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Geometry of Evolving Plane Curves Problem via Lie Group Analysis

Nassar H. Abdel-All[1]

M. A. Abdel-Razek[1]

H. S. Abdel-Aziz[2]

A. A. Khalil[2,*]

Abstract: The purpose of the present work is to construct new geometrical models for motion of plane curves. We have obtained nonlinear partial differential equations and have discussed the solutions of these equations using symmetry groups methods. Also, geometric interpretation for these solutions are given through the Gaussian and mean curvatures to the soliton surfaces attached to the solution of the evolving problem.

Key Words: Motion of curve; Symmetry groups; Monge form

1. INTRODUCTION

We shall consider motion of curves in a plane. The problem is interesting since we may set two different subjects in the same theoretical basis. One is a geometrical interpretation of integrable systems. Connections between the differential geometry of curve motions and the integrable systems have been discussed. It has been shown that the nonlinear Schrödinger equation describes the dynamics of a thin, non-stretching vortex lament[1]. The analysis is extended to more general types of motion and other integrable systems[2, 3]. The other is surface dynamics, the dynamics of shapes in physical and biological systems[4, 5]. Examples include crystal growth[4, 6] propagation ame fronts[7] and the Saman-Taylor problem[8]. With the intended application in both subjects, we shall present a general formulation of evolving curves in 2-dimensions.

We will describe the motion of curves by considering a smooth curve in three-dimensional, parameterized by u. Let $r(u,t)$ denotes the position vector of a point on the curve C at time t. There is a metric on the curve is given by

$$g(u,t) = < \frac{\partial r}{\partial u}, \frac{\partial r}{\partial u} >, \tag{1}$$

where $<,>$ is the Euclidean scalar product. The arc length along the curve is given by

$$s(u,t) = \int_0^u \sqrt{g(u,t)}dt. \tag{2}$$

[1]Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt.
[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt.
E-mail address: amalaboelwafay@yahoo.com (A. A. Khalil).
[*]Corresponding author.
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Let $F : I = [0, u] \subset \Re \rightarrow SO(n)$ be a linear mapping, where $F(s) = [T, n, b]^t$, $F(0) = I$, $s \in I$ and $F(s) \in SO(1, 3)$ is an orthogonal matrix. The matrix $F(s)$ is the Frenet matrix which define the Frenet frame and depends on one parameter s. The group $SO(1, 3)$ called one parameter Lie group along the curve in the space. We may use either $\{u, t\}$ or $\{s, t\}$ as coordinates of a point on the space curve. For the curve $r(u, t)$, let $\{T, n, b\}$ denote respectively the unit tangent, normal and binormal vectors of the Frenet frame. The unit tangent vector T is defined as

$$T = \frac{\partial r}{\partial s} = g^{-\frac{1}{2}} \frac{\partial r}{\partial u}, \tag{3}$$

$$\frac{\partial F}{\partial s} = AF, \tag{4}$$

in which A is called Cartan matrix which is given by

$$A = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \tag{5}$$

where $\frac{\partial}{\partial s} := \frac{\partial}{\partial s}|_t$ and $k(s, t)$ and $\tau(s, t)$ are respectively the curvature and torsion of the curve at the points r. Dynamics of the curve is expressed through the $\dot{r}$ of the curves as the following

$$\dot{r} := \frac{\partial r}{\partial t}|_\alpha = Un + Vb + WT. \tag{6}$$

The motion is said to be local if $\{U, V, W\}$ depends only on local values of $\{k, \tau\}$ and their derivatives[6]. For further discussions, it is important to notice that u and t are independent but s and t are not independent. As a consequence, while u and t derivatives commute, s and t derivatives in general do not commute as in the following

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\partial}{\partial u} = \frac{\partial}{\partial u}\frac{\partial}{\partial t}, \\ &\frac{\partial}{\partial t}\frac{\partial}{\partial s} - \frac{\partial}{\partial s}\frac{\partial}{\partial t} = -(\frac{\partial W}{\partial s} + kU)\frac{\partial}{\partial s}. \end{aligned} \tag{7}$$

