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New Exact Jacobi Elliptic Function Solutions for Nonlinear Equations Using F-expansion Method

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Abstract: In this work, Jacobi elliptic function solutions for integrable nonlinear equations using F-expansion method are represented. KdV and Boussinesq equations are considered and new results are obtained.

Key Words: Jacobi elliptic functions; F-expansion method; Solitary waves; Periodic solutions

1. INTRODUCTION

It is well known that nonlinear equations play a major role describing many phenomena in many fields of sciences such as fluid mechanics, mathematical physics, biology, hydrodynamics, solid state physics and optical fibers. A variety of powerful methods were used to obtain exact solutions for nonlinear equations for examples inverse scattering method[1], Hirota bilinear form[8], Painlevé analysis[4], tanh-function method[11] and it's extensions[5–7].

Recently, the F-expansion method[12] is used to obtain exact Jacobi elliptic function solutions which give many exact solitary wave and periodic solutions for nonlinear equations.

In this work, we present a new technique using the F-expansion method to obtain new exact Jacobi elliptic function solutions for integrable nonlinear equations. Here, we don't neglect the integration constants in the resulting integrable ODEs using the wave transformation, use a transformation in which we express the solution function as a sum of another independent function and a constant which are determined later, hence we obtain new exact Jacobi elliptic function solutions for the integrable nonlinear equations.

2. NEW EXACT SOLUTIONS USING F-EXPANSION METHOD

Consider the nonlinear evolution and wave equations in the following forms

$$P(u, u_t, u_x, u_{xx}, ...) = 0 \quad \text{and} \quad P(u, u_{tt}, u_x, u_{xx}, ...) = 0. \tag{1}$$

Introducing the wave transformation

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$$u(x,t) = U(\xi), \quad \xi = k(x - \omega t), \tag{2}$$

to change (1) into a nonlinear ODE

$$O(U, U', U'', U''', ...) = 0, \tag{3}$$

where $' = \frac{d}{d\xi}$, $k > 0$ is the wave number and ω is the travelling wave velocity.

Assuming (3) is integrated with respect to ξ as many times as possible without neglecting the integration constants. For the evolution equations the maximum number of integration is 1 and for the wave equations is 2. For reasons that will be explained below, we only leave the integration constant of the last integration.

To obtain the new exact solution, possibly having a determined constant term d, we introduce the transformation

$$U = \phi + d. \tag{4}$$

Substituting (4) into (3) and setting the constant part equals to zero in the resulting nonlinear ODE in ϕ assuming that the function ϕ and its derivatives have the following asymptotic values

$$\phi(\xi) \to \phi_{\pm} \text{ as } \xi \to \pm\infty, \tag{5}$$

and

$$\phi^{(n)}(\xi) \to 0 \text{ as } \xi \to \pm\infty \text{ for } n \geq 1, \tag{6}$$

where the superscripts denotes differentiation to the order n, with respect to ξ. Also we assume that $\phi_{\pm}$ satisfies an algebraic equation in ϕ, then we get the values of d.

Applying the F-expansion method, using the finite expansion

$$\phi(\xi) = S(F) = \sum_{i=0}^{r} a_i F^i, \tag{7}$$

where r and $a_i (i = 0, 1, ...r)$ are constants to be determined later and F is the general solution of the ODE equation in the form

$$F'^2 = A + BF^2 + CF^4, \tag{8}$$

in which A, B and C are given values.

The relations between A, B and C and the corresponding values of $F(\xi)$ are given in [12], for examples, if $A = 1$, $B = -(1 + m^2)$ and $C = m^2$ we obtain the general Jacobi sn-function solution in the form

$$F = sn(\xi + C_2), \tag{9}$$

and if $A = m^2$, $B = -(1 + m^2)$ and $C = 1$, we obtain the general Jacobi ns-function solution in the form

$$F = ns(\xi + C_2), \tag{10}$$

where C_2 is an arbitrary constant and $0 < m < 1$ is the modules.

