Studies in Mathematical Sciences **ISSN 1923-8444 [Print]**
Vol. 2, No. 2, 2011, pp. 78-89 **ISSN 1923-8452 [Online]**
www.cscanada.org **www.cscanada.net**

Optimal Investment and Portfolio Strategies with Minimum Guarantee and Inflation Protection for a Defined Contribution Pension Scheme*

Chukwuma R. Nwozo[1]
Charles I. Nkeki[2†]

Abstract: We study the optimal investment and optimal portfolio strategies with minimum guarantee and inflation protection in a defined contribution (DC) pension scheme. We assume a market structure that is characterized by a cash account, an indexed bond (i.e., inflation-linked bond) and stock. We obtain optimal share of portfolio values (that depend on the minimum guarantee) in the indexed bond and stock for the pension plan member (PPM) at time t. We find that in the presence of indexed bond in the investment strategy, the inflation risk that is associated with the PPM's contributions and minimum guarantee is hedged. Hence, indexed bond can be used to hedge inflation risk that is associated with a contributory pension funds scheme. We also find in our numerical result that, with effective management of the pension funds by pension fund administrators (PFAs), PPMs will have high returns from their investment. From our results, we find that the optimal terminal wealth that will accrue to the PPM and PFA to be $N1.97\times10^{22}$. We also find that the minimum guarantee that will accrue to the PPM at retirement to be $N3.05\times10^{8}$. The portfolio values in stock, indexed bond and cash account are 0.5, 3.1, and -2.6, respectively. Therefore, more investment should go to indexed bond and stock since they yield positive portfolio value.

Key words: Optimal Investment; Portfolio Strategies; Minimum Guarantee; Defined Contribution

1. INTRODUCTION

In this paper, we study the optimal investment and optimal portfolio strategies with minimum guarantee policies under inflation protection strategy. We consider a market that is characterized by a cash, an inflation-linked bond and stocks. In related literature, Zhang *et al* [9] considered the optimal management and inflation protection strategy for DC pension plans. They adopted Martingale approach in to compute an

* **Mathematics Subject Classification**: 91B28, 91B30, 93E20
1 Department of Mathematics, University of Ibadan, Ibadan, Oyo State. Nigeria. Email: crnwozo@yahoo.com
2 Department of Mathematics, University of Benin, P. M. B. 1154, Benin City, Edo State. Nigeria.
Email: nkekicharles2003@yahoo.com
† Corresponding Author
‡ Received Decemeber 15, 2010; accepted April 3, 2011

analytic expression for their optimal strategy. Hojgaard and Vigna [4] considered a mean-variance portfolio selection and they determined the efficient frontier for DC pension schemes. Vigna[8] studied efficiency of mean-variance based portfolio selection in DC pension schemes. Deelstra *et al* [2] considered the optimal design of the minimum guarantee in a DC pension scheme. They studied the investment in the financial market by assuming that the pension fund optimizes its retribution which is a part of the surplus. They developed a favourable sharing rules for the pension fund. The sharing rule allows for partial risk transfer between the PPM and the pension fund manager. The deterministic or stochastic minimum guarantee play a prominent role in determining the form of optimal asset allocation. Grossman and Zhou[3], Basak[1] considered the deterministic case of minimum guarantees. Tepla[7] considered the stochastic guarantee case and studied the optimal behaviour of an investor subject to a minimum performance constraint which state that the final wealth must be at least as high as the one generated by an investment in a stochastic benchmark portfolio. He concluded that the optimal policy corresponds to the one of an unconstrained investor having to pay a proportion of the benchmark value at the final period. Rudolf and Ziemba[6] considered a stochastic guarantee. The aim of the PPM is to maximize the expected utility of the surplus by maximizing the portfolio values and the investment strategies. Hence, our aim is to design an optimal portfolio with minimum guarantee that maximize the expected utility of the terminal surplus of a PPM. We assume in this paper, that the risky assets and the salary of the PPM are driven by a geometric Brownian motion with constant drifts and volatilities. Nkeki and Nwozo[5] considered optimal portfolio strategies with minimum guarantee maximizing the expected utility of terminal surplus for a DC pension plan. They considered a market structure that is characterized by a riskless asset and n risky assets. The investment and inflation risk associated with their portfolio are non-hedgeable. In this paper, the inflation and the investment risks are hedged with the presence of indexed bonds as one of the underlying assets.

This paper is organized as follows: in section 2, we present description of the financial market models, the dynamics of the PPM's salary as well as the dynamics of the PPM's minimum guarantee. In section 3, we present the wealth process of the PPM's. In section 4, we present the optimization program and expected terminal wealth of the PPM and portfolio processes using constant relative risks aversion (CRRA) utility function. Section 5, present the numerical result of our models. Finally, section 6 conclude the paper.

2. THE DESCRIPTION OF THE FINANCIAL MODELS

In this section, we describe the financial models, PPM's salary and the minimum guarantee.

2.1 Financial Model

The Brownian motion $W(t) = \left(W^S(t), W^I(t)\right)^{Tr}$ is a 2-dimensional processes defined on a given probability space $\left(\Omega, F, \{F_t^S\}_{t\geq 0}, \{F_t^I\}_{t\geq 0}, P\right)$, where P is the real world probability measure and σ^S and σ^I is the volatility of stock and volatility of the indexed bond respectively with respect to changes in $W^S(t)$ and $W^I(t)$ respectively. Tr denotes transpose. μ and α are the appreciation rate for stock and indexed bond respectively. Moreover, σ^S and σ^I are respectively, the coefficients of the markets and are progressively measurable with respect to the filtration $\{F_t^S\}_{t\geq 0}$ and $\{F_t^I\}_{t\geq 0}$, respectively such that $F_t^S \bigcup F_t^I \in F$ and $F_t^S \bigcap F_t^I = \phi$.