2. TWO-DIMENSIONAL MOTION

Motion in a plane is characterized by $V \equiv 0$ and $\tau \equiv 0$ and from (4) and (6) we have

$$\frac{\partial F}{\partial s} = AF, \tag{8}$$

where

$$A = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \quad F = \begin{pmatrix} T \\ n \end{pmatrix}, \tag{9}$$

$$\dot{r} = Un + WT. \tag{10}$$

We are in a position to derive time evolutions of geometrical quantities. Using (7) in (8) and (10), we obtain

$$\frac{\partial F}{\partial t} = EF, \tag{11}$$

$$\dot{g} = 2g(\frac{\partial W}{\partial s} - kU), \tag{12}$$

$$\dot{k} = 2g(\frac{\partial^2 U}{\partial s^2} + k^2 U + \frac{\partial k}{\partial s} W), \tag{13}$$

where E is called the evolution matrix and is given by

$$E = \begin{pmatrix} 0 & \frac{\partial U}{\partial s} + kU \\ -(\frac{\partial U}{\partial s} + kU) & 0 \end{pmatrix}. \tag{14}$$

Here, F may be a column vector or a square matrix,A, E belong to the Lie algebra g of certain linear Lie group[9, 10]. From (2) and (12), time development of the arc length is given by

$$\begin{aligned} \dot{s} &= \int_0^{\alpha} g^{\frac{1}{2}} (\frac{\partial W}{\partial s} - kU) d\alpha \\ &= W(s,t) - \int_o^s kU ds, \end{aligned} \tag{15}$$

provided $W(0,t) = 0$. Because $k = k(s(t),t)$, we have

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + \dot{s}\frac{\partial}{\partial s}, \\ \dot{k} &= \frac{\partial k}{\partial t} + (W(s,t) - \int_0^s kU ds)\frac{\partial k}{\partial s}. \end{aligned} \tag{16}$$

From (13), it follows that[11]

$$\frac{\partial k}{\partial t} = \frac{\partial^2 U}{\partial s^2} + k^2 U + \frac{\partial k}{\partial s} \int_0^s kU ds. \tag{17}$$

Notice that $k(s;t)$, and hence the $2D$ motion of the curve, follows from specifying $U(s;t)$ and then integration (17). The tangential component $W(s;t)$ determines how points parameterized by u move along the curve, but it does not affect the shape of the curve.

3. SYMMETRY GROUP

Now, we present the most general Lie group of point transformations, for obtaining solutions to the partial differential equations governing the evolving model for the considered problem[12, 13, 17].

Definition 3.1 *We consider a scalar $m-th$ order PDE represented by*

$$\Delta(s, k^{(m)}) = 0. \tag{18}$$

Now, we want to present the most general Lie group of point transformations, which apply on obtaining equations. Where $s = (s_i)$, $i = 1 \ldots p$ is a vector for which the components s_i are independent variables and $k = (k_j)\,(j = 1 \ldots q)$ is a vector coset of k_j dependent variables, and $k^{(m)} = \frac{\partial^m k}{\partial s^m}$. The infinitesimal generator of the one-parameter Lie group of transformations for equation (18) is

$$X = \sum_{i=1}^{p} \zeta^i(s,k)\frac{\partial}{\partial s^i} + \sum_{\chi=1}^{q} \phi^{\chi}(s, k^{(m)})\frac{\partial}{\partial k^{\chi}}, \tag{19}$$

where $\zeta^i(s,k), \phi^{\chi}(s,k)$ are the infinitesimals, and the m-th prolongation of the infinitesimal generator (19) is (see [5–8])

$$pr^{(m)}X = X + \sum_{\chi=1}^{q} \sum_{j} \phi_j^{\chi}(s, k^{(m)})\frac{\partial}{\partial k_j^{\chi}}, \tag{20}$$

where

$$\phi_j^\chi(s,k^{(m)}) = D_j(\phi^\chi - \sum_{i=1}^{p}\zeta^i k_i^\chi) + \sum_{i=1}^{p}\zeta^i k_{j,i}^\chi. \tag{21}$$

and D is the total derivative operator defined by

$$D_j = \frac{\partial}{\partial s^j} + k_j^\chi\frac{\partial}{\partial k^\chi} + k_{ij}^\chi + \ldots, \quad k_j = \frac{\partial k}{\partial s^j}, \quad j = 1,\ldots,p. \tag{22}$$

