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Numerical Solution of Overland Flow Model Using Finite Volume Method

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Abstract

Overland flow is one of Computational Fluid Dynamics (CFD) problem. In this paper we investigate the water level of overland flow that is often occured after rainfall on the land surface. Finite volume method is used to solve this problem. Quadratic upstream interpolation for convective kinetics (QUICK) sheme is used to have the discretitation of the overland flow model because this sheme have been proved its numerical stability. Numerical simulation of the solutions is presented to describe the behaviour of this model.

Key words

Overland flow; Finite volume method; QUICK

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1. INTRODUCTION

Rainfall is an aspect of the hydrologic cycle that is important in the role of supplying water in the world. But heavy rainfall with long duration can cause overland flow that it potentially occur flood. Overland flow is water on the the land surface that flow after rainfall. Overland flow take place if the precipitation level over the infiltration level to absorb water.

In order to know overland flow level, mathematical model and its numerical solution are needed to predict accurately. Many numerical methods were developed to solve the overland flow model. Mac Cormack and predictor corector methods was the method that was used to have the numerical solution of overland flow (Alhan *et al.,* 2005). Second-order Lax–Wendroff and the three-point centred finite difference schemes were used to get the numerical solution of overland flow (Gottardi & Venutelli, 2008). Finite element method was used to have the numerical solution of overland flow model (Jaber & Mohtar, 2003). Cubic-spline interpolation technique (CSMOC scheme) was used to have the numerical solution of overland flow model (Tsai & Yang, 2005).

In this paper, finite volume method is used to solve overland flow with QUICK sheme because this method suitable for Computational Fluid Dynamics (CFD) problem. Furthermore, we simulate several condition to show model performance.

2. NUMERICAL SOLUTION USING FINITE VOLUME METHOD

The physical model of overland flow can be seen in this figure.

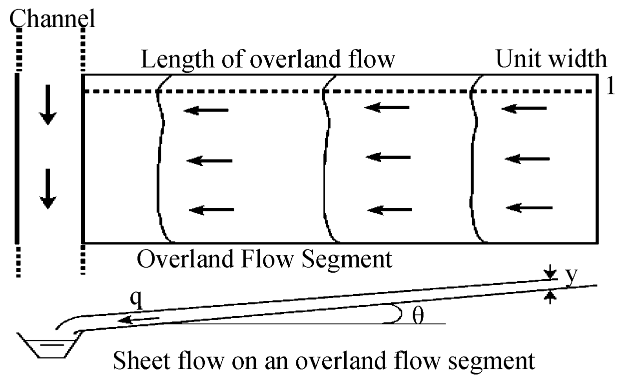

Figure 1
Overland Flow (Alhan *et al.*, 2005)

Where:
y = depth of water
q = the flow per unit width

Mathematical model of overland flow is governed from physical laws include continuity and momentum equations. This equations is called governing equation. This is based on Reynolds Transport Theorem (Chow, dkk., 1988).

2.1 Continuity Equations

Reynold Transport Theorem is used to get overland flow model is (Apsley, 2007):

$$\frac{d}{dt}(\rho\forall\phi)+\sum_{faces}(\underbrace{C\phi}_{adveksi}-\underbrace{\Gamma\frac{\partial\phi}{\partial n}}_{difusi}A)=Source \tag{1}$$

Where:
$\forall$ = volume of fluid
ρ = mass of fluid
ϕ = consentration
C = convektivity
Γ = diffusivity
A = wide of surface
Scalar transport of mass conservation overland flow is

$$\frac{d}{dt}(mass)+net\ outward\ mass\ flux=0 \tag{2}$$

$$\frac{\partial A}{\partial t}+\frac{\partial q}{\partial x}=(i-f) \tag{3}$$

Equation (3) is continuity overland flow in conservation form. Continuity equations in non-conservation form is

$$\frac{\partial y}{\partial t}+V\frac{\partial y}{\partial x}+y\frac{\partial V}{\partial x}=(i-f) \tag{4}$$

Where V is water velocity, i is rainfall intensity, x is distance, t is time and f is infiltration rate.

