Studies in Mathematical Sciences
Vol. 2, No. 1, 2011, pp. 80-87
www.cscanada.org

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net

Hybrid Function Projective Synchronization of Chaotic Systems with Fully Unknown Parameters

Zhanguo LI[1,*]
Wei XU[1]

Abstract: To compensate for projective synchronization (PS) and function projective synchronization (FPS), we propose a hybrid function projective synchronization (HFPS), which applies the different time-varying functions as the synchronization scaling factors. Based on the adaptive control method, we design a simple controller and a set of update laws of unknown parameters to carry out HFPS in identical and different chaotic systems with fully unknown parameters. According to the Lyapunov stability theorem and the Barbalat lemma, we prove the asymptotical stability of the error dynamical system at the origin. Then two numerical examples are given to validate the feasibility and effectiveness of the developed procedure in this paper.

Key Words: Hybrid function projective synchronization; Lyapunov stability theorem; Adaptive control; Unknown parameters

1. INTRODUCTION

Chaos control and chaos synchronization were believed to be impossible. However, Pecora and Carroll made a breakthrough in 1990[1]. They first synchronized two identical chaotic systems with different initial conditions. Thereafter, various synchronization schemes were developed such as complete synchronization[1], phase synchronization[2], anti-synchronization[3], lag synchronization[4], projective synchronization[5-8], etc. Researches on chaos synchronization in deterministic and stochastic dynamical systems attracted a growing interest in physics and other scientific fields due to its potential applications in secure communication, biology, economy, etc. In recent years, projective synchronization (PS) has been extensively studied because its proportional feature can be used to obtain faster communication[6-8]. Yet, the scaling factors of PS are often real numbers, which could induce the feeblish security in chaotic communication. So function projective synchronization (FPS) was proposed by selecting function as scaling factor[9,10]. Compared to PS, FPS could be used to get more secure chaotic communication. To the best of our knowledge, few theoretical results about FPS between different chaotic systems have been reported now. Motivated by the aforementioned reasons, we put forward an idea that the complexity and security of secure communication will be enhanced if we select different time-varying function as the scaling factors. This paper focuses on the study of hybrid function projective synchronization (HFPS), in which the synchronization scaling factors consist of different time-varying functions. Then we design a simple controller and a set of update laws, with which we prove that the error dynamical system is

[1]Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China.
E-mail address: lizhanguo@mail.nwpu.edu.cn.
*Corresponding author.
†Received 26 November 2010; accepted 19 December 2010.

asymptotically stable at the origin. Then the feasibility and effectiveness of the proposed procedure in this paper are illustrated with two numerical examples.

2. MATHEMATICAL MODEL AND PROBLEM STATEMENT

Consider two chaotic systems in the form of

$$\dot{x} = f(x) + F(x) \cdot p, \tag{1}$$
$$\dot{y} = g(y) + G(y) \cdot q + u, \tag{2}$$

where $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$ are the state vectors, $p = (p_1, p_2, \cdots, p_k)^T$ and $q = (q_1, q_2, \cdots, q_m)^T$ denote fully unknown parameter vectors, $f : R^n \to R^n$, $F : R^n \to R^{n\times k}$, $g : R^n \to R^n$ and $G : R^n \to R^{n\times m}$ are continuously differentiable vector functions and matrix functions, respectively. Notation u represents a synchronization controller to be designed to implement HFPS between systems (1) and (2). Many usual chaotic systems have the form of system (1) or (2) such as the well-known Lorenz system, Chen system, and so on.

Definition 2.1 Time-varying function $\alpha(t)$ is referred to as a scaling function if it is continuously differentiable, bounded and $\alpha(t) \neq 0$ for any t.

Definition 2.2 Matrix $A(t)$ is viewed as a scaling function matrix if its any element is a scaling function.

Definition 2.3 Systems (1) and (2) are referred to as achieving HFPS if $\lim_{t\to\infty} \| e(t) \| = \lim_{t\to\infty} \| y(t) - A(t)\cdot x(t) \| = 0$, where $e(t) = y(t) - A(t)\cdot x(t)$ denotes the synchronization error between systems (1) and (2), and $A(t)$ is a given scaling function matrix.