A vector field X is an infinitesimal symmetry of the system of differential equations (18) if and only if it satis.es the infinitesimal invariance condition

$$pr^{(m)}X(\Delta)|_{\Delta=0} = 0. \tag{23}$$

4. SOLITON GEOMETRY

In this paper, we construct the soliton surfaces associated with the single soliton solutions (similarity solution) of the equation (17). For this purpose, if $k = k(s,t)$ is a similarity solution of Eq. (17) we have a solution surface σ given from the Monge patch $f = (s,t,k(s,t))$. The tangent vectors f_s , f_t for the soliton surface σ are given by

$$\begin{aligned} f_s &= (1,0,k_s),\\ f_t &= (0,1,k_t). \end{aligned} \tag{24}$$

The normal unit vector field on the tangents plane $T_p\sigma$ is given by

$$N = \frac{f_s \wedge f_t}{|f_s \wedge f_t|}. \tag{25}$$

The first and second fundamental forms on σ are defined respectively by

$$\begin{aligned} I &=< df, df >= g_{11}ds^2 + 2g_{12}dsdt + g_{22}dt^2,\\ II &=< -df, dN >= L_{11}ds^2 + 2L_{12}dsdt + L_{22}dt^2, \end{aligned} \tag{26}$$

where the tensor g_{ij} and L_{ij} are given by

$$\begin{aligned} &g_{11} =< f_s, f_s >, \quad g_{12} =< f_s, f_t >, \quad g_{22} =< f_t, f_t >,\\ &L_{11} =< f_{ss}, N >, \quad L_{12} =< f_{st}, N >, \quad L_{22} =< f_{tt}, N > . \end{aligned} \tag{27}$$

The Gauss equations associated with σ are

$$\begin{aligned} f_{ss} &= \Gamma_{11}^1 f_s + \Gamma_{11}^2 f_t + L_{11}N,\\ f_{st} &= \Gamma_{12}^1 f_s + \Gamma_{12}^2 f_t + L_{12}N,\\ f_{tt} &= \Gamma_{22}^1 f_s + \Gamma_{22}^2 f_t + L_{22}N, \end{aligned} \tag{28}$$

while the Weingarten equations comprise

$$\begin{aligned} N_s &= \frac{g_{12}L_{12} - g_{22}L_{11}}{g}f_s + \frac{g_{12}L_{11} - g_{11}L_{12}}{g}f_t,\\ N_t &= \frac{g_{12}L_{22} - g_{22}L_{12}}{g}f_s + \frac{g_{12}L_{12} - g_{11}L_{22}}{g}f_t, \end{aligned} \tag{29}$$

where

$$g = |f_s \wedge f_t|^2 = g_{11}g_{22} - g_{12}^2, \tag{30}$$

The function $\Gamma^i_{j\lambda}$ in (28) are the usual Christoffel symbols given by the relations

$$\Gamma^i_{j\lambda} = \frac{1}{2}g^{il}(g_{il,\lambda} + g_{\lambda l,j} - g_{j\lambda l,l}). \tag{31}$$