The positive integer r in (7) is determined by the balancing procedure in the resulting nonlinear ODE in S. Thus, we have an algebraic system of equations from which the constants k, ω, a_i are obtained and the function ϕ is determined, hence we get the new exact solutions of (1).

3. APPLICATIONS

Now, we give two examples of integrable nonlinear equations to find their new exact solutions as an application of the suggested method.

3.1 KdV Equation

Consider the KdV equation in the form[2]

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{11}$$

where α and β are constants.

Using (2) to change (11) into a nonlinear ODE and integrating once with respect to ξ, we obtain

$$-\omega U + \frac{1}{2}\alpha U^2 + \beta k^2 U'' + C_1 d = 0, \tag{12}$$

where $C_1 d$ is the integration constant.

Substituting (4) into (12), we have

$$(\alpha d - \omega)\phi + \frac{1}{2}\alpha\phi^2 + \beta k^2 \phi'' + d(\frac{1}{2}\alpha d + C_1 - \omega) = 0. \tag{13}$$

Using the conditions (5), (6) and that $\phi_\pm$ satisfies the algebraic equation

$$(\alpha d - \omega)\phi_\pm + \frac{1}{2}\alpha\phi_\pm^2 = 0, \tag{14}$$

then the constant term in (13) equals zero, i.e.,

$$d(\frac{1}{2}\alpha d + C_1 - \omega) = 0. \tag{15}$$

Thus we have two cases according to the values of d as in the following

Case A: $d = 0$

In this case, (13) becomes

$$-\omega\phi + \frac{1}{2}\alpha\phi^2 + \beta k^2 \phi'' = 0. \tag{16}$$

Substituting (7) and making use of (8), we get

$$-\omega S + \frac{1}{2}\alpha S^2 + \beta k^2 \left[\left(BF + 2CF^3\right)\frac{dS}{dF} + \left(A + BF^2 + CF^4\right)\frac{d^2S}{dF^2}\right] = 0. \tag{17}$$

Balancing the nonlinear term S^2 with the derivative term $\frac{d^2S}{dF^2}$ to get $r = 2$ and using (7) to have

$$\phi(\xi) = S(F) = a_0 + a_1 F + a_2 F^2. \tag{18}$$

Substituting (18) into (17) and setting zero all the coefficients of $F^i (i = 0, 1, ..., 4)$, we get the system of the algebraic equations

$$\begin{aligned}
&-\omega a_0 + \tfrac{1}{2}\alpha a_0^2 + 2\beta k^2 a_2 A = 0,\\
&-\omega a_1 + \beta k^2 a_1 B + \alpha a_0 a_1 = 0,\\
&-\omega a_2 + \tfrac{1}{2}\alpha a_1^2 + \alpha a_0 a_2 + 4\beta k^2 a_2 B = 0,\\
&\alpha a_1 a_2 + 2\beta k^2 a_1 C = 0,\\
&\tfrac{1}{2}\alpha a_2^2 + 6\beta k^2 a_2 C = 0.
\end{aligned} \tag{19}$$

Solving the system (19), we obtain

$$\begin{aligned}
&a_0 = \frac{4\beta k^2\left(-B \pm \sqrt{B^2 - 3AC}\right)}{\alpha}, \quad && a_1 = 0,\\
&a_2 = -\frac{12\beta k^2 C}{\alpha}, && \omega = \pm 4\beta k^2 \sqrt{B^2 - 3AC}.
\end{aligned} \tag{20}$$

Using (4) and (18), we obtain the exact solutions

$$\begin{array}{l} u_{1,2} = \frac{4\beta k^2}{\alpha}\left[-B \pm \sqrt{B^2 - 3AC} - 3CF^2\right], \\ \xi = k\left(x \mp 4\beta k^2 \sqrt{B^2 - 3AC}t\right). \end{array} \tag{21}$$