Remark: HFPS includes the usual projective synchronization, complete synchronization and anti-synchronization. These are all special case of HFPS. For instance, HFPS will become the usual so-called projective synchronization when the scaling functions are selected as real numbers.

Suppose that $A(t)_{n\times n}$ is a given scaling matrix and $e(t) = y(t) - A(t)\cdot x(t)$ denotes the synchronization error between systems (1) and (2). Then we obtain the error dynamical system as follows:

$$\begin{aligned} \dot{e}(t) &= \dot{y}(t) - \dot{A}(t)\cdot x(t) - A(t)\cdot \dot{x}(t) \\ &= g(y) + G(y)\cdot q + u - \dot{A}(t)\cdot x(t) - A(t)[f(x) + F(x)\cdot p] \\ &= g(y) + G(y)\cdot q - \dot{A}(t)\cdot x(t) - A(t)f(x) - A(t)F(x)\cdot p + u. \end{aligned} \tag{3}$$

According to the feature of system (3), the controller u can be designed as

$$u = -g(y) - G(y)\cdot \hat{q} + \dot{A}(t)\cdot x(t) + A(t)\cdot f(x) + A(t)\cdot F(x)\cdot \hat{p} - K\cdot e, \tag{4}$$

where $\hat{q}$ and $\hat{p}$ denote the estimate values of q and p, respectively, and K is a positive definite matrix. Utilizing controller (4), the error dynamical system (3) can be simplified as

$$\dot{e}(t) = G(y)\cdot(q - \hat{q}) - A(t)\cdot F(x)\cdot(p - \hat{p}) - K\cdot e, \tag{5}$$

where $q - \hat{q}$ and $p - \hat{p}$ represent the deviation between the estimate value and the actual value of q and p, respectively.

Theorem 2.4 If the estimates of the unknown parameters contained in the adaptive controller (4) are on-line adjusted by the algorithms

$$\begin{cases} \dot{\hat{q}} = [G(y)]^T \cdot e, \\ \dot{\hat{p}} = -[A(t) \cdot F(x)]^T \cdot e, \end{cases} \tag{6}$$

then the synchronization error dynamical system (5) is asymptotically stable at the origin.

Proof. Combine the error system (5) and the system (6) which estimates the unknown parameters as an augmented non-autonomous system. From this augmented system it is easy to check that the point $O = \{(e, \hat{q}, \hat{p}) \mid e = 0, \hat{q} = q, \hat{p} = p\}$ is the unique fixed point in $n \times m \times k$ -dimensional phase space because $F(x)$ and $G(y)$ depend on the state vectors x and y, respectively, and $A(t)$ is time-varying. Construct a non-negative function in the neighborhood of the singular point O as follows:

$$V = \frac{1}{2}[e^T \cdot e + (q - \hat{q})^T \cdot (q - \hat{q}) + (p - \hat{p})^T \cdot (p - \hat{p})]. \tag{7}$$

Obviously, $V = 0$ if and only if $\hat{q} = q, \hat{p} = p, e = 0$, i.e., at the equilibrium point O, and $V > 0$ for the other points in the neighborhood of the fixed point O. By using the above augmented system, the time derivative of V with respect to time t is as follows:

$$\begin{aligned} \dot{V} &= \dot{e}^T \cdot e - (q - \hat{q})^T \cdot \dot{\hat{q}} - (p - \hat{p})^T \cdot \dot{\hat{p}} \\ &= [G(y) \cdot (q - \hat{q}) - A(t) \cdot F(x) \cdot (p - \hat{p}) - K \cdot e]^T \cdot e - (q - \hat{q})^T \cdot [G(y)]^T \cdot e + (p - \hat{p})^T \cdot [A(t) \cdot F(x)]^T \cdot e \\ &= (q - \hat{q})^T \cdot [G(y)]^T \cdot e - (p - \hat{p})^T \cdot [F(x)]^T \cdot [A(t)]^T \cdot e - e^T \cdot K^T \cdot e - (q - \hat{q})^T \cdot [G(y)]^T \cdot e \\ &\quad + (p - \hat{p})^T \cdot [F(x)]^T \cdot [A(t)]^T \cdot e \\ &= -e^T \cdot K^T \cdot e. \end{aligned} \tag{8}$$