The compatibility conditions $(f_{ss})_t = (f_{st})_s$and $(f_{st})_t = (f_{tt})_s$applied to the linear Gauss system (28) produce the nonlinear Mainardi-Codazzi system

$$\begin{aligned}
&(\frac{L_{11}}{\sqrt{g}})_t - (\frac{L_{12}}{\sqrt{g}})_s + \frac{L_{11}}{\sqrt{g}}\Gamma^2_{12} - 2\frac{L_{12}}{\sqrt{g}}\Gamma^2_{12} + \frac{L_{22}}{\sqrt{g}}\Gamma^2_{11} = 0,\\
&(\frac{L_{22}}{\sqrt{g}})_s - (\frac{L_{12}}{\sqrt{g}})_t - 2\frac{L_{12}}{\sqrt{g}}\Gamma^1_{22} + \frac{L_{11}}{\sqrt{g}}\Gamma^1_{12} + \frac{L_{22}}{\sqrt{g}}\Gamma^1_{11} = 0,
\end{aligned} \tag{32}$$

or, equivalently,

$$\begin{aligned}
L_{11t} - L_{12s} &= L_{11}\Gamma^1_{12} + L_{12}(\Gamma^2_{12} - \Gamma^1_{11}) - L_{22}\Gamma^2_{11},\\
L_{12t} - L_{22s} &= L_{11}\Gamma^1_{22} + L_{12}(\Gamma^2_{22} - \Gamma^1_{12}) - L_{22}\Gamma^2_{12}.
\end{aligned} \tag{33}$$

The Gaussian and mean curvatures at the regular points on the soliton surface are given by respectively

$$K = k_1k_2 = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \quad g \neq 0, \tag{34}$$

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\frac{L_{11}g_{22} - 2L_{12}^2g_{12} + L_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}, \tag{35}$$

where $g = det(g_{ij})$, $L = det(L_{ij})$ and k_1, k_2 are the principal curvatures. The surface for which $K = 0$ is called parabolic surface, but if $k_1 = 0$ and $k_2 =$ constant or $k_1 =$ constant and $k_2 = 0$, we have surface semi round semi .at (cylindrical like surface).The integrability conditions for the systems (11) and (8) are equivalent to the Mainardi-Codazzi system of PDE (32). This give a geometric interpretation for the surface defined by the variables s, t to be a soliton surface[9, 10, 15, 16].

5. APPLICATION

Here, we consider special types of evolving curves by choosing the normal velocity U as in the following categories.

5.1 Case I: $U = -k_s$

In equation (17), we get the known modified Korteweg-de Vries (mKdV) equation[18]

$$k_t + \frac{3}{2}k^2k_s + k_{sss} = 0, \tag{36}$$

the solution space M has coordinates (s, t, k) and its solutions $k = f(s, t)$ define a 2-dimensional surface of $M \subset \mathfrak{R}^3$. Using (18) , (19) the prolongation of a vector field

$$X = \xi\frac{\partial}{\partial s} + \eta\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial k}, \tag{37}$$

on M has the form

$$Pr^{(m)}X = \xi\frac{\partial}{\partial s} + \eta\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial k} + \sum_J \phi^J \frac{\partial}{\partial k_J}, \tag{38}$$

where ζ, η and ϕ are functions of the variables s, t and k, whose coefficients in view of (21) are given by the explicit formulas

$$\begin{aligned} \phi^s &= D_s(\phi - \zeta k_s - \eta k_t) + \zeta k_{ss} + \eta k_{st}, \\ \phi^t &= D_t(\phi - \zeta k_s - \eta k_t) + \zeta k_{st} + \eta k_{tt}, \\ \phi^{ss} &= D_{ss}(\phi - \zeta k_s - \eta k_t) + \zeta k_{sss} + \eta k_{sst}. \end{aligned} \tag{39}$$

From (36) and (38) it is easy to see that the vector field X is an infinitesimal symmetry of the mKdV equation (36) if and only if

$$\begin{aligned} \phi^s &= D_s(\phi - \zeta k_s - \eta k_t) + Pr^{(m)}X(k_t + \frac{3}{2}k^2 k_s + k_{sss}) \\ &= \phi^t + 3\phi k k_s + \frac{3}{2}k^2\phi^s + \phi^{sss} = 0, \end{aligned} \tag{40}$$

whenever $k_t + \frac{3}{2}k^2 k_s + k_{sss} = 0$.