We assume that the financial market is arbitrage-free, complete and continuously open between time 0 and T, i.e., there is only one process $\theta = \left(\theta^S, \theta^I\right)^{Tr}$ satisfying

$$\theta = \sigma^{-1}\begin{pmatrix} \mu - r \\ \alpha + \varpi - r \end{pmatrix} = \begin{pmatrix} \theta^S \\ \theta^I \end{pmatrix}, \text{ where, } \sigma = \begin{pmatrix} \sigma^S & 0 \\ 0 & \sigma^I \end{pmatrix},$$

and $\det(\sigma) = \sigma^S \sigma^I \neq 0.$ This comfirmed our assumption that the market is complete.

The first asset is a risk-free asset (cash account) whose price process, $Y(t)$ is given by the dynamics

$$\frac{dY(t)}{Y(t)} = rdt\ , \ Y(0) = 1, \tag{1}$$

where $r > 0$ represents the short term interest rate. The stocks, whose prices are denoted by $S(t).$ The dynamics of $S(t)$ given by

$$dS(t) = \mu S(t)dt + \sigma^S S(t)dW^S(t), \ \ S(0) = s > 0.. \tag{2}$$

The assets are the indexed bond, whose prices are denoted by $D(t).$ The dynamics of $D(t)$ given by

$$dD(t) = (\alpha + \varpi)D(t)dt + \sigma^I D(t)dW^I(t), \ \ D(0) = d > 0,, \tag{3}$$

where α is the real rate of return and $\varpi > 0$ is the expected rate of inflation. Hence, the indexed bond pays an expected rate of nominal return equal to the sum of the rate of real return and expected rate of inflation. Again, the volatility σ^I is caused by the source of inflation $W^I(t).$

Let the contirbutions of the PPM be invested into a risk-free asset with a nominal return r, stocks with instantaneous expected gross of return μ and indexed bond with instantaneous expected gross of return $\alpha + \varpi.$ Let $X(t)$ be the wealth process, where $\Delta(t)$ is the portfolio process at time t. Let $\Delta^S(t)$ be the proportion of wealth invested in stock at time t and $\Delta^I(t)$ be the proportion of wealth invested in the indexed bond at time t, then $1 - \Delta^S(t) - \Delta^I(t)$ is the proportion of wealth invested in the risk-free asset. Hence, the corresponding portfolio value process $X(t)$ satisfies

$$\frac{dX(t)}{X(t)} = \Delta^S(t)\frac{dS(t)}{S(t)} + \Delta^I(t)\frac{dD(t)}{D(t)} + \frac{1 - \Delta^S(t) - \Delta^I(t)}{Y(t)}dY(t). \tag{4}$$

2.2 The Dynamics of the PPM's Salary

We assume that the PPM's salary, $C(t)$ follows the dynamics:

$$\frac{dB(t)}{B(t)} = \kappa dt + \sigma_I dW^I(t), \ \ B(0) = c > 0, \tag{5}$$

where κ is the expected growth rate of salary and $W^I(t)$ is a Brownian motion which is the source of uncertainty associated with the salary which causes the values of the salary to fluctuate around the expected inflation with an instantaneous intensity of fluctuation σ_I. Both κ and σ_I are assume to be constants. Applying *Ito* Lemma to Eq.(8), we obtain the following as a solution to the stochastic differential equation.

$$B(t) = c\exp\left[\left(\kappa - \frac{1}{2}\sigma_I^2\right)t + \sigma_I W^I(t)\right]. \tag{6}$$

Let $q > 0$ be the fixed contribution rate, then the contributions of the PPM is given by $qB(t)$ at time t.

2.3 The Dynamics of PPM's Contribution and Minimum Guarantee

We describe in this subsection, the present value of expected contribution and flow of minimum guarantee processes as well as their dynamics.

Definition 1: The present value of expected future contribution process is defined as

$$\Phi(t) = E_t\left[\int_t^T \frac{\Lambda(s)}{\Lambda(t)} qB(s)ds\right], \tag{7}$$

where E_t is the conditional expectation with respect to the Brownian filtration $\{F(t)\}_{t\geq 0}$ and

$\Lambda(t) \equiv \exp\left[-rt - \frac{1}{2}\|\theta\|^2 t - \theta^{Tr}W(t)\right]$ is the stochastic discount factor which adjusts for norminal interest rate and market price of risk.

Theorem 1: The present value of expected future contribution process $\Phi(t)$ is proportional to the instantaneous contribution process $qB(.)$, that is,

$$\Phi(t) = \frac{1}{\eta}\left(\exp[\eta(T-t)-1]qB(t)\right), \text{ for all } t \in [0,T], \tag{8}$$

with $\eta = \kappa - r - \sigma_I\|\theta\|$.

Proof: (see [4]).

Taking the differential of Eq.(8), we obtain the following

$$d\Phi(t) = \Phi(t)\left[\left(r + \sigma_I\|\theta\|\right)dt + \sigma_I dW^I(t)\right] - qB(t)dt. \tag{9}$$

Definition 2: The present value of expected flow of minimum guarantee process is defined as

$$G(t) := E_t\left[\int_{T-t}^T \frac{\Lambda(s)}{\Lambda(t)} qB(s)\exp[\delta(T-s)]ds\right], T \geq t; \tag{10}$$

where, $\delta \in [0,r]$ is the instantaneous guaranteed rate of return; and $qB(t)$ is the flow of contributions into the pension funds at time t.

