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The Crossing Numbers of Cartesian Products of Stars with 5-Vertex Graphs

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Abstract: In this paper, the crossing number of Cartesian products of a specific 5-vertex graph with a star are given, and this fills up the crossing number list of Cartesian products of all 5-vertex graphs with stars (presented by Marian Klesc).

Key words: Crossing number; Star; Cartesian product; Graph

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1. INTRODUCTION

The graph theory terminology not defined here refer to [1], and with no special explanation all the related graphs are simple connected graphs. Let G be a simple graph with set vertex V and edge set E. The crossing number $cr(G)$ of a graph G is the minimum number of edge crossings in any drawing of G. It is well known that the crossing number of a graph is attained only in good drawings of the graph, which are those drawings with no edge crossing itself, no adjacent edges cross each other, no two edges intersecting more than once, and no three edges having a common point expected as their common vertex. Here make ϕ a good drawing of the graph

G, with the number of crossings in ϕ denoted by $cr_\phi(G)$. If ϕ is a good drawing of G satisfying $cr_\phi(G) = cr(G)$, then ϕ is called an optimal drawing of G. For more on the theory of crossing numbers, we refer to [2].

The calculation of the crossing number of graph is a classical problem, and yet it is also an elusive one. In fact, Garey and Johnson have proved that in general the problem of determining the crossing number of a graph is a NP-complete problem in [3].

The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \times G_2) = \{\{(u_i, v_j), (u_m, v_n)\}: \ u_i = u_m \ and \ \{v_j, v_m\} \in E(G_2)\}, \\ or \ \{\{u_i, u_n \in E(G_1)\} \ and \ v_j = v_m\}$$

At present, only few families of graphs with arbitrarily large crossing number for the plane are known. Most of them are Cartesian products of special graphs. Let c_n and P_n be the cycle and the path with n vertexes, and S_n denotes the star k_{1n}. The crossing numbers of Cartesian products of all 4-vertexes graphs with cycles are determined in [4] and with paths and stars in [5]. There are several known exact results on the crossing numbers of Cartesian products of paths, cycles and stars with 5-vertex graphs in [6–10], Marian Klesc gave a description of Cartesian products of all 5-vertex graphs with paths, cycles and stars by a table , whose crossing numbers are known. In this paper, the crossing number of $G_{18} \times S_n$ are given (see Figure 1(a) for G_{18}), and this fills up the crossing number list of Cartesian products of all 5-vertex graphs with stars. The main result of this paper is the following theorems.

Theorem 1: For $n \geq 1$, $cr\,(H_n) = Z\,(5, n) + n + \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 2: For $n \geq 1$, $cr\,(G_{18} \times S_n) = Z\,(5, n) + 2n + \left\lfloor \frac{n}{2} \right\rfloor$. (For any real number x, $\lfloor x \rfloor$ denotes the maximum integer not greater than x).

2. CROSSING NUMBER OF $\mathbf{G_{18} \times S_n}$

Firstly, let us denote by H_n the graph obtained by adding eight edges to the graph $k_{5,n}$ (containing n vertex of degree 5 and one vertex of degree $n+2$, two vertex of degree $n+3$, two vertexes of degree $n+4$, and $5n+8$ edges (see Figure 1(b)). Consider now the graph G_{18} in Figure 1(a). It is easy to see that $H_n = G_{18} \cup K_{5,n}$, where the five vertexes of degree n in $k_{5,n}$ and the vertexes of G_{18} are the same. For $i = 1, 2, ..., n$, let T^i denote the subgraph of $k_{5,n}$ which consists of the five edges incident with a vertex of degree five in $k_{5,n}$, thus we have

$$H_n = G_{18} \cup k_{5,n} = G_{18} \cup \left(T^1 \cup T^2 \cup \cdots \cup T^n\right)$$

Let ϕ be an optimal drawing of H_n (see Figure 1(b)). Under the drawing ϕ, we observe the supper bound of H_n, therefore we obtain that

Proposition 1

$$cr\,(Hn) \leq Z\,(5, n) + n + \left\lfloor \frac{n}{2} \right\rfloor$$

We now explain some notations. Let A and B be two sets of edges of a graph G. We use the sign $cr_\phi\,(A, B)$ to denote the number of all crossings whose two crossed edges are respectively in A and in B. Especially, $cr_\phi\,(A, A)$ is simply written as $cr_\phi\,(A)$. If G has the edge set E, the two signs $cr_\phi\,(G)$ and $cr_\phi\,(E)$ are essentially the same. The following Lemma 1, which can be shown easily, is usually used in the proofs of our theorem.

