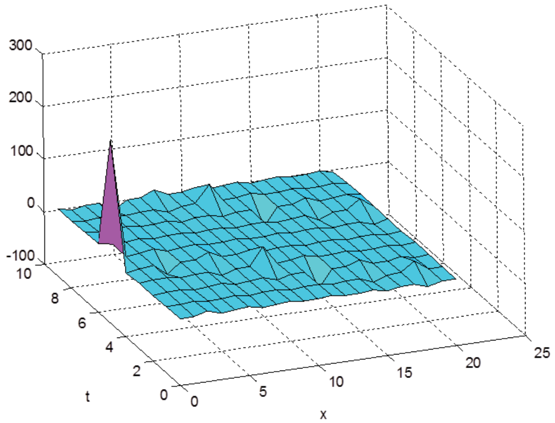


**Figure 1**
**Represents $u(x,t)$ in (34) for $-10 \leq x \leq 10$ and $0 \leq t \leq 1$**

compaction solitons with compact support. To find the traveling wave solutions of Equation (29), [19] used $G'/G$ expansion Method.

Let us now solve Equation (35) by the proposed method. We introduce the transformation $\xi = k(x - \lambda t)$, where $k$ and $\lambda$ are real constants. Equation (35) transforms to the ODE:

$$-k\lambda u' - \frac{\delta}{3}k(u^3)' + k^3 u''' = 0 \tag{36}$$

Integrating (36) once with zero constant to get the following ordinary differential equation:

$$\lambda u + \frac{\delta}{3}u^3 - k^2 u'' = 0 \tag{37}$$

Seeking the solution in (13)

$$\begin{aligned} &\lambda\alpha \cot^{\beta}(\mu\xi) + \frac{\delta}{3}\alpha^3 \cot^{3\beta}(\mu\xi) \\ &-\alpha\beta\mu^2 k^2 \left[(\beta - 1)\cot^{\beta-2}(\mu\xi) + 2\beta \cot^{\beta}(\mu\xi) + (\beta + 1)\cot^{\beta+2}(\mu\xi)\right] = 0 \end{aligned} \tag{38}$$

Equating the exponents and the coefficients of each pair of the cot functions we find the following algebraic system: $3\beta = \beta + 2 \;\rightarrow\; \beta = 1$

$$\begin{aligned} \cot^3(\mu\xi):&\ \frac{\delta}{3}\alpha^3 - \alpha\beta\mu^2 k^2(\beta + 1) = 0 \\ \cot^1(\mu\xi):&\ \lambda\alpha - \alpha\beta^2\mu^2 k^2 = 0 \end{aligned} \tag{39}$$

By solving the algebraic system (39), we get

$$\lambda = \mu^2 k^2, \quad \alpha = \mp\sqrt{\frac{6}{\delta}}k\mu \tag{40}$$

Then by substituting Equation (40) into Equation (13), the exact soliton solution of Equation (35) can be written in the form

$$u(x,t) = \mp\sqrt{\frac{6}{\delta}}k\mu \cot\left(\mu k(x-\mu^2k^2t)\right), \quad 0 < \mu k(x-\mu^2k^2t) < \pi \tag{41}$$

For $\mu = k = \delta = 1$, Equation (41) becomes

$$u(x,t) = \mp\sqrt{6}\cot(x-t) \tag{42}$$

$u(x,t)$ in (42) is represented in Figure 2 for $-10 \le x \le 10$ and $0 \le t \le 1$.