2.2 Momentum Equations

In a similar manner, momentum overland flow is derivated from Reynold Transport Theorem.

$$\frac{\partial q}{\partial t}+\frac{\partial(\frac{q^2}{A})}{\partial x})-gA\frac{\partial y}{\partial x}=gA(S_0-S_f) \tag{5}$$

Equation (5) is momentum overland flow in conservative form, and momentum overland flow in non-conservation form is

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}-g\frac{\partial y}{\partial x}=g(S_0-S_f)-\frac{V}{y}(i-f) \tag{6}$$

Where g is acceleration of gravity, t is time, S_f is friction slope and S_0 is bed slope.

3. FINITE VOLUME METHOD USING QUADRATIC UPSTREAM INTERPOLATION FOR CONVECTIVE KINETICS (QUICK) SCHEME

Continuity dan momentum equations are solved simultanously. Numerical solution of overland flow model using finite volume method is solved by integrating the differential equation that we have. The first step, we have to solve the governing equations.

If $q=VA$ then $A=\frac{q}{V}$, so the equation (3) become

$$\frac{\partial\left(\frac{q}{V}\right)}{\partial t}+\underbrace{\frac{\partial q}{\partial x}}_{Flux}=\underbrace{(i-f)}_{Source} \tag{7}$$

Continuity equation in (7) can be solved using QUICK scheme that be ilustrated in figure (2)

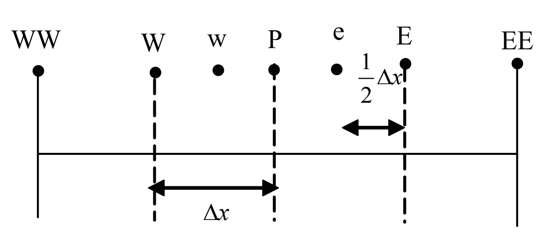

Figure 2
Control Face of Control Volume

The first step integrate equation (7) over the control volume and time interval from t to $t+\Delta t$, we have

$$\int_{CV}\int_{t}^{t+\Delta t}\frac{\partial\left(\frac{q}{V}\right)}{\partial t}dt d\forall+\int_{t}^{t+\Delta t}\int_{CV}\frac{\partial q}{\partial x}d\forall dt=\int_{t}^{t+\Delta t}\int_{CV}(i-f)d\forall dt \tag{8}$$

From equation (8) we have

$$\frac{1}{V}(q_P-q_P^0)\Delta\forall+\int_{t}^{t+\Delta t}(qA)_e-(qA)_w dt=(i-f)\Delta\forall\Delta t \tag{9}$$

In equation (9), A is face area of the control volume, $\Delta\forall$ is its volume which equal to $A\Delta x$ where Δx is the width of the control volume.

Using QUICK sheme, $q_e=q_P+\frac{1}{8}(3q_E-2q_P-q_W)$ and $q_w=q_W+\frac{1}{8}(3q_P-2q_W-q_{WW})$, equation (9) may be written as

$$\frac{1}{V}(q_P-q_P^0)\Delta\forall+\int_{t}^{t+\Delta t}(A(q_P+\frac{1}{8}(3q_E-2q_P-q_W)))-(A(q_W+\frac{1}{8}(3q_P-2q_W-q_{WW})))dt=(i-f)\Delta\forall\Delta t$$

(10)

To evaluate the left hand side of equation (10) we make an assumption the variation of q_P, q_E, q_W, and q_{WW} with time. We integrated the flow per unit width at time t or at time $t+\Delta t$ to calculate the time integral or combination of the flow per unit width at time t or at time $t+\Delta t$. We used weighted parameter q between 0 and 1 to approach the integral of the flow per unit width respect to time as

$$\int_{t}^{t+\Delta t}q_P dt=[\theta q_P+(1-\theta)q_P^0]\Delta t \tag{11}$$

Using (11), equation (10) we write as

$$\begin{aligned}&\frac{1}{V}(q_P-q_P^0)\Delta\forall+A\Delta t\{\left(\theta q_P+(1-\theta)q_P^0\right)+\frac{1}{8}[3\left(\theta q_E+(1-\theta)q_E^0\right)-2\left(\theta q_P+(1-\theta)q_P^0\right)-\\&\left(\theta q_W+(1-\theta)q_W^0\right)]-\left(\theta q_W+(1-\theta)q_W^0\right)-\frac{1}{8}[3\left(\theta q_P+(1-\theta)q_P^0\right)-2\left(\theta q_W+(1-\theta)q_W^0\right)-\\&\left(\theta q_{WW}+(1-\theta)q_{WW}^0\right)]\}=(i-f)\Delta\forall\Delta t\end{aligned} \tag{12}$$