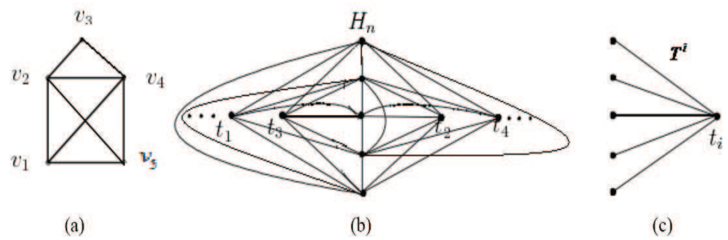

Figure 1
(a) G_{18}; (b) A Good Drawing of H_n; (c) T^i

Lemma 1 Let A, B, C be mutually disjoint subsets of E, then
(1)$cr_\phi(A \cup B) = cr_\phi(A) + cr_\phi(A, B) + cr_\phi(B)$;
(2)$cr_\phi(A \cup B, C) = cr_\phi(A, C) + cr_\phi(B, C)$,
where ϕ is a good drawing of E.

On the crossing numbers of complete bipartite graph $k_{m,n}$, Kleitman obtained the following result in [10].

Lemma 2 If $m \le 6$, then $cr\,(k_{m,n}) = Z\,(m, n)$,
where $Z\,(m, n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

Lemma 3 $cr(H_1) = 1$, $cr(H_2) = 3$

Proof. On the one hand, because H_1 contains the subgraph $k_{3,3}$ and $cr(k_{3,3}) = 1$, we have that $cr(H_1) \ge 1$. On the other hand, by Proposition 1, $cr(H_1) \le 1$. This implies that $cr(H_1) = 1$.

Then we prove that $cr(H_2) = 3$. On the one hand, by Proposition 1, $cr(H_2) \le 3$. On the other hand, we prove that $cr(H_2) \ge 3$; we may assume that $cr_\phi(H_2) = x$, then the graph H_2 in the drawing ϕ will turn into a planar graph by removing x edges. According to Euler Theorem, $v - e + f = 2$, that is $7 - (18 - x) + f = 2$, and $f = 13 - x$, and because $2e \ge 3f$, that is $2(18 - x) \ge 3(13 - x)$, $x \ge 3$. Hence $cr(H_2) = 3$. □

Lemma 4 Let ϕ be a good drawing of H_n, if there exist T^1 and T^2, such that $cr_\phi(T^1 \cup T^2) = 0$, then $cr_\phi(G_{18}, T^1 \cup T^2) \ge 3$, $cr_\phi(T^3, T^1 \cup T^2) \ge 4$.

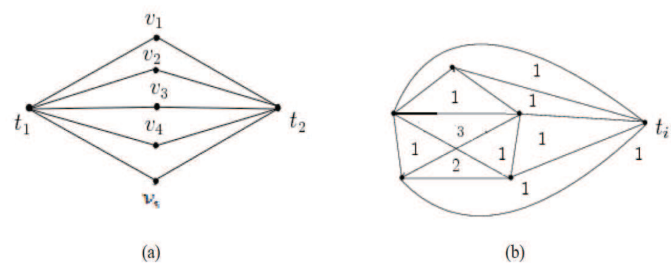

Figure 2
(a)A Good Drawing of $T^1 \cup T^2$ when $cr_\phi(T^1 \cup T^2) = 0$; (b) A Good Drawing of $G_{18} \cup T^i$ when $cr_\phi(G_{18}, T^i) = 0$

Let $H = \langle\{T^1 \cup T^2\}\rangle$ be the edge-induced subgraph of H_3. Obviously, H is isomorphic to the complete bipartite graph with two division $\{t_1,\ t_2\}$ and $\{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$. Since $cr_\phi(T^1 \cup T^2) = 0$, the subdrawing ϕ^* of H induced by ϕ must be isomorphic to Figure 2(a). As shown in Figure 2(a), each edge of the subset of edges $\{v_1v_4,\ v_2v_5,\ v_2v_4\}$ and the edge set $T^1 \cup T^2$ has at least one cross, so $cr_\phi(G_{18}, T^1 \cup T^2) \geq 3$. Because $T^1 \cup T^2 \cup T^3$ is isomorphic to the complete two bipartite graph $K_{3,5}$, then $cr_\phi(T^3, T^1 \cup T^2) \geq cr\,(K_{3,5}) = 4$.