It is easy to check that $\dot{V} = -e^T \cdot K^T \cdot e \le 0$ because K^T is a positive definite matrix which can be deduced from the fact that K is a positive definite matrix. According to the Lyapunov stability theorem for non-autonomous differential equations, we can get the fact that the fixed-point O of the augmented system is Lyapunov stable, i.e., $e \in L_\infty$. On the other hand, due to $\dot{V} = -e^T \cdot K^T \cdot e \le 0$ and K^T is a positive definite matrix, we have $\int_0^t \lambda_{\min}(K^T) \cdot \|e\|^2 \cdot dt \le \int_0^t e^T \cdot K^T \cdot e \cdot dt \le \int_0^t -\dot{V} \cdot dt = V(0) - V(t) \le V(0)$, where $\lambda_{\min}(K^T)$ denotes the minimum characteristic root of matrix K^T. Hence, we know $e \in L_2$. Besides, the error dynamical system (5) implies $\dot{e} \in L_\infty$. Based on the Barbalat's lemma, the above conditions indicate $e(t) \to 0$ as $t \to \infty$ for any initial conditions. Then the error dynamical system (5) is asymptotically stable at the origin. This means that HFPS between master system (1) and slave system (2) has achieved globally and asymptotically. This completes the proof.

3. NUMERICAL EXAMPLES

In this section, we present two numerical examples to demonstrate the effectiveness of HFPS between identical and different chaotic systems with different initial conditions, respectively.

Example 1. The well-known Lorenz system[11] is of the following form

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = bx_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - c x_3. \end{cases} \tag{9}$$

We view system (9) as the master system and introduce the slave system in the form:

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1) + u_1, \\ \dot{y}_2 = by_1 - y_2 - y_1 y_3 + u_2, \\ \dot{y}_3 = y_1 y_2 - c y_3 + u_3, \end{cases} \tag{10}$$

where a, b, c denote the model parameters which are fully unknown in the simulation progress. u_1, u_2 and u_3 represent the synchronization controllers. For convenience, we choose the scaling function matrix $A(t) = diag(\alpha_1(t), \alpha_2(t), \alpha_3(t))$, where $\alpha_i(t)$ is time-varying scaling factors and $\alpha_i(t) \neq \alpha_j(t)\,(i \neq j)$. Thus, the synchronization error $e(t) = y(t) - A(t)\cdot x(t)$ can be described by

$$e_k(t) = y_k(t) - \alpha_k(t)\cdot x_k(t), \quad k = 1, 2, 3. \tag{11}$$

According to systems (4) and (6), the synchronization controllers can be selected as below:

$$\begin{cases} u_1 = -\hat{a}\cdot[y_2 - y_1 - \alpha_1(t)\cdot(x_2 - x_1)] + \dot{\alpha}_1(t)\cdot x_1 - k\cdot e_1, \\ u_2 = -\hat{b}\cdot[y_1 - \alpha_2(t)\cdot x_1] + [y_2 - \alpha_2(t)\cdot x_2] + y_1\cdot y_3 - \alpha_2(t)\cdot x_1\cdot x_3 + \dot{\alpha}_2(t)\cdot x_2 - k\cdot e_2, \\ u_3 = \hat{c}\cdot[y_3 - \alpha_3(t)\cdot x_3] - y_1\cdot y_2 + \dot{\alpha}_3(t)\cdot x_3 + \alpha_3(t)\cdot x_1\cdot x_2 - k\cdot e_3, \end{cases} \tag{12}$$

where $\hat{a}, \hat{b}, \hat{c}$ denote the estimate values for the unknown parameters a, b, c and obey the following update laws:

$$\begin{cases} \dot{\hat{a}} = [y_2 - y_1 - \alpha_1(t)\cdot(x_2 - x_1)]\cdot e_1, \\ \dot{\hat{b}} = [y_1 - \alpha_2(t)\cdot x_1]\cdot e_2, \\ \dot{\hat{c}} = -[y_3 - \alpha_3(t)\cdot x_3]\cdot e_3. \end{cases} \tag{13}$$