Proposition 1: Let $G(t)$ be the present value of expected flow of minimum guarantee process, then,

$$G(t) = \frac{qB(t)}{\psi}\exp[\delta T]\left(\exp[\psi t]-1\right), T \geq t, \tag{11}$$

with $\psi = \kappa - r - \delta - \sigma_I\|\theta\|$.

Proof: By definition 2,

$$G(t) = E_t\left[\int_{T-t}^{T} \frac{\Lambda(s)}{\Lambda(t)} qB(s)\exp[\delta(T-s)]ds\right], T \geq t.$$

$$= qB(t)E_t\left[\int_{T-t}^{T} \frac{\Lambda(s)}{\Lambda(t)}\frac{B(s)}{B(t)}\exp[\delta(T-s)]ds\right]$$

The processes $\Lambda(.)$ and $B(.)$ are geometric Brownian motions. It therefore follows that $\frac{\Lambda(s)}{\Lambda(t)}\frac{B(s)}{B(t)}$ is independent of $\{F(t)\}_{s\geq t}$. Hence,

$$G(t) = qB(t)E\left[\int_{0}^{t} \frac{\Lambda(s)}{\Lambda(0)}\frac{B(s)}{B(0)}\exp[\delta(T-s)]ds\right]$$

$$= qB(t)\exp[\delta T]E\left[\int_0^t \exp[\psi s]ds\right]$$

$$= \frac{qB(t)\exp[\delta T]}{\psi}(\exp[\psi t]-1), T \geq t.$$

where, $\psi = \kappa - r - \delta - \sigma_I\|\theta\|$.

At time T, we have

$$G(T) = \frac{qB(T)\exp[\delta T]}{\psi}(\exp[\psi T]-1). \tag{12}$$

Lemma 1: The dynamics of the flow of minimum guarantee is given by

$$dG(t) = G(t)(\beta(t)dt + \sigma_I dW^I(t)), \tag{13}$$

where $\beta(t) = \kappa + \frac{\psi\exp[\psi t]}{\exp[\psi t]-1}, t > 0.$

Proof: Taking differential of bothsides of Eq.(11), we obtain the following:

$$dG(t) = \frac{qB(t)\exp[(\delta)T]}{\psi}(\exp[\psi t]-1)[(\beta(t)dt + \sigma_I dW^I(t))]$$

$$= G(t)\left(\left(\kappa + \frac{\psi\exp[\psi t]}{\exp[\psi t]-1}\right)dt + \sigma_I dW^I(t)\right)$$

$$= G(t)(\beta(t)dt + \sigma_I dW^I(t)).$$

3. THE WEALTH PROCESS OF THE PPM

If the PPM contributes continuously into his DC pension fund with a fixed contribution rate $q > 0$ and $(1-\Delta(t))$, $\Delta^S(t)$ and $\Delta^I(t)$ shares of the pension fund invested in a cash account stocks and indexed bonds respectively, then, the corresponding wealth process with an initial value $x > 0$, denoted by $X(t)$, is governed by the following equation

$$dX(t) = X(t)\left[\left(r + \Delta^S(t)(\mu - r) + \Delta^I(t)(\alpha + \varpi - r)\right)dt + \sigma^S \Delta^S(t) dW^S(t) + \sigma^I \Delta^I(t) dW^I(t)\right] + qB(t)dt \tag{14}$$

Definition 4: The surplus process $V(t)$, $t \geq 0$ is defined by

$$V(t) = X^{\Delta}(t) + \Phi(t) - G(t), \tag{15}$$

Proposition 2: Let $V(t)$ be the surplus process of a PPM and $X(t)$, $\Phi(t)$ and $G(t)$ satisfies Eq.(4), (9) and (11), respectively, then

$$dV(t) = \left(rX(t) + (\mu - r)\Delta^S(t)X(t) + \Delta^I(t)X(\alpha + \varpi - r) - G(t)\beta(t) + r\Phi(t) + \sigma_I \theta^I \Phi(t)\right)dt + \sigma^S \Delta^S(t)X(t)dW^S(t) + \sigma^I \Delta^I(t)X(t)dW^I(t) + \sigma_I \Phi(t)dW^I(t) \tag{16}$$

Proof: Finding the differential of both sides of Eq.(16) and substitute in Eq.(4), (9) and (11), the result follows.

4. THE OPTIMIZATION PROGRAM AND EXPECTED TERMINAL WEALTH OF A PPM'S

In this section, we consider the optimization of PPM's surplus using HJB equation. From the HJB equation, we derived the optimal portfolios in the stock and indexed bond markets at time t using CRRA utility function.

Let $$U(V,t) = \sup_{\{\Delta\}\in\Pi_V} E\left[U(V(T))\middle| V(t) = V\right] \tag{17}$$

be the value function, where $\Pi_V[0,T]$ is the set of admissible policy that is F_V −progressively measurable and satisfy the integrability conditions

$$E\left[\int_t^T \Delta^S(u)^2 du\right] < \infty, \quad E\left[\int_t^T \Delta^I(u)^2 du\right] < \infty.$$