Theorem 1: $cr\,(H_n) = Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor$.

The drawing in Figure 1(b) shows that $cr\,(H_n) \leq Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor$. Thus, in order to prove Theorem 1, we need only to prove that $cr\,(H_n) \geq Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor$ for any drawing ϕ of H_n. We prove the reverse inequality by induction on n. The cases $n = 1$ and $n = 2$ are trivial.

Case 1 If there exist $1 \leq i \neq j \leq n$, such that $cr_\phi(T^i \cup T^j) = 0$, then $cr\,(H_n) \geq Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor$.

Without loss of generality, we may assume that $cr_\phi(T^{n-1} \cup T^n) = 0$. By Lemma 3, $cr_\phi(G_{18}, T^{n-1} \cup T^n) \geq 3$ and $cr_\phi(T^i, T^{n-1} \cup T^n) \geq 4$ (for any $i = 1, 2, ..., n-2$). This implies that $cr_\phi(H_{n-2}, T^{n-1} \cup T^n) \geq 4\,(n-2) + 3 = 4n - 5$, since $H_n = H_{n-2} \cup (T^{n-1} \cup T^n)$, we have

$$
\begin{aligned}
&cr_\phi\,(H_n) = cr_\phi(G_{18} \cup T^1 \cup T^2 \cdots \cup T^{n-2} \cup T^{n-1} \cup T^n) \\
&= cr_\phi(H_{n-2} \cup T^{n-1} \cup T^n) = cr_\phi(H_{n-2}) + cr_\phi(H_{n-2}, T^{n-1} \cup T^n) \\
&\geq z\,(5, n-2) + (n-2) + \left\lfloor \frac{n-2}{2} \right\rfloor + (4n-5) = z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}
$$

Case 2 If every pair of T^i and T^j cross each other such that $cr_\phi(T^i \cup T^j) \geq 1$ for all $1 \leq i \neq j \leq n$, then $cr\,(H_n) \geq Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor$.

Subcase 2.1 If there is at least one subgraph T^i which does not cross G_{18}, Let us suppose $cr_\phi(G_{18} \cup T^n) = 0$ and let F be the subgraph $G_{18} \cup T^n$ of the graph H_n.

Consider the subdrawing ϕ^* and ϕ^{**} of G_{18} and F, respectively, induced by ϕ. Since $cr_\phi(G_{18} \cup T^n) = 0$, the subdrawing ϕ^* divides the plane in such a way that all vertices are the boundary of one "region" and all edges of G_{18} are in the "region". The subdrawing ϕ^{**} of F induced by ϕ is only one (see Figure 2(b)).

Consider now the subdrawing $F \cup T^i$ of the subdrawing ϕ and let $t_i (1 \leq i \leq n-1)$ be the vertex of T_i of degree five. If t_i is in the region marked with 2, then $cr_\phi(T^i, G_{18} \cup T^n) \geq 3$, using $cr_\phi(T^i \cup T^j) \geq 1$, thus $cr_\phi(T^i, G_{18} \cup T^n) \geq 4$. If t_i is in the region marked with 1 and 3, $cr_\phi(T^i, G_{18} \cup T^n) \geq 4$, then

$$
\begin{aligned}
&cr_\phi\,(H_n) = cr_\phi(G_{18} \cup T^n \cup T^1 \cup T^2 \cdots T^{n-1}) \\
&= cr_\phi\left(G_{18} \cup T^n, \left(T^1 \cup T^2 \cdots T^{n-1}\right)\right) + cr_\phi(T^1 \cup T^2 \cdots \cup T^{n-1}) \\
&\geq 4\,(n-1) + z\,(5, n-1) \geq Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}
$$