Equation (12) dividing by $A\Delta t$ throughouth, we have

$$\begin{aligned}&\frac{\Delta x}{V\Delta t}(q_P-q_P^0)+\theta(q_P+\frac{3}{8}q_E-\frac{2}{8}q_P-\frac{1}{8}q_W-q_W-\frac{3}{8}q_P+\frac{2}{8}q_W+\frac{1}{8}q_{WW})+\\&(1-\theta)(q_P^0+\frac{3}{8}q_E^0-\frac{2}{8}q_P^0-\frac{1}{8}q_W^0-q_W^0-\frac{3}{8}q_P^0+\frac{2}{8}q_W^0+\frac{1}{8}q_{WW}^0=(i-f)\Delta x\end{aligned} \tag{13}$$

We can write equation (13) as

$$\begin{aligned}&a_P(q_P-q_P^0)+\theta(q_P+\frac{3}{8}q_E-\frac{2}{8}q_P-\frac{1}{8}q_W-q_W-\frac{3}{8}q_P+\frac{2}{8}q_W+\frac{1}{8}q_{WW})+\\&(1-\theta)(q_P^0+\frac{3}{8}q_E^0-\frac{2}{8}q_P^0-\frac{1}{8}q_W^0-q_W^0-\frac{3}{8}q_P^0+\frac{2}{8}q_W^0+\frac{1}{8}q_{WW}^0)=b\end{aligned} \tag{14}$$

Where:

$$a_P=\frac{\Delta x}{V\Delta t},\ b=(i-f)\Delta x$$

We can write equation (14) as

$$a_P(q_P - q_P^0) + \theta(\frac{3}{8}q_P + \frac{3}{8}q_E - \frac{7}{8}q_W + \frac{1}{8}q_{WW}) +$$
$$(1-\theta)(\frac{3}{8}q_P^0 + \frac{3}{8}q_E^0 - \frac{7}{8}q_W^0 + \frac{1}{8}q_{WW}^0) = b \qquad (15)$$

After we have numerical solution of continuity equation, in a similar manner we do the discretion of momentum equation in conservatif form. From equation (5) by replacing $A = \frac{q}{V}$, we have

$$\frac{\partial q}{\partial t} + \frac{\partial (Vq)}{\partial x} - gA\frac{\partial y}{\partial x} = gA(S_0 - S_f) \qquad (16)$$

$gA\frac{\partial y}{\partial x}$ is source from momentum equation, it be moved to right hand side, then we have

$$\frac{\partial q}{\partial t} + \underbrace{\frac{\partial (Vq)}{\partial x}}_{Flux} = \underbrace{gA(S_0 - S_f + \frac{\partial y}{\partial x})}_{Source} \qquad (17)$$

define $S = gA(S_0 - S_f + \frac{\partial y}{\partial x})$, we have

$$\frac{\partial q}{\partial t} + \underbrace{\frac{\partial (Vq)}{\partial x}}_{Flux} = S \qquad (18)$$

The equation (18) is integrated to t and to the control volume, we have

$$\int_t^{t+\Delta t}\int_{CV} \frac{\partial q}{\partial t} d\forall dt + \int_t^{t+\Delta t}\int_{CV} \frac{\partial (Vq)}{\partial x} d\forall dt = \int_t^{t+\Delta t}\int_{CV} S d\forall dt \qquad (19)$$

Equation (19) is integrated

$$(q_P - q_P^0)\Delta\forall + \int_t^{t+\Delta t} (VAq)_e - (VAq)_w dt = S\Delta\forall\Delta t \qquad (20)$$