From the prolongation formulas (39), and equating the coefficients of the independent derivative monomials to zero, leads to the infinitesimal determining equations which together with their differential consequences reduce to the following system of PDEs

$$\begin{aligned} &\phi_{sss} + \phi_t + \frac{3}{2}k^2\phi_s = 0, \\ &\eta_t - 3\zeta_s = 0, \\ &\zeta_k = \eta_k = \eta_s = \phi_{kk} = 0, \\ &3k\phi + 3\phi_{ss}k - \zeta_{sss} + 3k^2\zeta_s - \zeta_t = 0, \\ &\phi_{sk} - \zeta_{ss} = 0. \end{aligned} \tag{41}$$

Then the solutions of system (41) are of the following form

$$\zeta = \frac{1}{3}c_1 s + c_2, \quad \eta = c_1 t + c_3, \quad \phi = -\frac{1}{3}c_1 k. \tag{42}$$

From (37) if follows the solutions (42) define the three-dimensional mKdV symmetry algebra with the basis given by

$$X_1 = \frac{1}{3}s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - \frac{1}{3}k\frac{\partial}{\partial k}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = \frac{\partial}{\partial t}. \tag{43}$$

The combination of space and time translations $(cX_2 + X_3)$ lead to a reduction of (36) to an ordinary differential equation (ODE) through the transformation $y = s - ct$ and $w(y) = k$ where c is the speed of the traveling wave. This reduction is given by

$$-c\omega' + \frac{3}{2}\omega^2\omega' + \omega''' = 0, \quad \left(' \equiv \frac{d}{dy}\right). \tag{44}$$

By integrating (44) twice to obtain the following ODE (we take the integration constant to be zero if boundary conditions ω, ω' and ω'' tend to zero at $y \to \pm\infty$)

$$-c\omega^2 + \frac{1}{4}\omega^4\omega'^2 = 0, \tag{45}$$

then

$$\omega' = \pm\sqrt{c\omega^2 - \frac{1}{4}\omega^4}. \tag{46}$$

Thus, we have a solution as in the form

$$k = \pm 2\sqrt{c}sech(\sqrt{c}(s - ct)). \tag{47}$$

This solution is a similar solution to the PDE (36), This solution can be written in the Monge form $k = k(s, t)$ which define a regular surface as shown in Figure (1). This surface is a soliton surface $(1 + 1)$. From (34) and (35), one can see that the Gaussian and mean curvatures of the soliton surface (47) are given by respectively

$$\begin{aligned} K &= 0, \\ H &= \frac{c^{\frac{3}{2}}(1 + c^2)(-3 + cosh(\sqrt{c}(s - ct)))sech(\sqrt{c}(s - ct))^2}{2(1 + 4c^2(1 + c^2)sech(\sqrt{c}(s - ct))^2 tanh(\sqrt{c}(s - ct))^2)^{\frac{3}{2}}}. \end{aligned} \tag{48}$$

In the following Figure 1, a soliton surface is portrayed where $c = 1$, $-5 \le s \le 5$, $0.1 \le t \le 2$.

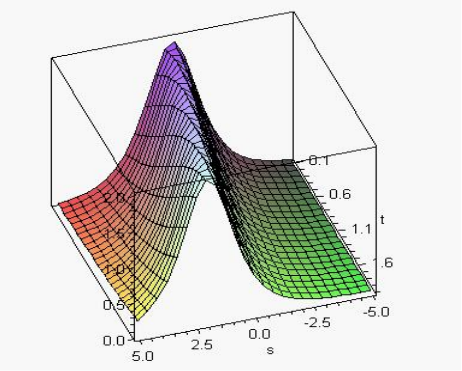

Figure 1: Soliton surface of (36)

The symmetry generator $X_1 = \frac{1}{3}s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + \frac{1}{3}\frac{\partial}{\partial k}$ leads to the invariants $y = \frac{t}{s^3}$ and $\omega = sk$. After some detailed and tedious calculations, (36) becomes an ODE in the form

$$-\omega' + \frac{9}{2}y\omega^2\omega' + \frac{3}{2}\omega^3 + 27\omega'''y^3 + 135\omega''y^2 + 6\omega + 114y\omega' = 0, \tag{49}$$

which can be solved numerically as show in Figure 2, with initial conditions ($\omega(1) = 1, \omega'(1) = 2$ and $\omega''(1) = 3$).