Based on controller (12) and update law (13), the error dynamical system between systems (9) and (10) can be written as:

$$\begin{cases} \dot{e}_1 = [y_2 - y_1 - \alpha_1(t)\cdot(x_2 - x_1)]\cdot(a - \hat{a}) - k\cdot e_1, \\ \dot{e}_2 = [y_1 - \alpha_2(t)\cdot x_1]\cdot(b - \hat{b}) - k\cdot e_2, \\ \dot{e}_3 = -[y_3 - \alpha_3(t)\cdot x_3]\cdot(c - \hat{c}) - k\cdot e_3. \end{cases} \tag{14}$$

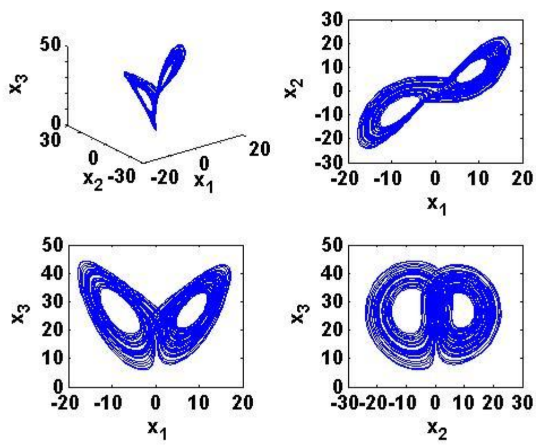

Figure 1: Lorenz chaotic attractor and its two-dimension projections on $x_1 - x_2$, $x_1 - x_3$ and $x_2 - x_3$ plane

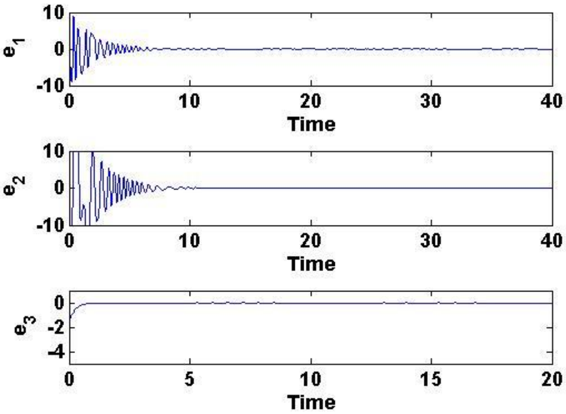

Figure 2: Time history of HFPS synchronization errors of e_1, e_2 and e_3 between systems (9) and (10)

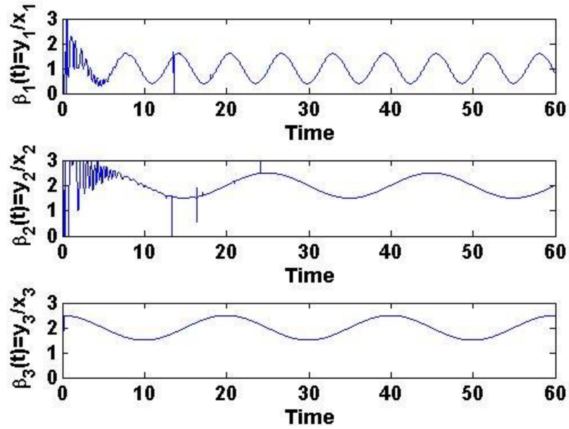

Figure 3: Time history of scaling functions reconstructed by state vectors of systems (9) and (10)