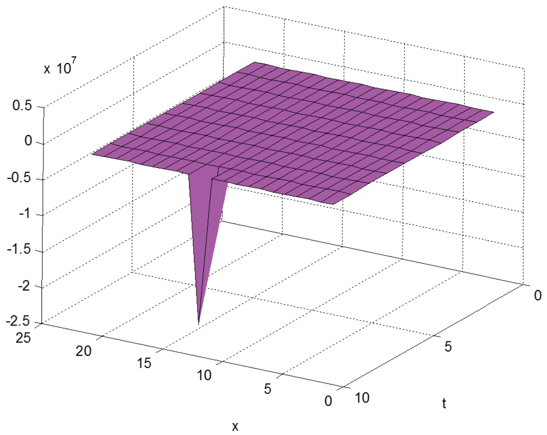


**Figure 2**
**Represents $u(x,t)$ in (42) for $-10 \le x \le 10$ and $0 \le t \le 1$**

## *3.4. Perturbed Burgers Equation*

In this section the study is going to be focused on the perturbed Burgers equation. The solitary wave ansatz method will be adopted to obtain the exact 1-soliton solution of the Burgers equation in (1+1) dimensions. The search is going to be for a topological 1-soliton solution. The perturbed Burgers equation that is given by the following form [20]:

$$u_t + auu_x + bu_{xx} = cu^2u_x + \beta uu_{xx} + \gamma(u_x)^2 + \delta u_{xxx} \tag{43}$$

Equation (43) appears in the study of gas dynamics and also in free surface motion of waves in heated fluids. The perturbation terms are obtained from long-wave perturbation theory. Equation (43) shows up in the long-wave small-amplitude limit of extended systems dominated by dissipation, where dispersion is also present at a higher order [20].

To solve Equation (43) by the proposed method. We introduce the transformation $\xi = k(x-\lambda t)$, where $k$, and $\lambda$ are real constants. Equation (43) transforms to the ODE:

$$-[\lambda - au + cu^2]u' + [bk - dku]u'' - \gamma k(u')^2 - \delta k^2 u''' = 0 \tag{44}$$

Seeking the solution in (12)

$$
\begin{aligned}
&-\alpha\beta\mu\left[\lambda\tan^{\beta-1}(\mu\xi)-a\alpha\tan^{2\beta-1}(\mu\xi)+c\alpha^2\tan^{3\beta-1}(\mu\xi)\right]\\
&-\alpha\beta\mu\left[\lambda-a\alpha\tan^{2\beta+1}(\mu\xi)+c\alpha^2\tan^{3\beta+1}(\mu\xi)\right]\\
&+\alpha\beta\mu^2bk\left[(\beta-1)\tan^{\beta-2}(\mu\xi)+2\beta\tan^{\beta}(\mu\xi)+(\beta+1)\tan^{\beta+2}(\mu\xi)\right]\\
&-\alpha\beta\mu^2dka\left[(\beta-1)\tan^{2\beta-2}(\mu\xi)+2\beta\tan^{2\beta}(\mu\xi)+(\beta+1)\tan^{2\beta+2}(\mu\xi)\right]\\
&-\gamma k\alpha^2\beta^2\mu^2\left[\tan^{2\beta-2}(\mu\xi)+2\tan^{2\beta}(\mu\xi)+\tan^{2\beta+2}(\mu\xi)\right]\\
&-\delta k^2\beta\mu^3\alpha\left[(\beta-1)(\beta-2)\tan^{\beta-3}(\mu\xi)+(3\beta^2-3\beta+2)\tan^{\beta-1}(\mu\xi)\right.\\
&\left.+(\beta+1)(\beta+2)\tan^{\beta}(\mu\xi)+2\beta^2\tan^{\beta+1}(\mu\xi)+(\beta+1)(\beta+2)\tan^{\beta+2}(\mu\xi)\right]=0
\end{aligned}
\tag{45}
$$

From (45), equating exponents $2\beta-2$ and $3\beta-1$ yield

$$2\beta-2=3\beta-1,\ \text{so that}\ \beta=-1 \tag{46}$$

It needs to be noted that the same value of $\beta$ is obtained when the exponent pairs $\beta-2=2\beta-1$, $2\beta-2=\beta-3$ are equated, thus setting their coefficients to zero yields:

$$
\begin{aligned}
&\lambda+c\alpha^2-2\mu dk\alpha-2\gamma\alpha k\mu+8\delta k^2\mu^2=0\\
&c\alpha^2-2\mu dk\alpha-\gamma\alpha k\mu+6\delta k^2\mu^2=0\\
&-a\alpha+2\mu bk=0\\
&\lambda-\gamma\alpha k\mu=0
\end{aligned}
\tag{47}
$$