By the application of *Ito-Doeblin* lemma to Eq.(16), we obtain the following HJB equation

$$U_t(V,t) + \max_{\Delta\in\Pi_V}\left\{\begin{array}{l} \left(rX + \Delta^S(t)X(\mu - r) + \Delta^I(t)X(\alpha + \varpi - r)\right)U_X(V,t) + \sigma_I\theta^I \Phi U_\Phi(V,t) + \\ r\Phi U_\Phi(V,t) + \frac{1}{2}\Delta^S(t)^2 X^2(\sigma^S)^2 U_{XX}(V,t) + \frac{1}{2}\Delta^I(t)^2 X^2(\sigma^I)^2 U_{XX}(V,t) \\ + \frac{1}{2}\sigma_I{}^2\Phi^2 U_{\Phi\Phi}(V,t) - \frac{1}{2}\sigma_I{}^2 G^2 U_{GG}(V,t) - \sigma_I{}^2 G\Phi U_{\Phi G}(V,t) - \alpha(t)GU_G(V,t) - \\ \sigma^I\sigma_I XG\Delta^I(t)U_{XG}(V,t) + \sigma^I\sigma_I X\Phi\Delta^I(t)U_{X\Phi}(V,t) - G\Phi\sigma_I^2 U_{G\Phi}(V,t) \end{array}\right\} = 0 \tag{18}$$

where $\Delta(t) = \{\Delta^S(t), \Delta^I(t)\}$ is set of admissible portfolio strategies.

Let the utility function be concave and the value function be smooth, that is $U(V,t) \in C^{1,2}(\Re \times [0,T])$, then Eq.(18) is well-defined. From Eq.(18), we obtain

$$\Delta^{S*}(t) = -\frac{U_X(V,t)(\mu - r)}{X(t)(\sigma^S)^2 U_{XX}(V,t)}, \tag{19}$$

$$\Delta^{I*}(t) = -\frac{U_X(V,t)(\alpha + \varpi - r)}{X(t)(\sigma^I)^2 U_{XX}(V,t)} - \frac{U_{X\Phi}(V,t)\Phi\sigma_I}{X(t)(\sigma^I)^2 U_{XX}(V,t)} + \frac{U_{XG}(V,t)G\sigma_I}{X(t)(\sigma^I)^2 U_{XX}(V,t)}. \tag{20}$$

Proposition 4: Suppose that $U(V(t)) = \frac{(X(t)+\Phi(t)-G(t))^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1$, then

$$\Delta^{S*}(t) = \frac{(\mu - r)}{\gamma(\sigma^S)^2} + \frac{(\mu - r)\Phi(t)}{\gamma X(t)(\sigma^S)^2} - \frac{(\mu - r)G(t)}{\gamma X(t)(\sigma^S)^2}, \tag{21}$$

$$\Delta^{I*}(t) = \frac{(\alpha + \varpi - r)}{\gamma(\sigma^I)^2} + \frac{(\alpha + \varpi - r)\Phi(t)}{\gamma X(t)(\sigma^I)^2} - \frac{(\alpha + \varpi - r)G(t)}{\gamma X(t)(\sigma^I)^2} - \frac{\sigma_I\Phi(t)}{\gamma X(t)\sigma^I} - \frac{\sigma_I G(t)}{\gamma X(t)\sigma^I}, \tag{22}$$

and

$$\Delta_0^*(t) = 1 + \frac{(\alpha + \varpi - r)G(t)}{\gamma X(t)(\sigma^I)^{2^2}} + \frac{\sigma_I\Phi(t)}{\gamma X(t)\sigma^I} + \frac{\sigma_I G(t)}{\gamma X(t)\sigma^I} + \frac{(\mu - r)G(t)}{\gamma X(t)\sigma^I}$$
$$-\left(\frac{(\alpha + \varpi - r)}{\gamma X(t)(\sigma^I)^2} + \frac{(\alpha + \varpi - r)\Phi(t)}{\gamma X(t)(\sigma^I)^2} + \frac{(\mu - r)}{\gamma(\sigma^I)^2} + \frac{(\mu - r)\Phi(t)}{\gamma X(t)(\sigma^I)^2}\right), \tag{23}$$

Proof: Determining the following partial derivatives

$$U_X(V,t) = V^{-\gamma}, U_{XX}(V,t) = -\gamma V^{-\gamma-1}, U_{XG}(V,t) = -\gamma V^{-\gamma-1}, U_{X\Phi}(V,t) = -\gamma V^{-\gamma-1} \tag{24}$$

and substitute into Eq.(24), Eq.(19) and Eq.(20), we obtain the desired result. Therefore, the elements of the optimal portfolio value vector is a function of the wealth process, PPM's contributions and minimum guarantee.

Observe that our assumption of concavity of U turns out to be true, as

$U_{XX}(V,t) = U_{XG}(V,t) = U_{X\Phi}(V,t) = -\gamma V^{-\gamma-1} < 0,$ since $\gamma > 0$.

The optimal portfolio rule of Eq.(19) and Eq.(20), referred to as the adjusted classical Merton rule, tells us that under the CRRA utility function, the proportion of wealth invested in stocks, and indexed bonds at time t, depend on the wealth process, contribution process and minimum guarantee. Hence, we have that the proportion of the portfolio values that should be transferred to cash account over time against investment and inflation risks is

$$\frac{(\alpha + \varpi - r)G(t)}{\gamma X(t)(\sigma^I)^2} + \frac{\sigma_I\Phi(t)}{\gamma X(t)\sigma^I} + \frac{\sigma_I G(t)}{\gamma X(t)\sigma^I} + \frac{(\mu - r)G(t)}{\gamma X(t)(\sigma^S)^2}.$$

These terms are the inter-temporal hedging strategy that offset any shock to the contributions of the PPM that should be transferred to cash account at time t.