In the numerical simulations, the fourth-order Runge-Kutta method with time step of 0.01 is used throughout. The initial conditions are chosen as $x_1(0)=1$, $x_2(0)=1$, $x_3(0)=1$, $y_1(0)=-2$, $y_2(0) =-1$, $y_3(0)=1$, $\hat{a}(0)=0$, $\hat{b}(0)=0$, $\hat{c}(0)=0$. Letting the feedback gain $k=1$ and assuming the "unknown" parameters $a=10, b=28, c=8/3$ to ensure chaotic behavior, we choose the desired scaling functions $\alpha_1(t)=1+0.6\sin(t)$, $\alpha_2(t)=2+0.5\sin(\frac{\pi}{10}t)$, $\alpha_3(t)=2+0.5\cos(\frac{\pi}{10}t)$. Then we know the initial conditions of the error dynamical system (14) are $e_1(0)=-3$, $e_2(0)=-3$, $e_3(0)=-1.5$. The results obtained by simultaneously solving systems (9), (10), (13) and (14), combined with the controller (12), are shown in Figures 1-3, respectively. Figure 1 shows that the Lorenz system is chaotic under the above parametric conditions. Figure 2 displays the time history of synchronization errors. From Figure 2 one can see clearly that the synchronization error vectors converge to zero as time evolving. Figure 3 depicts that the predefined scaling functions $\alpha_i(t)\,(i=1,2,3)$ can be accurately recovered by computing $\alpha_1(t)=y_1(t)/x_1(t)$, $\alpha_2(t)=y_2(t)/x_2(t)$ and $\alpha_3(t)=y_3(t)/x_3(t)$ as time evolving. Figures 2 and 3 show that HFPS between the master system (9) and the slave system (10) has been achieved.

Example 2. We still use the Lorenz system (9) as the master system and introduce a new chaotic system (15)[3] as the slave system:

$$\begin{cases} \dot{y}_1 = -y_2 y_3 + a_1 y_1 + u_1, \\ \dot{y}_2 = y_1 y_3 + b_1 y_2 + u_2, \\ \dot{y}_3 = \frac{1}{3} y_1 y_2 + c_1 y_3 + u_3, \end{cases} \tag{15}$$

where a_1, b_1 and c_1 represent the unknown parameters of system (15), and u_1, u_2, u_3 are synchronization controllers. Similar to Example 1, we define the synchronization error $e_i(t) = y_i(t) - \alpha_i(t) \cdot x_i(t)$, $i = 1,2,3$. Then we present the controllers with the following form

$$\begin{cases} u_1 = y_2 \cdot y_3 + \dot{\alpha}_1(t) \cdot x_1 - \hat{a}_1 \cdot y_1 + \alpha_1(t) \cdot (x_2 - x_1) \cdot \hat{a} - k \cdot e_1, \\ u_2 = -y_1 \cdot y_3 + \dot{\alpha}_2(t) \cdot x_2 - \alpha_2(t) \cdot x_2 - \alpha_2(t) \cdot x_1 \cdot x_3 - \hat{b}_1 \cdot y_2 + \alpha_2(t) \cdot x_1 \cdot \hat{b} - k \cdot e_2, \\ u_3 = -\frac{1}{3} y_1 \cdot y_2 - \hat{c}_1 \cdot y_3 + \dot{\alpha}_3(t) \cdot x_3 + \alpha_3(t) \cdot x_1 \cdot x_2 - \alpha_3(t) \cdot x_3 \cdot \hat{c} - k \cdot e_3. \end{cases} \tag{16}$$

The estimation values $\hat{a}_1, \hat{b}_1, \hat{c}_1$ for a_1, b_1, c_1 obey the update laws below

$$\begin{cases} \dot{\hat{a}} = -\alpha_1(t) \cdot (x_2 - x_1) \cdot e_1, \\ \dot{\hat{b}} = -\alpha_2(t) \cdot x_1 \cdot e_2, \\ \dot{\hat{c}} = \alpha_3(t) \cdot x_3 \cdot e_3, \\ \dot{\hat{a}}_1 = y_1 \cdot e_1, \\ \dot{\hat{b}}_1 = y_2 \cdot e_2, \\ \dot{\hat{c}}_1 = y_3 \cdot e_3. \end{cases} \tag{17}$$