If $A = 1$, $B = -(1 + m^2)$ and $C = m^2$, we obtain the Jacobi sn-function solutions

$$\begin{array}{l} u_{1,2} = \frac{4\beta k^2}{\alpha}\left[1 + m^2 \pm \sqrt{1 - m^2 + m^4} - 3m^2 sn^2(\xi + C_2)\right], \\ \xi = k\left(x \mp 4\beta k^2 \sqrt{1 - m^2 + m^4}t\right). \end{array} \tag{22}$$

Let $m \to 1$, we obtain the following solitary wave solutions

$$\begin{array}{l} u_1 = \frac{12\beta k^2}{\alpha}\mathrm{sech}^2\left(k(x - 4\beta k^2 t) + C_2\right), \\ u_2 = \frac{4\beta k^2}{\alpha}\left[1 - 3\tanh^2\left(k(x + 4\beta k^2 t) + C_2\right)\right], \end{array} \tag{23}$$

these solutions are exact solutions ($\alpha = 1$, $\omega = \pm 4\beta k^2$ and $C_2 = 0$) which are obtained in [10].

If $A = m^2$, $B = -(1 + m^2)$ and $C = 1$, we obtain the Jacobi ns-function solutions

$$\begin{array}{l} u_{1,2} = \frac{4\beta k^2}{\alpha}\left[1 + m^2 \pm \sqrt{1 - m^2 + m^4} - 3ns^2(\xi + C_2)\right], \\ \xi = k\left(x \mp 4\beta k^2 \sqrt{1 - m^2 + m^4}t\right). \end{array} \tag{24}$$

Let $m \to 1$, we obtain the singular solitary wave solutions

$$\begin{array}{l} u_1 = -\frac{12\beta k^2}{\alpha}\mathrm{csch}^2\left(k(x - 4\beta k^2 t) + C_2\right), \\ u_2 = \frac{4\beta k^2}{\alpha}\left[1 - 3\coth^2\left(k(x + 4\beta k^2 t) + C_2\right)\right]. \end{array} \tag{25}$$

Let $m \to 0$, we obtain the periodic solutions

$$\begin{array}{l} u_1 = \frac{4\beta k^2}{\alpha}\left[2 - 3\csc^2\left(k(x - 4\beta k^2 t) + C_2\right)\right], \\ u_2 = -\frac{12\beta k^2}{\alpha}\csc^2\left(k(x + 4\beta k^2 t) + C_2\right), \end{array} \tag{26}$$

Case B: $d = \frac{2}{\alpha}(\omega - C_1)$

In this case, (13) becomes

$$(\omega - 2C_1)\phi + \frac{1}{2}\alpha\phi^2 + \beta k^2\phi'' = 0. \tag{27}$$

Substituting (7) into (27) and making use of (8), we obtain

$$(\omega - 2C_1)S + \frac{1}{2}\alpha S^2 + \beta k^2\left[\left(BF + 2CF^3\right)\frac{dS}{dF} + \left(A + BF^2 + CF^4\right)\frac{d^2S}{dF^2}\right] = 0. \tag{28}$$

Substituting (18) into (28), setting zero all the coefficients of $F^i (i = 0, 1, ..., 4)$ and solving the resulting algebraic equations system, we obtain

$$\begin{array}{ll} a_0 = \frac{4\beta k^2}{\alpha}\left(-B \mp \sqrt{B^2 - 3AC}\right), & a_1 = 0, \\ a_2 = -\frac{12\beta C}{\alpha}, & \omega = 2C_1 \pm 4\beta k^2\sqrt{B^2 - 3AC}. \end{array} \tag{29}$$

Now, we find the values of d corresponding to the values of ω

$$\omega = 2C_1 \pm 4\beta k^2\sqrt{B^2 - 3AC} \quad \text{gives} \quad d = \frac{2C_1 \pm 8\beta k^2\sqrt{B^2 - 3AC}}{\alpha}. \tag{30}$$

Using (4) and (18), we obtain the exact solutions

$$\begin{aligned} u_{1,2} &= \tfrac{2C_1}{\alpha} + \tfrac{4\beta k^2}{\alpha}\left[-B \pm \sqrt{B^2 - 3AC} - 3CF^2\right], \\ \xi &= k\left(x - \left(2C_1 \pm 4\beta k^2 \sqrt{B^2 - 3AC}\right)t\right), \end{aligned} \tag{31}$$

where C_1 is an arbitrary constant.