Using QUICK sheme, equation (20) can be write

$$(q_P - q_P^0)\Delta\forall + VA\Delta t\{\left(\theta q_P + (1-\theta)q_P^0\right) + \frac{1}{8}[3\left(\theta q_E + (1-\theta)q_E^0\right) - 2\left(\theta q_P + (1-\theta)q_P^0\right) -$$
$$\left(\theta q_W + (1-\theta)q_W^0\right)] - \left(\theta q_W + (1-\theta)q_W^0\right) - \frac{1}{8}[3\left(\theta q_P + (1-\theta)q_P^0\right) - 2\left(\theta q_W + (1-\theta)q_W^0\right) -$$
$$\left(\theta q_{WW} + (1-\theta)q_{WW}^0\right)] = S\Delta\forall\Delta t \qquad (21)$$

Dividing by AΔt, we have

$$\frac{\Delta x}{\Delta t}(q_P - q_P^0) + V\{\left(\theta q_P + (1-\theta)q_P^0\right) + \frac{1}{8}[3\left(\theta q_E + (1-\theta)q_E^0\right) - 2\left(\theta q_P + (1-\theta)q_P^0\right) -$$
$$\left(\theta q_W + (1-\theta)q_W^0\right)] - \left(\theta q_W + (1-\theta)q_W^0\right) - \frac{1}{8}[3\left(\theta q_P + (1-\theta)q_P^0\right) - 2\left(\theta q_W + (1-\theta)q_W^0\right) -$$
$$\left(\theta q_{WW} + (1-\theta)q_{WW}^0\right)] = S\Delta x \qquad (22)$$

We can write equation (22) as

$$a_P(q_P - q_P^0) + \theta(q_P + \frac{3}{8}q_E - \frac{2}{8}q_P - \frac{1}{8}q_W - q_W - \frac{3}{8}q_P + \frac{2}{8}q_W + \frac{1}{8}q_{WW}) + (1-\theta)(q_P^0 + \frac{3}{8}q_E^0 - \frac{2}{8}q_P^0 - \frac{1}{8}q_W^0 - q_W^0 - \frac{3}{8}q_P^0 + \frac{2}{8}q_W^0 + \frac{1}{8}q_{WW}^0) = \frac{S\Delta x}{V} \quad (23)$$

Or we can write as

$$a_P(q_P - q_P^0) + \theta(\frac{3}{8}q_P + \frac{3}{8}q_E - \frac{7}{8}q_W + \frac{1}{8}q_{WW}) + (1-\theta)(\frac{3}{8}q_P^0 + \frac{3}{8}q_E^0 - \frac{7}{8}q_W^0 + \frac{1}{8}q_{WW}^0) = \frac{S\Delta x}{V} \quad (24)$$

Substitute equation (15) to (25) , we have

$$b = \frac{S\Delta x}{V} \quad (25)$$

We evaluate $q = 1$. This sheme is called **fully implicit**. From equation (24), we have

$$a_P(q_P - q_P^0) + (\frac{3}{8}q_P + \frac{3}{8}q_E - \frac{7}{8}q_W + \frac{1}{8}q_{WW}) = b \quad (26)$$

Or we can write as

$$(a_P + \frac{3}{8})q_P + \frac{3}{8}q_E - \frac{7}{8}q_W + \frac{1}{8}q_{WW} = b + a_P q_P^0 \quad (27)$$

Equation (27) is numerical solution of overland flow. To get numerical solution, domain is devided into 5 nodes that it describe number of node in control volume. The number of variabel equal to the equations. The equation change to matrix equation $Mq = H$, where M is coefisien of q , q is the flow per unit width that we want to find, and H is value in right hand side equation (27). The matrix form is

$$\begin{pmatrix} a_P + 3/8 & 3/8 & 0 & 0 & 0 \\ -7/8 & a_P + 3/8 & 3/8 & 0 & 0 \\ 1/8 & -7/8 & a_P + 3/8 & 3/8 & 0 \\ 0 & 1/8 & -7/8 & a_P + 3/8 & 3/8 \\ 0 & 0 & 1/8 & -7/8 & a_P + 3/8 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} = \begin{pmatrix} b + a_P q_P^0 \\ b + a_P q_P^0 \\ b + a_P q_P^0 \\ b + a_P q_P^0 \\ b + a_P q_P^0 \end{pmatrix} \quad (28)$$

4. SIMULATION OF OVERLAND FLOW MODEL

Simulation of overland flow model using synthetic case can be seen in the example to demonstrate the theory that is presented in the previous section.