Now, taking the expectation of bothsides of Eq.(14), we have the following ordinary differential equation

$$\begin{cases} dE(X^*(t)) = E\left[X^*(t)\left(r + \Delta^{S^*}(t)(\mu - r) + \Delta^{I^*}(t)(\alpha + \varpi - r)\right) + qB(t)\right]dt \\ E(X^*(0)) = x_0. \end{cases} \tag{25}$$

Substituting Eq.(21) and Eq.(22) into Eq.(25), we have the following:

$$\begin{cases} dE\left(X^*(t)\right) = \begin{cases} (\rho + r)E\left(X^*(t)\right) + \dfrac{(\theta^S)^2}{\gamma} cq\exp[kt]\left(\dfrac{1}{\eta}\left(\exp[\eta(T-t)]-1\right) - \dfrac{1}{\psi}\exp[\delta T]\left(\exp[\psi t]-1\right)\right) + \\ \dfrac{(\theta^I)^2}{\gamma} cq\exp[kt]\left(\dfrac{1}{\eta}\left(\exp[\eta(T-t)]-1\right)\left(1-\sigma_I\sigma^I\right) - \dfrac{1}{\psi}\exp[\delta T]\left(\exp[\psi t]-1\right)\left(1+\sigma_I\sigma^I\right)\right) \\ + cq\exp[kt] \end{cases} \\ E\left(X^*(0)\right) = x_0. \end{cases} \quad (26)$$

where, $\rho = \dfrac{(\theta^S)^2 + (\theta^I)^2}{\gamma}$.

Solving Eq.(26), we have the following

$$E\left(X^*(t)\right) = x_0\exp(r+\rho)t + \frac{cq\left((\theta^S)^2 + (\theta^I)^2\left(1-\sigma_I\sigma^I\right)\right)}{\gamma\eta(r-\kappa+\rho+\eta)}\left(\exp\left(\eta T + (r+\rho)t\right) - \exp\left(\eta T + (\kappa-\eta)t\right)\right)$$
$$-\frac{cq\left((\theta^S)^2 + (\theta^I)^2\left(1+\sigma_I\sigma^I\right)\right)}{\gamma(\kappa-r-\rho-\psi)}\left(\frac{\exp\left(\delta T + (\kappa+\psi)t\right)}{\psi} + \frac{\exp\left(\delta T + (r+\rho)t\right)}{(r-\kappa+\rho)} - \frac{(\kappa-r-\rho-\psi)\exp\left(\delta T + \kappa t\right)}{\psi(\kappa-r-\rho)}\right)$$
$$+\frac{cq\left(\gamma\eta - (\theta^S)^2 + (\theta^I)^2\left(\sigma_I\sigma^I - 1\right)\right)}{\gamma\eta}\left(\frac{\exp(\kappa t)}{(\kappa-r-\rho)} - \frac{\exp((r+\rho)t)}{(r-\kappa+\rho)}\right). \quad (27)$$

At $t = T$, we have the expected terminal wealth to be

$$E\left(X^*(T)\right) = x_0\exp(r+\rho)T + \underbrace{\left\{\begin{array}{l} \dfrac{cq\left((\theta^S)^2 + (\theta^I)^2\left(1-\sigma_I\sigma^I\right)\right)}{\gamma\eta(r-\kappa+\rho+\eta)}\left(\exp\left((\eta+r+\rho)T\right) - \exp\left((\eta+\kappa-\eta)T\right)\right) \\ -\dfrac{cq\left((\theta^S)^2 + (\theta^I)^2\left(1+\sigma_I\sigma^I\right)\right)}{\gamma(\kappa-r-\rho-\psi)}\left(\dfrac{\exp\left((\delta+\kappa+\psi)T\right)}{\psi} + \dfrac{\exp\left((\delta+r+\rho)T\right)}{(r-\kappa+\rho)} - \dfrac{(\kappa-r-\rho-\psi)\exp\left((\delta+\kappa)T\right)}{\psi(\kappa-r-\rho)}\right) \\ +\dfrac{cq\left(\gamma\eta - (\theta^S)^2 + (\theta^I)^2\left(\sigma_I\sigma^I - 1\right)\right)}{\gamma\eta}\left(\dfrac{\exp(\kappa T)}{(\kappa-r-\rho)} - \dfrac{\exp((r+\rho)T)}{(r-\kappa+\rho)}\right) \end{array}\right\}}_{H}. \quad (28)$$

It is easy to see that the expected optimal terminal wealth is the sum of the wealth that one would get for investing the portfolios both in cash account and the risky ones, plus H that depend on the goodness of the risky asset with respect to the risk-free one, contributions of the PPM and minimum guarantee.

5. NUMERICAL RESULTS

In this section, we present the numerical results of our models. The parameters and their corresponding values are presented in Table 1. We assume in this paper that the PPM will retire at the age of 35 years. In figure 1, displayed the optimal expected wealth of the PPM up to the terminal period. Figure 2 displayed the minimum guarantee to be received by the PPM at retirement. Figure 3, 4 and 5 displayed respectively, the portfolio values of a PPM in stock market, indexed bond and cash account up to the terminal period. From our results, we found that the optimal terminal wealth that will accrued to the PPM and the Fund Manager to be $N1.97 \times 10^{22}$, where N denotes Naira. We also found that the minimum guarantee that will accrued to the PPM at retirement to be $N3.05 \times 10^{8}$. The portfolio values in stock, indexed bond and cash account are 0.5, 3.1, and -2.6, respectively. Therefore, more investment should go to indexed bond since it has the highest portfolio value. This is an expected result since one of the factors that affect the growth of investment (i.e., inflation) have be hedge with the investment in inflation-linked bond. Again, the fund manager should invest some proportion of the PPM's wealth into stock. The negative portfolio value in cash account implies that the risky assets are yielding good returns and therefore, more funds should be borrowed from cash account to finance the investment and such funds should only be invested in inflation-linked bond. The result shows that under high inflation rate, more of the PPM's contributions should be invested in inflation-linked bond.