If $A = 1$, $B = -(1 + m^2)$ and $C = m^2$, we obtain the Jacobi sn-function solutions

$$\begin{aligned} u_{1,2} &= \tfrac{2C_1}{\alpha} + \tfrac{4\beta k^2}{\alpha}\left[1 + m^2 \pm \sqrt{1 - m^2 + m^4} - 3m^2 sn^2(\xi + C_2)\right], \\ \xi &= k\left(x - \left(2C_1 \pm 4\beta k^2 \sqrt{1 - m^2 + m^4}\right)t\right). \end{aligned} \tag{32}$$

Let $m \to 1$, we obtain the solitary wave solutions

$$\begin{aligned} u_1 &= \tfrac{2C_1}{\alpha} + \tfrac{12\beta k^2}{\alpha}\operatorname{sech}^2\left(k(x - (2C_1 + 4\beta k^2)t) + C_2\right), \\ u_2 &= \tfrac{2C_1}{\alpha} + \tfrac{4\beta k^2}{\alpha}\left[1 - 3\tanh^2\left(k(x - (2C_1 - 4\beta k^2)t) + C_2\right)\right]. \end{aligned} \tag{33}$$

If $A = m^2$, $B = -(1 + m^2)$ and $C = 1$, we obtain the Jacobi ns-function solutions

$$\begin{aligned} u_{1,2} &= \tfrac{2C_1}{\alpha} + \tfrac{4\beta k^2}{\alpha}\left[1 + m^2 \pm \sqrt{1 - m^2 + m^4} - 3ns^2(\xi + C_2)\right], \\ \xi &= k\left(x - \left(2C_1 \pm 4\beta k^2 \sqrt{1 - m^2 + m^4}\right)t\right). \end{aligned} \tag{34}$$

Let $m \to 1$, we obtain the singular solitary wave solutions

$$\begin{aligned} u_1 &= \tfrac{2C_1}{\alpha} - \tfrac{12\beta k^2}{\alpha}\operatorname{csch}^2\left(k\left(x - \left(2C_1 + 4\beta k^2\right)t\right) + C_2\right), \\ u_2 &= \tfrac{2C_1}{\alpha} + \tfrac{4\beta k^2}{\alpha}\left[1 - 3\coth^2\left(k\left(x - \left(2C_1 - 4\beta k^2\right)t\right) + C_2\right)\right]. \end{aligned} \tag{35}$$

Let $m \to 0$, we obtain the periodic solutions

$$\begin{aligned} u_1 &= \tfrac{2C_1}{\alpha} + \tfrac{4\beta k^2}{\alpha}\left[2 - 3\csc^2\left(k\left(x - \left(2C_1 + 4\beta k^2\right)t\right) + C_2\right)\right], \\ u_2 &= \tfrac{2C_1}{\alpha} - \tfrac{12\beta k^2}{\alpha}\csc^2\left(k\left(x - \left(2C_1 - 4\beta k^2\right)t\right) + C_2\right). \end{aligned} \tag{36}$$

Remark 1: When $C_1 = 0$ in (31), we get the exact solutions (21).

3.2 Boussinesq Equation

Consider the Boussinesq equation in the form[3]

$$u_{tt} - \omega_0^2 u_{xx} - \alpha u_{xxxx} - \beta(u^2)_{xx} = 0, \tag{37}$$

where ω_0, α and β are constants.

Using (2) to change (37) into the following nonlinear ODE

$$(\omega^2 - \omega_0^2)U'' - \alpha k^2 U'''' - \beta(U^2)'' = 0. \tag{38}$$

Integrating twice and leave the integration constant of the last integration to obtain

$$(\omega^2 - \omega_0^2)U - \alpha k^2 U'' - \beta U^2 + C_1 d = 0, \tag{39}$$

where $C_1 d$ is an integration constant.