Synthetics Example

Rainfall continues with the intensity 3.2 cm/h over a 600 ft. The slope of the land is 0.0016. We want to evaluate the flow per unit width of overland flow in 5, 20, 30 and 90 minutes.

If we used Δx is 6 ft and time step is 1 minutes, The numerical solution can be seen into figure 3.

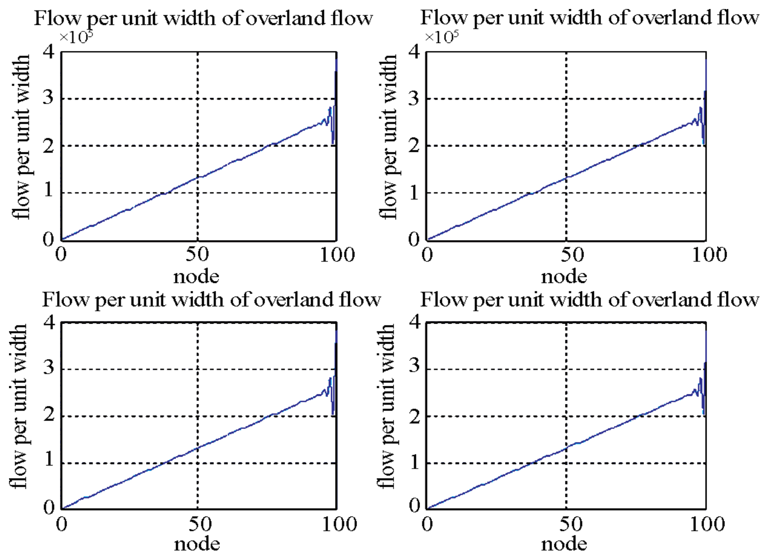

Figure 3
Flow Per Unit Width of Overland Flow

From the figure 3 we can see the flow per unit width for each time is increased, and at the end of the area we can see that the flow per unit witdh is in great quantities. It means that water flow to the lower land, and it can cause much water accumulation at the lower land.

5. CONCLUSION

In this paper, finite volume method can be applied to get the numerical solution of overland flow model because this method suitable for CFD problem. Quadratic Upstream Interpolation for Convective Kinetics (QUICK) sheme is used to have discretitation of overland flow model that have been proved its stablility. And also, finite volume method is good method to solve CFD problem, specially for fluid problem because this model show the behavior of overland flow in the reality problem.

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REFERENCES

[1] Alhan, C. M. K., Medina, M. A. dan Rao, P. (2005). On Numerical Modeling of Overland Flow. *Applied Mathematics and Computation*, *166*, 724-740.

[2] Gottardi, G. and Venutelli, M. (2008). An Accurate Time Integration Method for Simplified Overland Flow Models. *Advances in Water Resources*, *31*(1), 173-180.

[3] Jaber, F., H. And Mohtar, R., H. (2003). Stability and Accuracy of Two-Dimensional Kinematic Wave Overland Flow Modeling. *Advances in Water Resources*, 26(11), 1189-1198.

[4] Tsai, T., L. And Yang, J., C. (2005). Kinematic Wave Modeling of Overland Flow Using Characteristics Method with Cubic-Spline Interpolation. *Advances in Water Resources*, *28*(7), 661-670.

[5] Apsley, D.D. (2007). *Quantitative Properties of F.D. Schemes.* Lecture Handout: CFD, University of Manchester, Manchester.

[6] Chow , V.T., Maidment , D.R., Mays, L.W. (1988). *Applied Hydrology*. New York: McGraw-Hill.

[7] Ferziger, J.H. dan Peric, M. (2002). *Computational Methods for Fluid Dynamics* (Third, Rev. Edition). New York: Springer.

[8] Henderson, F.M. (1966). *Open Channel Flow*. New York: The MacMillan Company.

[9] Mays, L.W. (2001). *Water Resources Engineering*. New York: John Wiley.
[10] Versteg, H.K. dan Malalasekera, W. (1995). *An Introduction to Computational Fluid Dynamics the Finite Volume Method*. London: Longman Scientific & Technical.
[11] White, F.M. (1986). *Fluid Mechanics* (2nd ed.). New York : McGraw-Hill, Inc.