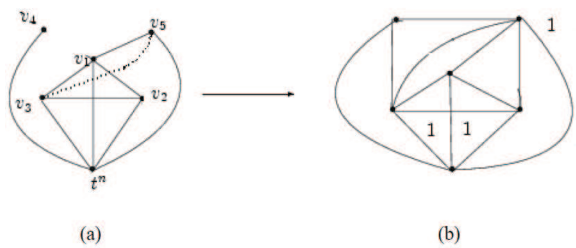

Figure 3
The Case when the Edge $\mathbf{t_n v_1}$ of $\mathbf{T^n}$ and the Edge $\mathbf{v_3 v_4}$ of $\mathbf{G_{18}}$ Cross

Subcase 2.2 If for any $T^i (1 \le i \le n)$, there is $cr_\phi(G_{18}, T^i) \ge 1$, and there exists one vertex t_i, $cr_\phi(G_{18}, T^n) = 1$, then $cr\,(H_n) \ge Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor$.

Without loss of generality, we may assume that $cr_\phi(G_{18}, T^n) = 1$. Let ϕ^* be the subdrawing of $G_{18} \cup T^n$ induced by ϕ. First we will illustrate the drawing of $G_{18} \cup T^n$ only has the following two kinds of circumstances:

Subcase 2.2.1: If the edge $t_n v_1$ of T^n and the edge $v_3 v_4$ of G_{18} cross, then it can be assumed that the vertexes t_n, v_1, v_3, v_4 distribute on the plane illustrated in Figure 3(a), and then the other three vertexes can be arbitrary distributed around the above vertexes. As there is no cross on the other four edges connected to the vertex t_n, the edges can be connected in the way illustrated in Figure 3(b). The edge $v_3 v_5$ can not connect along the dotted line in the graph, otherwise $cr_\phi(G_{18}, T^n) \ge 1$. Because $cr_\phi(T^i \cup T^j) \ge 1$ $(1 \le i \ne j \le n)$, in Figure 3(b), no matter the vertex t_i $(1 \le i \le n-1)$ is located, there is $cr_\phi(T^i, G_{18} \cup T^n) \ge 4$. Both $cr_\phi(G_{18} \cup T^n) = 1$ and $T^1 \cup T^2 \cdots T^{n-1}$ are isomorphic to $k_{5,n-1}$, hence

$$
\begin{aligned}
& cr_\phi\,(H_n) = cr_\phi(G_{18} \cup T^n \cup T^1 \cup T^2 \cdots T^{n-1}) \\
& = cr_\phi\left(G_{18} \cup T^n,\ \left(T^1 \cup T^2 \cdots T^{n-1}\right)\right) + cr_\phi\left(G_{18} \cup T^n\right) + cr_\phi\left(T^1 \cup T^2 \cdots \cup T^{n-1}\right) \\
& \ge 4\,(n-1) + 1 + z\,(5, n-1) \ge Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}
$$

Subcase 2.2.2: If the edge $t_n v_1$ of T^n and the edge $v_3 v_5$ of G_{18} cross as in Figure 4(a), this case can be analyzed and proved in a similar way in Case 1.

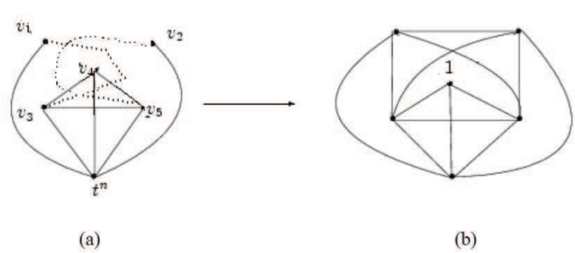

Figure 4
The Case when the Edge $\mathbf{t_n v_1}$ of $\mathbf{T^n}$ and the Edge $\mathbf{v_3 v_5}$ of $\mathbf{G_{18}}$ Cross

Subcase 2.3 If for any T^i $(1 \le i \le n)$, there is $cr_\phi(G_{18}, T^i) \ge 2$, then $cr\,(H_n) \ge Z\,(5,n) + n + \left\lfloor \dfrac{n}{2} \right\rfloor$.

$$\begin{aligned}
&cr_\phi\,(H_n) = cr_\phi(G_{18} \cup T^1 \cup T^2 \cdots \cup T^n) \\
&= cr_\phi\left(G_{18}, \left(T^1 \cup T^2 \cdots \cup T^n\right)\right) + cr_\phi\,(G_{18}) + cr_\phi(T^1 \cup T^2 \cdots \cup T^n) \\
&\ge 2n + z\,(5,n) > Z\,(5,n) + n + \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}$$

Let H be a graph ismorphic to G_{18}. Consider a graph G_H obtained by joining all vertexes of a connected graph G such that every vertex of H will only be adjacent to exactly one vertex of G. Let G_H^* be the graph obtained form by contracting the edges of H.