Remarks: $K = 0$ and $H \neq 0$ characterize a parabolic surface of cylindrical type. In the following cases using the same technique used in case one.

5.2 Case II: $U = -kk_s$

In this case, equation (17) becomes

$$k_t + 3k_sk_{ss} + kk_{sss} + \frac{4}{3}k_s = 0. \tag{50}$$

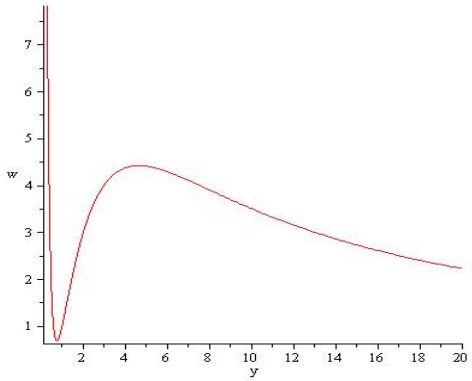

Figure 2: Numerical solution of (49)

Lie point symmetry for this equation is given by

$$X_1 = \frac{1}{4}s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - \frac{1}{4}k\frac{\partial}{\partial k}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = \frac{\partial}{\partial t}. \tag{51}$$

The traveling wave (soliton type) solution is obtained by $X = cX_2 + X_3 = c\frac{\partial}{\partial s} + \frac{\partial}{\partial s}$, by which (50) becomes the ODE (with the new independent variable $y = s - ct$, c being the speed of the wave)[14]

$$-c\omega' + 3\omega'\omega'' + \omega\omega''' + \frac{4}{3}\omega^3\omega' = 0, \tag{52}$$

which after some manipulations can be shown to have the conserved form

$$D(-c\omega^2 + \omega'^2 + \omega\omega'' + \frac{1}{3}\omega^4) = 0. \tag{53}$$

Thus, we have the second-order ODE (we take the integration constant to be zero)

$$-c\omega^2 + \omega'^2 + \omega\omega'' + \frac{1}{3}\omega^4 = 0. \tag{54}$$

Eq. (54) has the Lie point symmetry $\frac{\partial}{\partial y}$ which leads to invariants $\theta = \omega$ and $\delta = \omega'$ so that $\frac{d\delta}{d\theta} = \frac{\omega''}{\omega'}$ by which (54) can be written as

$$\frac{d\delta}{d\theta} = -\frac{1}{\delta}(\frac{\delta^2}{\theta} + \frac{\theta^3}{3} - c), \tag{55}$$

which has solution

$$\delta(\theta) = \pm\sqrt{-\frac{\theta^4}{9} + \frac{2c\theta}{3} + \frac{C_1}{\theta^2}}. \tag{56}$$

Substituting θ and δ, we get

$$\frac{d\omega}{dy} = \pm\sqrt{-\frac{\omega^4}{9} + \frac{2c\omega}{3} + \frac{C_1}{\omega^2}}. \tag{57}$$

Thus, a traveling wave solution of (57) is given by

$$s - ct = \int^k \frac{d\omega}{\pm\sqrt{-\frac{\omega^4}{9} + \frac{2c\omega}{3} + \frac{C_1}{\omega^2}}}. \tag{58}$$

The integral can be found in terms of special functions. For e.g. if the constant is set to zero, the integral is Jacobi F. If we take $c = 1$ the solution becomes

$$\omega = \pm \frac{216^{\frac{1}{3}} SN(\frac{1}{12}\sqrt{216^{\frac{1}{3}}\sqrt{3} - 66^{\frac{2}{3}}}(s-t), \frac{1}{2}\sqrt{2+2I\sqrt{3}})^2}{-\sqrt{3} - 3I + 216^{\frac{1}{3}} SN(\frac{1}{12}\sqrt{216^{\frac{1}{3}}\sqrt{3} - 66^{\frac{2}{3}}}(s-t), \frac{1}{2}\sqrt{2+2I\sqrt{3}})^2}, \tag{59}$$