Substituting (4) into (39) to get

$$(\omega^2 - \omega_0^2 - 2\beta d)\phi - \beta\phi^2 - \alpha k^2\phi'' + d\left(\omega^2 - \omega_0^2 - \beta d + C_1\right) = 0. \tag{40}$$

Using the conditions (5), (6) and the algebraic equation

$$(\omega^2 - \omega_0^2 - 2\beta d)\phi_\pm - \beta\phi_\pm^2 = 0, \tag{41}$$

then the constant term in (40) equals zero, i.e.,

$$d\left(\omega^2 - \omega_0^2 - \beta d + C_1\right) = 0. \tag{42}$$

Thus we have the following two cases according to the values of d.

Case A: $d = 0$

Using the same way as in this case, we obtain the exact solutions

$$\begin{aligned} u_{1,2} &= \tfrac{2\alpha k^2}{\beta}\left[-B + \sqrt{B^2 - 3AC} - 3CF^2\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2\sqrt{B^2 - 3AC}}t\right), \\ u_{3,4} &= \tfrac{2\alpha k^2}{\beta}\left[-B - \sqrt{B^2 - 3AC} - 3CF^2\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2\sqrt{B^2 - 3AC}}t\right). \end{aligned} \tag{43}$$

If $A = 1$, $B = -(1 + m^2)$ and $C = m^2$, we obtain the Jacobi sn-function solutions

$$\begin{aligned} u_{1,2} &= \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 + \sqrt{1 - m^2 + m^4} - 3m^2 sn^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2\sqrt{1 - m^2 + m^4}}t\right), \\ u_{3,4} &= \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 - \sqrt{1 - m^2 + m^4} - 3m^2 sn^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2\sqrt{1 - m^2 + m^4}}t\right), \end{aligned} \tag{44}$$

when $\omega = \pm\sqrt{\omega_0^2 + 4\alpha k^2\sqrt{B^2 - 3AC}}$ and $C_2 = 0$ in (44), we get the Jacobi sn-function solutions which are obtained in [9].

Let $m \to 1$, we obtain the following solitary wave solutions

$$\begin{aligned} u_{1,2} &= \tfrac{6\alpha k^2}{\beta}\operatorname{sech}^2\left(k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2}t\right) + C_2\right), \\ u_{3,4} &= \tfrac{2\alpha k^2}{\beta}\left[1 - 3\tanh^2\left(k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2}t\right) + C_2\right)\right]. \end{aligned} \tag{45}$$

If $A = m^2$, $B = -(1 + m^2)$ and $C = 1$, we obtain the Jacobi ns-function solutions

$$\begin{aligned} u_{1,2} &= \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 + \sqrt{1 - m^2 + m^4} - 3ns^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2\sqrt{1 - m^2 + m^4}}t\right), \\ u_{3,4} &= \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 - \sqrt{1 - m^2 + m^4} - 3ns^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2\sqrt{1 - m^2 + m^4}}t\right). \end{aligned} \tag{46}$$

Let $m \to 1$, we obtain the singular solitary wave solutions

$$\begin{aligned} u_{1,2} &= -\tfrac{6\alpha k^2}{\beta}\operatorname{csch}^2\left(k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2}t\right) + C_2\right), \\ u_{3,4} &= \tfrac{2\alpha k^2}{\beta}\left[1 - 3\coth^2\left(k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2}t\right) + C_2\right)\right]. \end{aligned} \tag{47}$$

Let $m \to 0$, we obtain the periodic solutions

$$\begin{aligned} u_{1,2} &= \tfrac{2\alpha k^2}{\beta}\left[2 - 3\csc^2\left(k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2}t\right) + C_2\right)\right], \\ u_{3,4} &= -\tfrac{6\alpha k^2}{\beta}\csc^2\left(k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2}t\right) + C_2\right). \end{aligned} \tag{48}$$