Table 1: Parameters and Their Values for Numerical Experiment

Symbol	Definition of the Parameter	Numerical Value
r	Nominal interest rate	0.02
μ	Expected stock return	0.10
σ^S	Volatility of stock	0.55
σ^I	Volatilty of indexed bond	0.35
σ_I	Volatility of inflation index	0.21
α	Expected indexed bond return	0.09
ϖ	Expected rate of inflation	0.12
κ	Expected growth of salary	0.03
q	Contribution rate	0.075
γ	Parameter for risk aversion	0.50
x_0	Intial wealth	N1000
c	Initial salary of PPM	N20000
δ	Expected growth of minimum guarantee	0.025

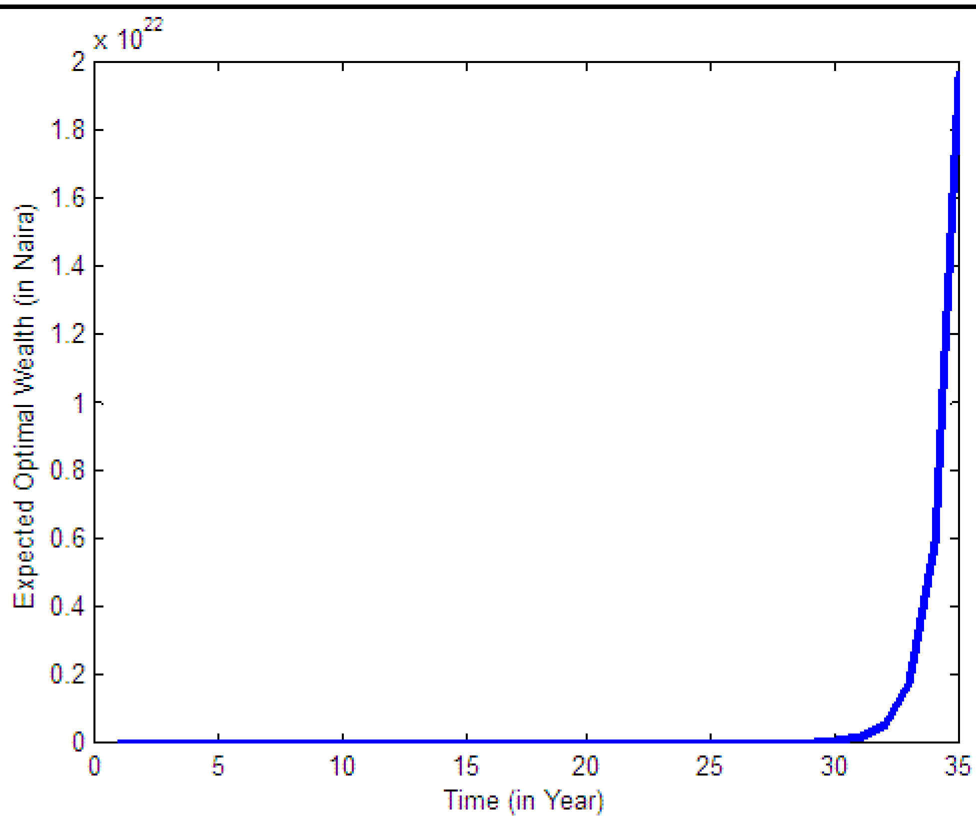

Figure 1: The Expected Optimal Wealth of PPM

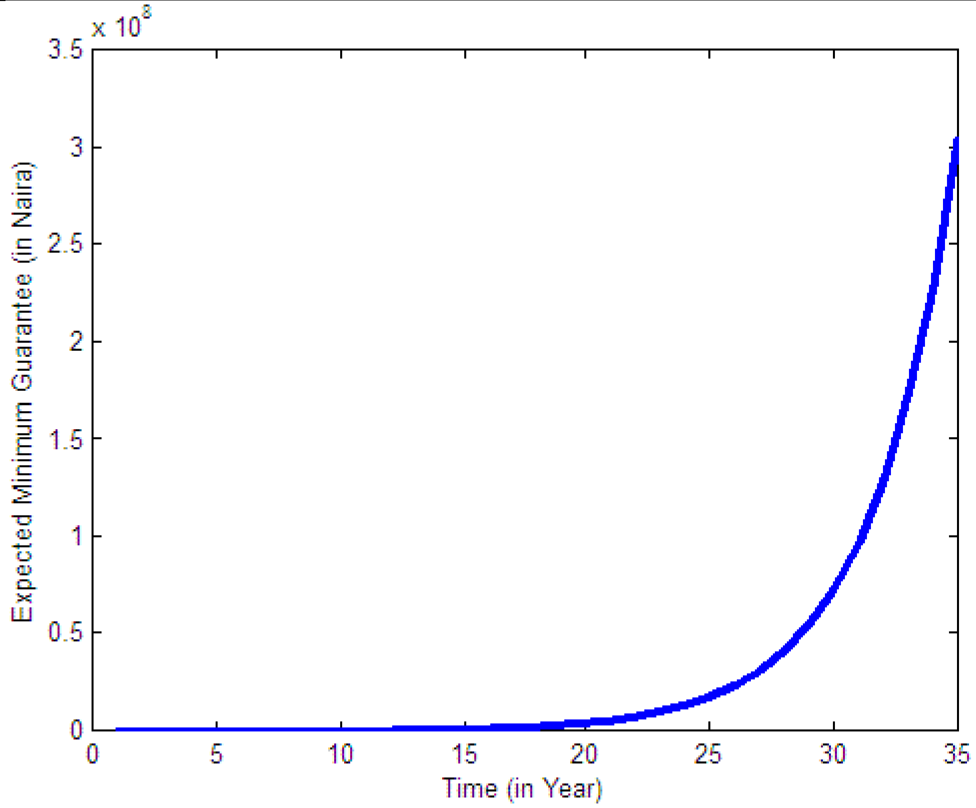

Figure 2: The Expected Minimum Guarantee of a PPM

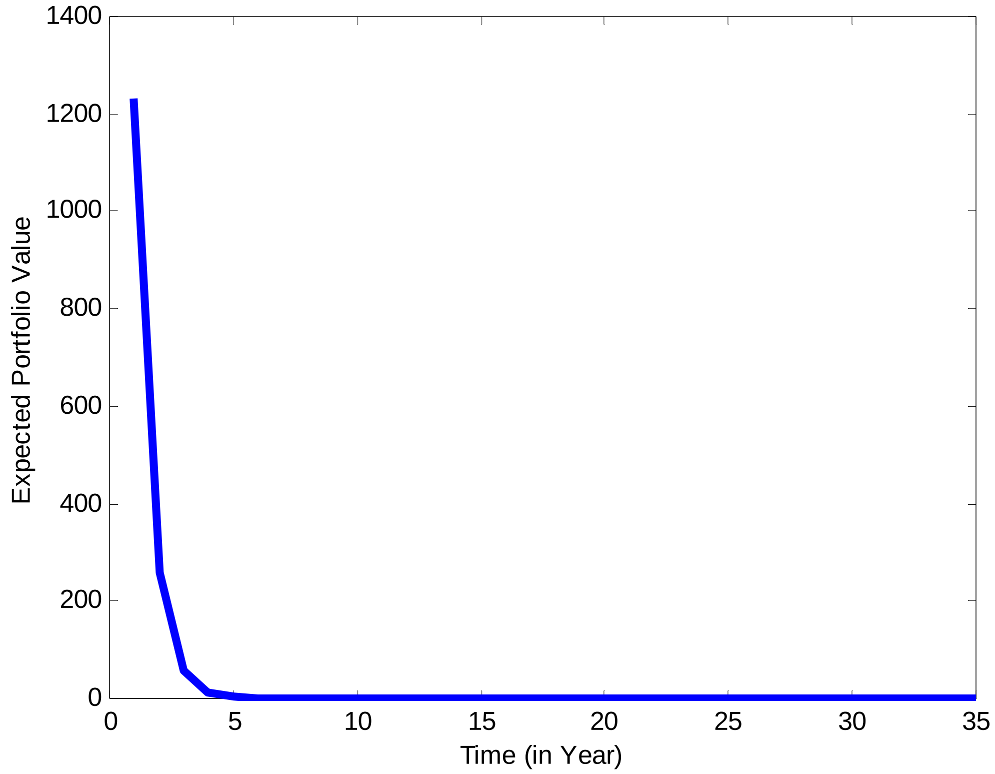

Figure 3: Expected Portfolio Value in Stock Market

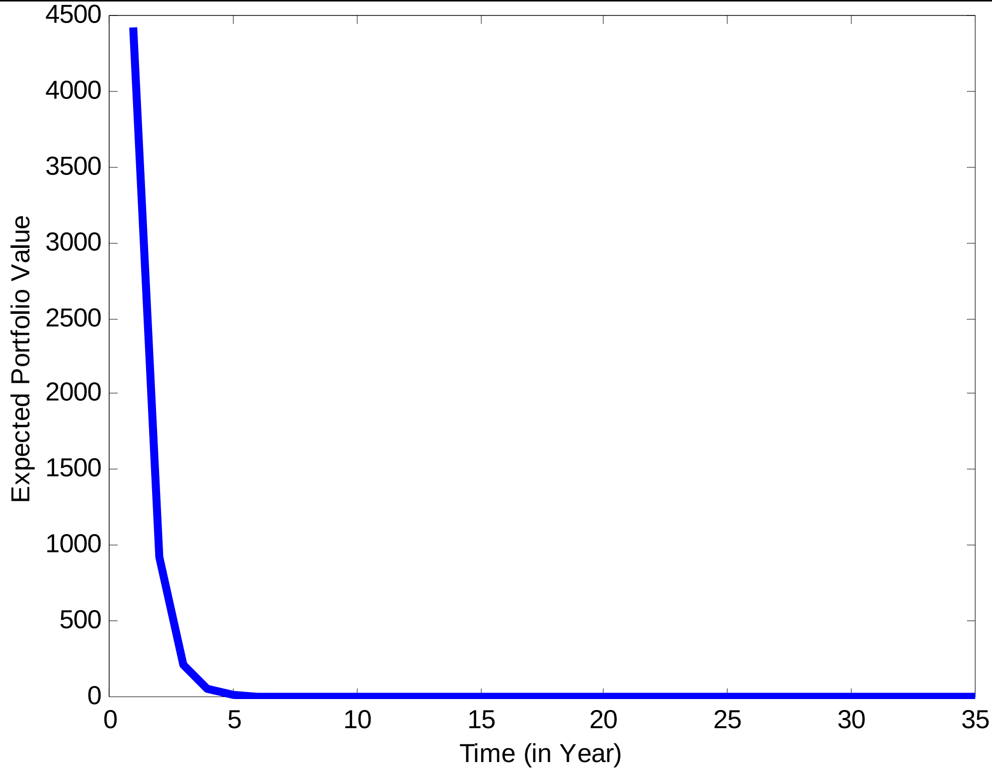

Figure 4: Expected Portfolio Value in Indexed Bond

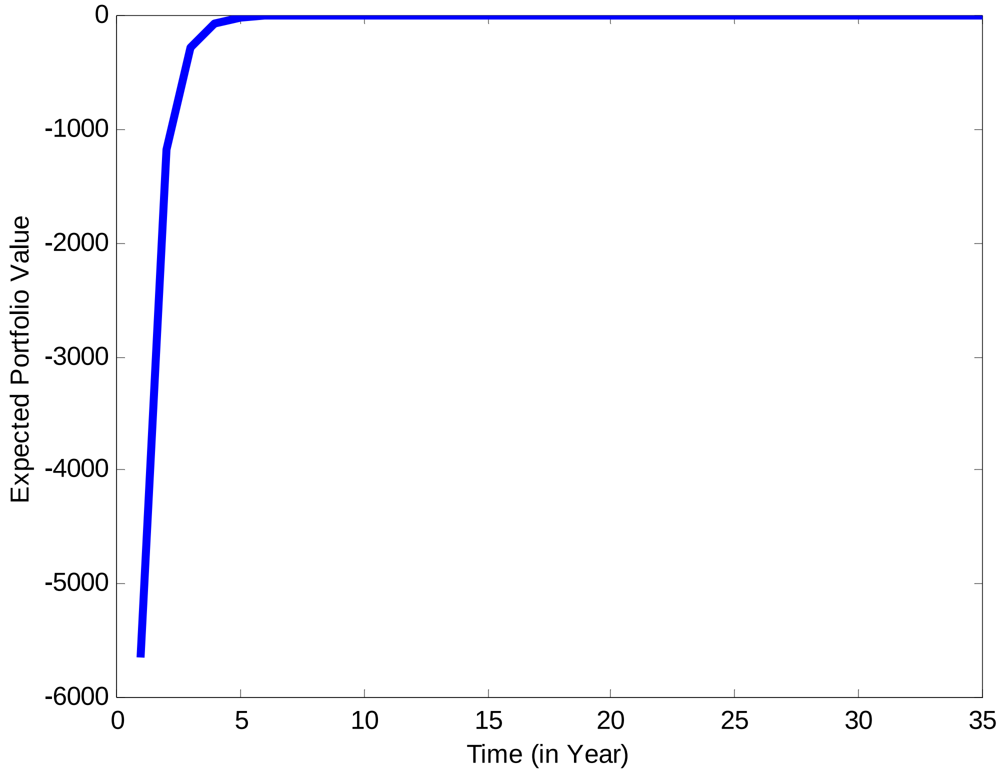

Figure 5: Expected Portfolio Value in Cash Account

6. CONCLUSION