Thus, we have Figure 3: Then Gaussian and mean curvatures are

$$\begin{aligned} K &= 0, \\ H &= Jacobi(F). \end{aligned} \tag{60}$$

In Figure 3, a soliton surface is portrayed where $c = 1$, $-5 \le s \le 5$, $0.1 \le t \le 2$.

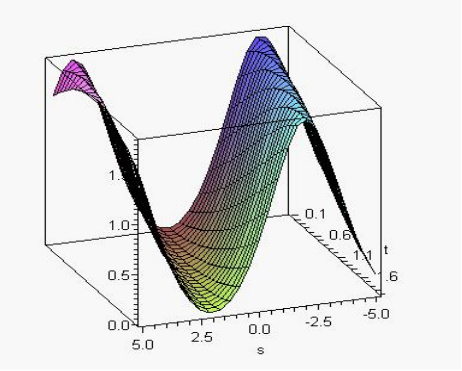

Figure 3: Soliton surface of (50)

The symmetry generator $X_1 = \frac{1}{4}s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - \frac{1}{4}k\frac{\partial}{\partial k}$ leads to the invariants $y = \frac{t}{s^4}$ and $\omega = sk$. After some detailed and tedious calculations, (50) becomes an ODE in the form

$$\begin{aligned} -\omega' + 312y\omega\omega' + 12\omega^2 + \frac{4}{3}\omega^4 + 64y^3\omega\omega''' + 336y^2\omega\omega'' \\ + \frac{16}{3}y^3\omega^3\omega' + 192y^3\omega'\omega'' + 336y^2\omega'^2 = 0. \end{aligned} \tag{61}$$

The numerical solution of Eq. (61) is shown in Figure 4 (with initial conditions $\omega(1) = 1, \omega'(1) = 2$ and $\omega''(1) = 3$).

5.3 Case III: $U = -k^2 k_s$

In this case, equation (17) becomes

$$k_t + 23k_s^3 + 6kk_6k_{ss} + k^2k_{sss} + \frac{5}{4}k_s = 0. \tag{62}$$

Lie point symmetry for this equation is given by

$$X_1 = \frac{1}{5}s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - \frac{1}{5}k\frac{\partial}{\partial k}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = \frac{\partial}{\partial t}. \tag{63}$$

It admits the symmetries X_2 and X_2, traveling wave solutions are obtainable by the substitution $y = s - ct$ (c is the wave speed), so that (62) becomes

$$\omega^2\omega''' + 6\omega\omega'\omega'' + (2\omega'^2 - c + \frac{5}{4}\omega^4)\omega' = 0, \tag{64}$$

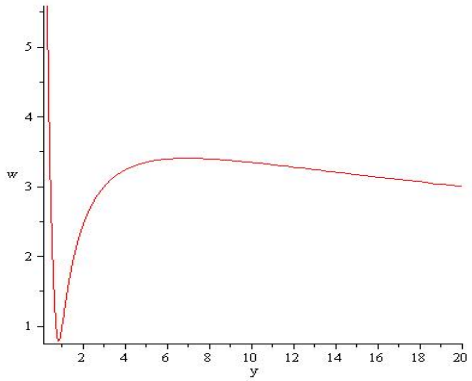

Figure 4: Numerical solution of (61)

by integrating which, we get

$$-c\omega + 2\omega\omega'^2 + \omega^2\omega'' + \frac{1}{4}\omega^5 = 0, \tag{65}$$

where we take the integration constant to be zero. Now solving the equation (65) with the lie symmetry $\frac{\partial}{\partial y}$ having invariants $\theta = \omega$ and $\delta = \omega'$. That is, by (54) $\frac{d\delta}{d\theta} = \frac{\omega''}{\omega'}$ can be written as

$$\frac{d\delta}{d\theta} = -\frac{1}{\delta}\left(\frac{2\delta^2}{\theta} + \frac{\theta^3}{4} - \frac{c}{\theta}\right), \tag{66}$$

which leads to

$$\delta(\theta) = \pm\sqrt{-\frac{\theta^4}{16} + \frac{c}{2} + \frac{C_1}{\theta^4}}. \tag{67}$$