Hence, the error dynamical system between systems (9) and (15) with controller (16) can be described as

$$\begin{cases} \dot{e}_1 = y_1 \cdot (a_1 - \hat{a}_1) - \alpha_1(t) \cdot (x_2 - x_1) \cdot (a - \hat{a}) - k \cdot e_1, \\ \dot{e}_2 = y_2 \cdot (b_1 - \hat{b}_1) - \alpha_2(t) \cdot x_1 \cdot (b - \hat{b}) - k \cdot e_2, \\ \dot{e}_3 = y_3 \cdot (c_1 - \hat{c}_1) + \alpha_3(t) \cdot x_3 \cdot (c - \hat{c}) - k \cdot e_3. \end{cases} . \tag{18}$$

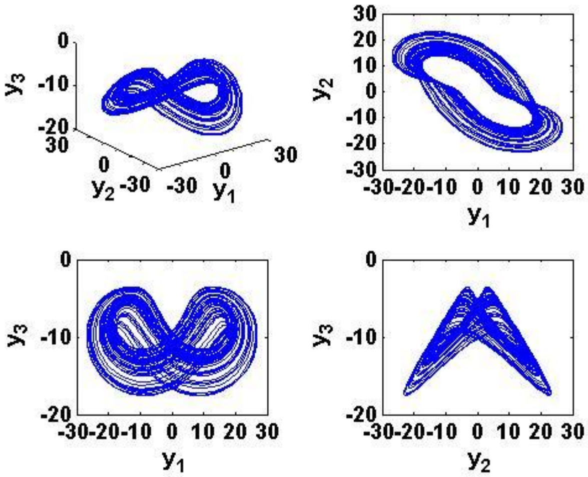

Figure 4: Chaotic attractor of system (15) without control and its two-dimension projections on $y_1 - y_2$, $y_1 - y_3$ and $y_2 - y_3$ plane

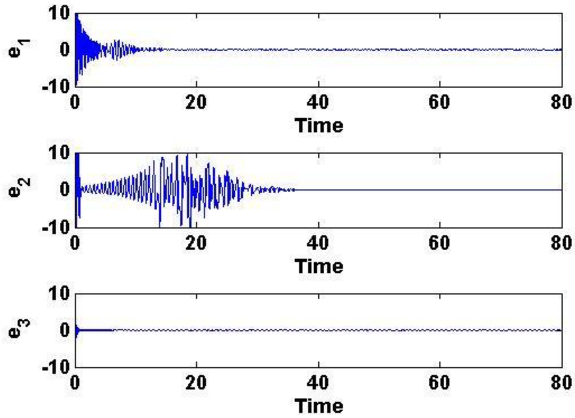

Figure 5: Time history of HFPS synchronization errors of e_1, e_2 and e_3 between systems (9) and (15)

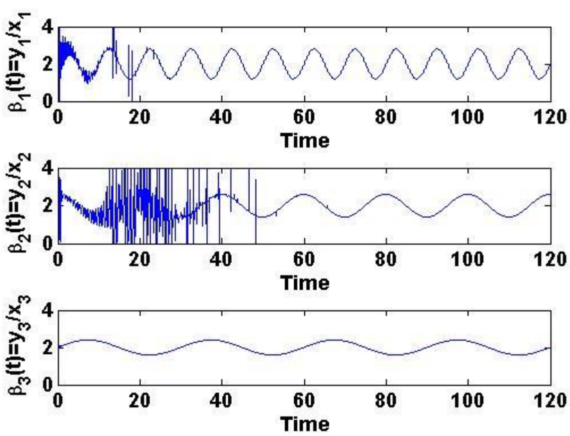

Figure 6: Time history of scaling functions reconstructed by state vectors of systems (9) and (15)