This paper studied the optimal investment and optimal portfolio strategies with minimum guarantee and inflation protection in a DC pension scheme. We derived the proportion of wealth that should be transferred into the cash account from the indexed bonds and stocks portfolio values in order to hedged the investment and inflation risk that is associated with the PPM's portfolios. We found that with effective management of the pension funds by PFAs, PPMs will have high returns from their investment. We derived an inter-temporal hedging strategy that offset any shock to the contributions of PPM in pension funds. From our results, we found that the optimal terminal wealth that will accrued to the PPM and PFA to be $N1.97 \times 10^{22}$. We also found that the minimum guarantee that will accrued to the PPM at retirement to be $N3.05 \times 10^{8}$. The portfolio values in stock, indexed bond and cash account are 0.5, 3.1, and -2.6, respectively. Therefore, more investment should go to indexed bond and stock since they yield positive portfolio value.

REFERENCES

[1] Basak, S. (2002). A Comparative Study of Portfolio Insurance. *Journal of Economic Dynamics and Control, 26*, 1217-1241.

[2] Deelstra, G., Grasselli, M. and Koehl, P. (2003). Optimal Investment Strategies in the Presence of a Minimum Guarantee. *Insurance: Mathematics and Economics*, *33*, 189-207.

[3] Grossman, S. J. and Zhou, J. (1996). Equilibrium Analysis of Portfolio Insurance. *Journal of Finance, 51*, 1379-1403.

[4] Hojgaard, B. and Vigna, E. (2007). Mean-variance Portfolio Selection and Efficient Frontier for Defined Contribution Pension Schemes. *Collegio Carlo Alberto*.

[5] Nkeki, C. I. and Nwozo, C. R. (2011). Optimal Portfolio Strategies with Minimum Guarantee Maximizing the Expected Utility of Terminal Surplus for a Defined Contribution Pension Plan. *Int. J. Appl. Math. Stat., 20*, 60-72.

[6] Rudolf, M. and Ziemba, W. T. (2004). Intertemporal Surplus Management. *Journal of Economic Dynamics and Control*, *28*, 975-990.

[7] Tepla, L. (2001). Optimal Investment with Minimum Performance Constraints. *Journal of Economic Dynamics and Control, 25*, 1629-1645.

[8] Vigna, E. (2010). On Efficiency of Mean-Variance based Portfolio Selection in DC Pension Schemes. *Collegio Carlo Alberto.*

[9] Zhang, A., Korn, R. and Ewald, C. O. (2007). Optimal Management and Inflation Protection for Defined Contribution Pension Plan. *MPRA paper*.