By solving the algebraic system (47), we get

$$\delta=\frac{b}{3a^2}\left[2da-2cb+\gamma a\right],\ \alpha=\frac{2\mu bk}{a},\ \lambda=2\frac{b}{a}\gamma k^2\mu^2 \tag{48}$$

Then by substituting Equation (48) into Equation (12), the exact soliton solution of equation (43) can be written in the form

$$u(x,t)=\frac{2bk}{a}\mu\cot\left[\mu k(x-2\frac{b}{a}\gamma k^2\mu^2t)\right] \tag{49}$$

for $\mu=k=a=b=1$, $\gamma=-1$

$$u(x,t)=2\cot(x+2t) \tag{50}$$

Figure 3 represents $u(x,t)$ in (50) for $-10\le x\le 10$ and $0.1\le t\le 1$.

### *3.5. The General Burgers-Fisher Equation*

Consider the following general Burger's-Fisher equation [21]

$$u_t-au^nu_x+bu_{xx}+cu(1-u^n)=0 \tag{51}$$

where $a$, $b$ and $c$ are nonzero constants. We introduce the transformation $\xi=k(x-\lambda t)$, where $k$, and $\lambda$ are real constants. The traveling wave variable $\xi$ permits us converting Equation (51) into the following ODE:

$$-\lambda ku'+aku^nu'+bk^2u''+cu-cu^{n+1}=0 \tag{52}$$

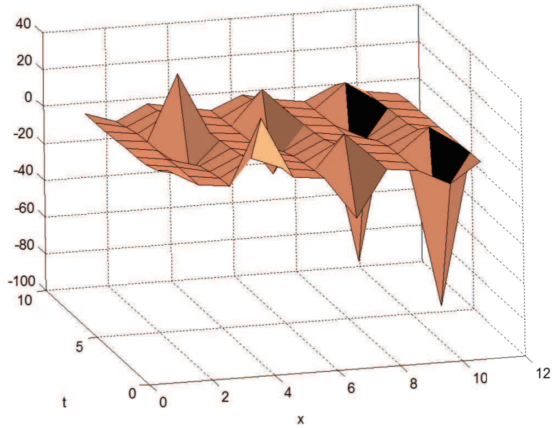


**Figure 3**
**Represents $u(x,t)$ in (50) for $-10 \le x \le 10$ and $0.1 \le t \le 1$**

Seeking the solution in (12)

$$\begin{aligned}&-\lambda k\alpha\beta\mu\left[\tan^{\beta-1}(\mu\xi)+\tan^{\beta+1}(\mu\xi)\right]\\&+ak\alpha^{n+1}\beta\mu\left[\tan^{(n+1)\beta-1}(\mu\xi)+\tan^{(n+1)\beta+1}(\mu\xi)\right]\\&+bk^2\alpha\beta\mu^2\left[(\beta-1)\tan^{\beta-1}(\mu\xi)+2\beta\tan^{\beta}(\mu\xi)+(\beta+1)\tan^{\beta+2}(\mu\xi)\right]\\&+c\alpha\tan^{\beta}(\mu\xi)-c\alpha^{n+1}\tan^{(n+1)\beta}(\mu\xi)=0\end{aligned} \tag{53}$$

From (53), equating exponents $(n+1)\beta+1$ and $\beta+2$ yield

$$(n+1)\beta+1=\beta+2 \tag{54}$$

so that

$$\beta=\frac{1}{n} \tag{55}$$

Equating exponents $\beta+1$ and $(n+1)\beta$ and the pairs $(n+1)\beta-1$ and $\beta$ gives the same value of $\beta$ in (55), so that substitute (55) in (53) then