Let the "unknown" parameters a,b,c and initial conditions of the master Lorenz system (9) taking the same values as that in Example 1. The initial conditions of system (15) are arbitrarily located at $y_1(0)=-1$, $y_2(0)=-2$, $y_3(0)=-3$, and the "unknown" parameters are chosen as $a_1=5$, $b_1=-10$, $c_1=-3.8$ to ensure the chaotic behavior. The chaotic attractor of (15) and its two-dimension projections on y_1-y_2 , y_1-y_3 and y_2-y_3 plane are shown in Figure 4. Taking $\hat{a}(0)=0$, $\hat{b}(0)=0$, $\hat{c}(0)=0$, $\hat{a}_1(0)=0$, $\hat{b}_1(0)=0$, $\hat{c}_1(0)=0$, $k=1$ and choosing the scaling functions as $\alpha_1(t)=2+0.8\sin(\frac{2\pi}{10}t)$, $\alpha_2(t)=2+0.6\cos(\frac{2\pi}{20}t)$, $\alpha_3(t)=2+0.4\sin(\frac{2\pi}{30}t)$, we see that the error system (18) has initial conditions $e_1(0)=-3$, $e_2(0)=-4.6$, $e_3(0)=-5$. The simulated results obtained via fourth-order Runge-Kutta method with time step of 0.01 are shown in Figures 5 and 6, respectively. We can see clearly from Figure 5 that the synchronization error converges to zero as time evolving. Figure 6 presents the fact that $y_i(t)/x_i(t)$ tend to the predefined scaling function $\alpha_i(t)$ ($i=1,2,3$) as time gone. Figures 5 and 6 show that HFPS between the Lorenz chaotic system (9) and the new chaotic system (15) has been achieved.

4. CONCLUSIONS

In this paper, we have investigated the HFPS problem in identical and different chaotic systems with fully unknown parameters. A simple controller and a set of update laws of unknown parameters are designed to implement HFPS. According to Lyapunov Stability Theorem and Barbalat Lemma, we have proved the asymptotical stability of error dynamical system at the origin with the designed controller and update laws. The effectiveness and the feasibility of the proposed procedure are verified via two numerical examples. Since the complete synchronization, anti-synchronization, projective synchronization are all embodied in HFPS, the synchronization scheme of HFPS presented in this paper goes deeper into the current works.

ACKNOWLEDGEMENTS

This work was supported in part by the National Science Foundation of China under grant Nos. 10872165 and 10932009.

REFERENCES

[1] Pecora, L. M., & Carroll, T. L. (1990). Synchronization in chaotic systems. *Physical Review Letters, 64*(8), 821-824.

[2] Rosenblum, M. G., Pikovsky, A. S., & Kurths, J. (1996). Phase synchronization of chaotic oscillators. *Physical Review Letters, 76*(11), 1804-1807.

[3] Chen, H. K., & Lee, C. I. (2004). Anti-control of chaos in rigid body motion. *Chaos, Solitons and Fractals, 21*(4), 957-965.

[4] Li, C. D., Liao, X. F., & Wong, K. W. (2004). Chaotic lag synchronization of coupled time-delayed systems and its applications in secure communication. *Physica D, 194*(3-4), 187-202.

[5] Li, R. H. (2008). A special full-state hybrid projective synchronization in symmetrical chaotic systems. *Applied Mathematics and Computation, 200*(1), 321-329.

[6] Hu, M. F., & Xu, Z. Y. (2008). Adaptive feedback controller for projective synchronization. *Nonlinear Analysis: Real World Applications, 9*(3), 1253-1260.

[7] Li, Z. G., & Xu, D. L. (2004). A secure communication scheme using projective chaos synchronization. *Chaos, Solitons and Fractals, 22*(2), 477-481.

[8] Chee, C. Y., & Xu, D. L. (2005). Secure digital communication using controlled projective synchronisation of chaos. *Chaos, Solitons and Fractals, 23*(3), 1063-1070.

[9] Luo, R. Z., & Wei, Z. M. (2009). Adaptive function projective synchronization of unified chaotic systems with uncertain parameters. *Chaos, Solitons and Fractals , 42*(2), 1266-1272.

[10] Du, H. Y., Zeng, Q. S., Wang, C. H., & Ling, M. X. (2010). Function projective synchronization in coupled chaotic systems. *Nonlinear Analysis: Real World Applications, 11*(2), 705-712.

[11] Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences, 20*(2), 130-141.