Lemma 5 $cr\,(G_H^*) \le cr\,(G_H) - 1$.

Proof. Consider a graph F obtained from G_H by joining all vertexes of H to the vertex Z. Because F contains the subgraph $k_{3,3}$ and $cr(k_{3,3}) = 1$, we have that $cr_\phi(F) \ge 1$. So there exists at least one crossing on the edges of H in F. Let ϕ be the good drawing of G_H, it is obviously that $cr_\phi(G_H) \ge 1$ and contains one subgraph S_4. Then, according to the methods shown in Figure 5, it must reduce at least one crossing by contracting G_H to the vertex h. This proves the Lemma 5. □

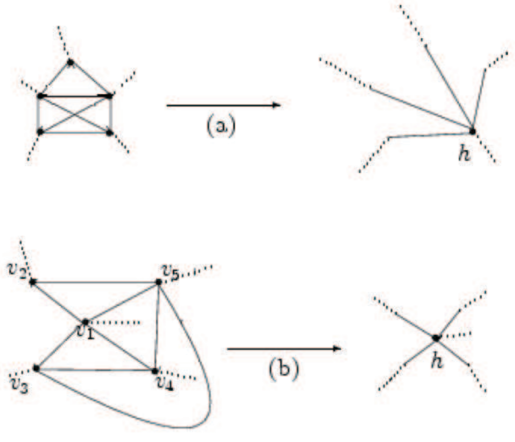

Figure 5
There are at Least 1 Crossing on F

Consider now the graph $cr(G_{18} \times S_n)$. For $n \ge 1$ it has $5(n+1)$ vertexes and edges that are the edges in $n+1$ copies G_{18}^i for $i = 0, 1, ..., n$ and in the five stars (see Figure 6), where the vertexes of are the central vertexes of the stars S_n.

Theorem 2: For $n \ge 1$, $Cr\,(G_{10} \times S_n) = Z\,(5,n) + 2n + \left\lfloor \dfrac{n}{2} \right\rfloor$.

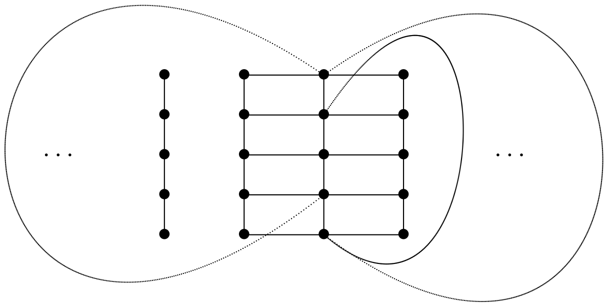

Figure 6

A God Drawing of $\mathbf{G_{18}} \times \mathbf{S_n}$

The drawing in Figure 6 shows that $cr\left(G_{18} \times S_n\right) \leq Z\left(5, n\right) + 2n + \left\lfloor \frac{n}{2} \right\rfloor$. To complete the proof, assume that there is an optimal drawing of $G_{18} \times S_n$ with fewer than $Z\left(5, n\right) + 2n + \left\lfloor \frac{n}{2} \right\rfloor$ crossings. Contracting the edges of G_{18}^i for all $i = 0, 1, ..., n$ in ϕ results in a graph isomorphic to H_n. In accordance with Lemma 4, we have $cr\left(H_n\right) < Z\left(5, n\right) + n + \left\lfloor \frac{n}{2} \right\rfloor$. This is impossible because in Theorem 1 it is shown that $cr\left(H_n\right) = Z\left(5, n\right) + n + \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, the graph has crossing number $Z\left(5, n\right) + 2n + \left\lfloor \frac{n}{2} \right\rfloor$.

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