$$\begin{aligned}&-\lambda k\alpha\beta\mu\left[\tan^{\frac{1}{n}-1}(\mu\xi)+\tan^{\frac{1}{n}+1}(\mu\xi)\right]\\&+ak\alpha^{n+1}\beta\mu\left[\tan^{\frac{1}{n}}(\mu\xi)+\tan^{\frac{1}{n}+2}(\mu\xi)\right]\\&+bk^2\alpha\beta\mu^2\left[(\beta-1)\tan^{\frac{1}{n}-2}(\mu\xi)+2\beta\tan^{\frac{1}{n}}(\mu\xi)+(\beta+1)\tan^{\frac{1}{n}+2}(\mu\xi)\right]\\&+c\alpha\tan^{\frac{1}{n}}(\mu\xi)-c\alpha^{n+1}\tan^{\frac{1}{n}+1}(\mu\xi)=0\end{aligned} \tag{56}$$

Thus setting the coefficients of the same pairs to zero yields:

$$\begin{aligned}&c\alpha^{n+1}+\lambda k\alpha\beta\mu=0\\&2bk^2\alpha\beta^2\mu^2+ak\alpha^{n+1}\beta\mu+c\alpha=0\\&bk^2\alpha\beta(\beta+1)\mu^2+ak\alpha^{n+1}\beta\mu=0\end{aligned} \tag{57}$$

By solving the algebraic system (57), we get

$$\lambda = \frac{b^2k^2\mu^2}{an^2}(n^2-1), \quad c = \frac{b\mu^2k^2}{n^2}(n-1), \quad \alpha = \left(-\frac{bk(n+1)\mu}{an}\right)^{\frac{1}{n}} \tag{58}$$

Then by substituting (58) into Equation (12), the exact soliton solution of Equation (51) can be written in the form

$$u(x,t) = \left[-\frac{bk(n+1)\mu}{an}\tan\left\{\mu k\left(x - \frac{b^2k^2\mu^2}{an^2}(n^2-1)t\right)\right\}\right]^{\frac{1}{n}} \tag{59}$$

It is worth noting that the proposed Tan-Cot function method is applicable here if $n$ does not equal to 1.

### *3.6. Benjamin-Bona-Mahony (BBM) Equation*

Consider the BBM equation [22]

$$u_t - u_{xxt} + u_x + uu_x = 0 \tag{60}$$

We introduce the transformation $\xi = k(x-\lambda t)$, where $k$, and $\lambda$ are real constants. The traveling wave variable $\xi$ permits us converting Equation (60) into the following ODE:

$$k^2\lambda u''' + [1+u-\lambda]u' = 0 \tag{61}$$

Seeking the solution in (12)

$$\begin{aligned}&k^2\lambda\beta\mu^3\alpha\left[(\beta-1)(\beta-2)\tan^{\beta-3}(\mu\xi) + (3\beta^2-3\beta+2)\tan^{\beta-1}(\mu\xi)\right.\\&+\left.(\beta+1)(\beta+2)\tan^{\beta}(\mu\xi) + 2\beta^2\tan^{\beta+1}(\mu\xi) + (\beta+1)(\beta+2)\tan^{\beta+2}(\mu\xi)\right]\\&+\alpha\beta\mu(1-\lambda)\{\tan^{\beta-1}(\mu\xi)+\tan^{\beta+1}(\mu\xi)\} + \alpha^2\beta\mu\{\tan^{2\beta-1}(\mu\xi)+\tan^{2\beta+1}(\mu\xi)\} = 0\end{aligned} \tag{62}$$

From Equation (62), equating exponents $\beta-3$ and $2\beta-1$ yield

$$\beta - 3 = 2\beta - 1 \tag{63}$$

so that

$$\beta = -2 \tag{64}$$

equating exponents of the pairs $\beta-1$ and $2\beta+1$ gives the same value of $\beta$ in (58), so that substitute (64) in (62) then setting coefficients of the same pairs in (62) to zero yields:

$$\begin{aligned}&k^2\lambda\mu^2(\beta-1)(\beta-2)+\alpha = 0\\&k^2\lambda\mu^2(3\beta^2-3\beta+2)+1-\lambda+\alpha = 0\\&2\beta^2k^2\lambda\mu^2+1-\lambda = 0\end{aligned} \tag{65}$$