Substituting θ and δ gives

$$s - ct = \int^{k} \frac{d\omega}{\pm\sqrt{-\frac{\theta^4}{16} + \frac{c}{2} + \frac{C_1}{\theta^4}}}. \tag{68}$$

then, we get solution

$$k = \pm\frac{2SN(\frac{1}{4}\sqrt{2}\sqrt{\sqrt{2}\sqrt{c}}(s-ct), I)}{\sqrt{\frac{\sqrt{2}Root\,of(Z^2-c, index=1)}{c}}}. \tag{69}$$

Thus, we have Figure 5. At $c = 1$, the Gaussian and mean curvatures are

$$\begin{aligned} K &= 0 \\ H &= \frac{-5I(CN(\frac{1}{2}(s-2t), I)^2 - IDN(\frac{1}{2}(s-2t), I)^2)SN(\frac{1}{2}(s-2t), I)}{2^{\frac{3}{4}}(10\sqrt{2}CN(\frac{1}{2}(s-2t), I)^2(DN\frac{1}{2}(s-2t), I)^2 + 2\sqrt{2})^{\frac{3}{2}}}. \end{aligned} \tag{70}$$

In the following Figure 5, a soliton surface is portrayed where $c = 1$, $-5 \le s \le 5$, $0.1 \le t \le 2$.

Also, the scaling symmetry $\frac{1}{5}s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - \frac{1}{5}k\frac{\partial}{\partial k}$ with invariants $y = \frac{t}{s^5}$,$\omega = sk$, leads to the reduced equation

$$\begin{aligned} &\omega' - 20\omega^3 - 660y\omega^2\omega' - 675y^2\omega^2\omega'' - 1300y^2\omega\omega'^2 - 750y^3\omega\omega'\omega'' \\ &- \frac{5}{4}\omega^5 - \frac{25}{4}y\omega^4\omega' - 125y^3\omega^2\omega'' - 250y^3\omega'^3 = 0. \end{aligned} \tag{71}$$

We have solved the equation (71) with with initial conditions $\omega(1) = 1, \omega'(1) = 2$ and $\omega''(1) = 3$). Numerical simulation is demonstrated in Figure 6.

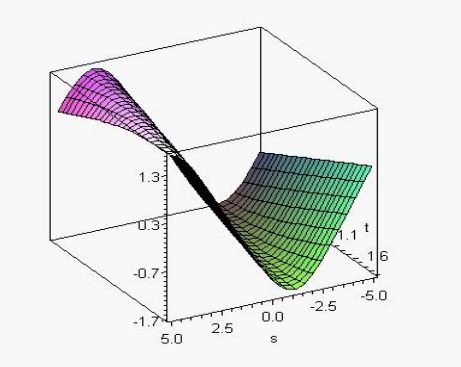

Figure 5: Soliton surface of (62)

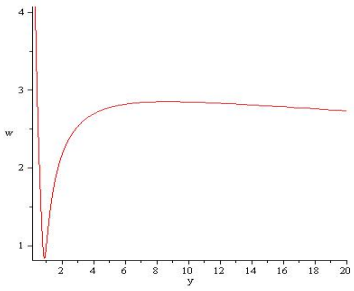

Figure 6: Numerical solution of (71)

6. CONCLUSIONS

We have constructed two new geometrical models for motion of plane curves other than the mKdV equation, which is known before and we have created solutions using symmetry methods and conclude that these equations represent cylindrical surfaces as the Gaussian curvature of these surfaces equal to zero.

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