Case B: $d = \frac{\omega^2 - \omega_0^2 + C_1}{\beta}$

Using the same way as in this case, we obtain the exact solutions

$$\begin{aligned} u_{1,2} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[-B + \sqrt{B^2 - 3AC} - 3CF^2\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2\sqrt{B^2 - 3AC} - 2C_1}t\right), \\ u_{3,4} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[-B - \sqrt{B^2 - 3AC} - 3CF^2\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2\sqrt{B^2 - 3AC} - 2C_1}t\right). \end{aligned} \tag{49}$$

If $A = 1$, $B = -(1 + m^2)$ and $C = m^2$, we obtain the Jacobi sn-function solutions

$$\begin{aligned} u_{1,2} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 + \sqrt{1 - m^2 + m^4} - 3m^2 sn^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2\sqrt{1 - m^2 + m^4} - 2C_1}t\right), \\ u_{3,4} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 - \sqrt{1 - m^2 + m^4} - 3m^2 sn^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2\sqrt{1 - m^2 + m^4} - 2C_1}t\right). \end{aligned} \tag{50}$$

Let $m \to 1$, we obtain the solitary wave solutions

$$\begin{aligned} u_{1,2} &= -\tfrac{C_1}{\beta} + \tfrac{6\alpha k^2}{\beta}\operatorname{sech}^2\left(k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2 - 2C_1}t\right) + C_2\right), \\ u_{3,4} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[1 - 3\tanh^2\left(k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2 - 2C_1}t\right) + C_2\right)\right]. \end{aligned} \tag{51}$$

If $A = m^2$, $B = -(1 + m^2)$ and $C = 1$, we obtain the Jacobi ns-function solutions

$$\begin{aligned} u_{1,2} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 + \sqrt{1 - m^2 + m^4} - 3ns^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2\sqrt{1 - m^2 + m^4} - 2C_1}t\right), \\ u_{3,4} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[1 + m^2 - \sqrt{1 - m^2 + m^4} - 3ns^2(\xi + C_2)\right], \\ \xi &= k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2\sqrt{1 - m^2 + m^4} - 2C_1}t\right). \end{aligned} \tag{52}$$

Let $m \to 1$, we obtain the singular solitary wave solutions

$$\begin{aligned} u_{1,2} &= -\tfrac{C_1}{\beta} - \tfrac{6\alpha k^2}{\beta}\operatorname{csch}^2\left(k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2 - 2C_1}t\right) + C_2\right), \\ u_{3,4} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[1 - 3\coth^2\left(k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2 - 2C_1}t\right) + C_2\right)\right]. \end{aligned} \tag{53}$$

Let $m \to 0$, we obtain the periodic solutions

$$\begin{aligned} u_{1,2} &= -\tfrac{C_1}{\beta} + \tfrac{2\alpha k^2}{\beta}\left[2 - 3\csc^2\left(k\left(x \mp \sqrt{\omega_0^2 + 4\alpha k^2 - 2C_1}t\right) + C_2\right)\right], \\ u_{3,4} &= -\tfrac{C_1}{\beta} - \tfrac{6\alpha k^2}{\beta}\csc^2\left(k\left(x \mp \sqrt{\omega_0^2 - 4\alpha k^2 - 2C_1}t\right) + C_2\right). \end{aligned} \tag{54}$$

Remark 2: When $C_1 = 0$ in (49), we get the exact solutions (43).

4. CONCLUSIONS

A new technique is presented, by adding integration constants, using the transformation (4) and the F-expansion method, to obtain new exact solutions for the integrable nonlinear equations.

By this technique, we obtained the new exact solutions of the KdV equation in (31)-(36) and the Boussinesq equation in (49)-(54) which all give the exact solutions obtained before by the F-expansion method as special cases.

Moreover, new results can be obtained if we use rest of the relations between the values A, B and C and the corresponding values of $F(\xi)$ given in [12].

The presented technique can be applied for many integrable nonlinear equations in which the odd- and even-order derivative terms don't coexist.

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