By solving the algebraic system (65), we get

$$\alpha = -\frac{12k^2\mu^2}{1-8k^2\mu^2}, \quad \lambda = \frac{1}{1-8k^2\mu^2} \tag{66}$$

Then by substituting Equation (66) into Equation (12), the exact soliton solution of equation (60) can be written in the form

$$u(x,t) = -\frac{12k^2\mu^2}{1-8k^2\mu^2}\cot^2\left[\mu k\left(x - \frac{1}{1-8k^2\mu^2}t\right)\right] \tag{67}$$

For $k = \mu = 1$, Equation (67) becomes:

$$u(x,t) = \frac{12}{7}\cot^2\left(x + \frac{1}{7}t\right) \tag{68}$$

$u(x,t)$ in (68) is represented in Figure 4 for $-4 \le x \le 4$ and $0.1 \le t \le 1$.

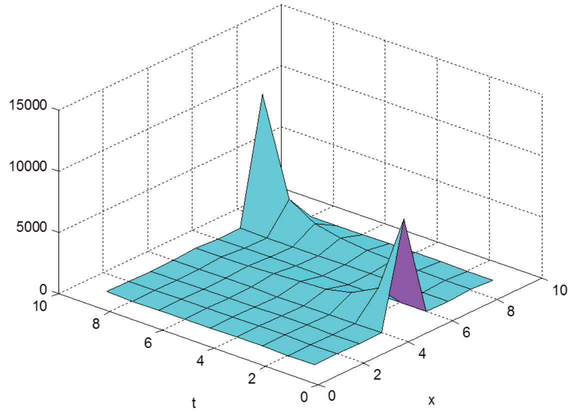


**Figure 4**
**Represents $u(x,t)$ in (68) for $-4 \le x \le 4$ and $0.1 \le t \le 1$**

## 4. CONCLUSION

In this paper, new method called the Tan-Cot function method has been successfully implemented to establish new solitary wave solutions for various types of nonlinear PDEs. We can say that the new method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas [23–29].

### REFERENCES

[1] Alquran, M., Al-Khaled, K., & Ananbeh, H. (2011). New soliton solutions for systems of nonlinear evolution equations by the rational sine-cosine method. *Studies in Mathematical Sciences, 3*(1), 1-9.
[2] Malfliet, W. (1992). Solitary wave solutions of nonlinear wave equations. *Am. J. Phys., 60*(7), 650-654.
[3] Khater, A. H., Malfliet, W., Callebaut, D. K., & Kamel, E. S. (2002). The tanh method, a simple transformation and exact analytical solutions for nonlinear reaction-diffusion equations. *Chaos Solitons Fractals, 14*(3), 513-522.

[4] Wazwaz, A. M. (2006). Two reliable methods for solving variants of the KdV equation with compact and noncompact structures. *Chaos Solitons Fractals, 28*(2), 454-462.

[5] El-Wakil, S. A., & Abdou, M. A. (2007). New exact travelling wave solutions using modified extended tanh-function method. *Chaos Solitons Fractals, 31*(4), 840-852.

[6] Fan, E. (2000). Extended tanh-function method and its applications to nonlinear equations. *Phys. Lett. A, 277*(4), 212-218.

[7] Wazwaz, A. M. (2005). The tanh-function method: Solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations. *Chaos Solitons and Fractals, 25*(1), 55-63.

[8] Xia, T. C., Li, B., & Zhang, H. Q. (2001). New explicit and exact solutions for the Nizhnik-Novikov-Vesselov equation. *Appl. Math. E-Notes, 1*, 139-142.

[9] Yusufoglu, E., & Bekir, A. (2006). Solitons and periodic solutions of coupled nonlinear evolution equations by using Sine-Cosine method. *Inter. J. Comput. Math., 83*(12), 915-924.

[10] Inc, M., & Ergut, M. (2005). Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method. *Appl. Math. E-Notes, 5*, 89-96.

[11] Zhang, S. (2006). The periodic wave solutions for the (2+1) - dimensional Konopelchenko Dubrovsky equations. *Chaos Solitons Fractals, 30*, 1213-1220.

[12] Feng, Z. S. (2002). The first integer method to study the Burgers-Korteweg-de Vries equation. *J. Phys. A. Math. Gen., 35*(2), 343-349.

[13] Ding, T. R., & Li, C. Z. (1996). *Ordinary differential equations.* Beijing: Peking University Press.

[14] Mitchell, A. R., & Griffiths, D. F. (1980). *The finite difference method in partial differential equations.* John Wiley & Sons.

[15] Parkes, E. J., & Duffy, B. R. (1998). An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations. *Comput. Phys. Commun., 98*, 288-300.

[16] Zhou, Y., Wang, M., & Miao, T. (2004). The periodic wave solutions and solitary for a class of nonlinear partial differential equations. *Phys. Lett. A, 323*, 77-88.

[17] Wazwaz, A. M. (2007). New solitons and kink solutions for the Gardner equation. *Communications in Nonlinear Science and Numerical Simulation, 12*, 1395-1404.

[18] Biswas, A. (2008). Soliton perturbation theory for the Gardner equation. *Adv. Studies Theor. Phys., 16*(2), 787-794.

[19] Elsayed, M., Zayed, E., & Al-Joudi, Shorog. (2009). The traveling wave solutions for nonlinear partial differential equations using the $(G'/G)$-expansion method. *International Journal of Nonlinear Science, 8*(4), 435-447.

[20] Jawad, A. J. M., Petkovic, M. D., & Biswas, A. (2010). Soliton solutions of Burgers equations and perturbed Burgers equation. *Applied Mathematics and Computation, 216*, 3370-3377.

[21] Javidi, M. (2006). Spectral collocation method for the solution of the generalized Burgers-Fisher equation. *Appl. Math. Comput., 174*, 345-352.

[22] Alquran, M., & Al-Khaled, K. (2011). Sine and solitary wave solution to the general Benjamin-Bona-Mahony-Burgers equation. *Physica Scripta, 83*, 065010.

[23] Biswas, A., Yildirim, A., Hayat, T., Aldossary, O. M., & Sassaman, R. (2012). Soliton perturbation theory of the generalized Klein-Gordon equation with full nonlinearity. *Proceedings of the Romanian Academy (Series A), 13*(1), 32-41.

[24] Triki, H., Yildirim, A., Hayat, T., Aldossary, O. M., & Biswas, A. (2012). Topological and non-topological soliton solutions of the Bretherton equation. *Proceedings of the Romanian Academy (Series A), 13*(2), 103-108.

[25] Ebadi, G., Kara, A. H., Petkovic, M. D., Yildirim, A., & Biswas, A. (2012). Solitons and conserved quantities of the Ito equation. *Proceedings of the Romanian Academy (Series A), 13*(3), 215-224.

[26] Johnpillai, A. G., Yildirim, A., & Biswas, A. (2012). Chiral solitons with Bohm potential by Lie group analysis and traveling wave hypothesis. *Romanian Journal of Physics, 57*(3-4), 545-554.

[27] Triki, H., Crutcher, S., Yildirim, A., Hayat, T., Aldossary, O. M., & Biswas, A. (2012). Bright and dark solitons of the modified complex Ginzburg Landau equation with parabolic and dual-power law nonlinearity. *Romanian Reports in Physics, 64*(2), 357-366.

[28] Crutcher, S., Oseo, A., Yildirim, A., & Biswas, A. (2012). Oscillatory parabolic law spatial optical solitons. *Journal of Optoelectronics and Advanced Materials, 14*(1-2), 29-40.

[29] Biswas, A., Khan, K., Rahaman, A., Yildirim, A., Hayat, T., & Aldossary, O. M. (2012). Bright and dark optical solitons ion birefringent fibers with Hamiltonian perturbations and Kerr law nonlinearity. *Journal of Optoelectronics and Advanced Materials, 14*(7-8), 571-576 .