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Time-optimal Control of Petrowsky Systems with Infinitely Many Variables and Control-state Constraints

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Abstract: In this paper, the time-optimal control problem for $n \times n$ differential systems of Petrowsky type with infinitely many variables and control-state constraints are considered. For some different cases of the observation, the necessary optimality conditions of optimal control are obtained by using the generalized Dubovitskii-Milyutin Theorem (Theorem 1.8.1, [1]).

Key Words: Time-optimal control problem; Petrowsky systems; Dubovitskii-Milyutin method; Canonical approximations; Optimality conditions

1. INTRODUCTION

The most widely studies of the problems in the mathematical theory of control are the "*time optimal*" control problems. The simple version, is the following optimization problem:

$$\text{find } (u, y) \in C([0,\infty); U \times Y) :$$

$$\left.\begin{array}{l} 0 < t \to \min \\ u(t) \in U_{ad} \\ y(t) \in Y_{ad}. \end{array}\right\}$$

where U_{ad}, Y_{ad} are spaces of admissible control and states respectively.

In order to explain the results we have in mind, it is convenient to consider the abstract form of the Dubovitskii-Milyutin theorem.

Let X be Banach space, $Q_k \subset X$, $int\, Q_k \neq \emptyset, k = 1, \ldots, p$ represent inequality constraints, $Q_k \subset X$, $k = p+1, \ldots, m$ represent equality constraints and $I : X \to R$ is given functional.

Theorem 1.1 *(Theorem 1.8.1, [1]) Assume that*

(i) $I : X \to R$ *is convex and continuous,*

(ii) the cones Q_k, $k = 1, \ldots, m$ *are convex,*

(iii) $\hat{x} \in \left(\bigcap_{k=1}^{p} int Q_k\right) \bigcap \left(\bigcap_{k=p+1}^{m} Q_k\right)$,

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(iv) the cones $[RTC(Q_k, x^0)]^*$, $k = p+1, \dots, m$ are either of the same sense or of the opposite sense, then x^0 is a solution of the problem

$$\min\left\{ I(x), x \in \bigcap_{k=1}^{m} Q_k \right\},$$

if and only if the following equation (Euler-Lagrange equation) must hold:

$$f_0 + \sum_{k=1}^{m} f_k = 0.$$

where $f_0 \in [RFC(I, x^0)]^*$, $f_k \in [RAC(Q_k, x^0)]^*, k = 1, \dots p$ and $f_k \in [RTC(Q_k, x^0)]^*, k = p+1, \dots m$ with not all functionals equal to zero simultaneously.

The above generalization of the Dubovitskii-Milyutin theorem is based on the definitions of the regular cones RTC, RFC, RAC and cones of the same sense and of the opposite sense which are introduced there (see Ref. [1]). But for the purpose of our problems we are going to use the following sufficient condition for two cones to be of the same sense.

Theorem 1.2 *(Theorem 3.3, [2]) Let C_1 be a cone of the form $C_1 = \{(x_1, y_1) \in X \times Y : x_1 = My_1\}$, $C_2 = X \times \hat{C}_2$, where $\hat{C}_2$ is a cone in Y (X, Y-normed spaces). If the operator M is linear and continuous, then*

$$C_1^* = \{(x_1^*, y_1^*) \in (X \times Y)^* : y_1^* = -M^* x_1^*\},$$
$$C_2^* = \left\{(0, y_2^*) \in (X \times Y)^* : y_2^* \in \hat{C}_2\right\},$$

and the cones C_1^*, C_2^* are of the same sense.

Various optimization problems associated with the optimal control of distributed parameter systems have been studied in [1, 3–6].

The problem of time-optimal control associated with the hyperbolic systems have been discussed in some papers (see, e.g., Ref. [7]) in which the existence of a time-optimal control of system governed by a hyperbolic equation systems involving Laplace operator has been discussed. In [3], the maximum principles for the time optimal control for wave equation is given by reduction of the second-order wave equation to a first-order system. All these results concerned the time optimal control problems of systems governed by only one hyperbolic equation and only control constraints.

In Refs. [8, 9], the above results for systems governed by one hyperbolic equation are extended to the case of $n \times n$ co-operative hyperbolic systems and Petrowsky systems respectively with only control constraints.

In the present paper, the above results are extended to the case of $n \times n$ differential systems of Petrowsky type with infinitely many variables and control-state constraints, the necessary optimality conditions of optimal control for $n \times n$ Petrowsky systems with control-state constraints are obtained. First, time optimal control problem is replaced by an equivalent one with fixed time, then, for a different cases of the observation (position, velocity and position-velocity) observation, the necessary optimality conditions are derived by using some generalization of the Dubovitskii-Milyutin Method for the case of m equality and inequality constraints[1].

2. SOBOLEV SPACES OF INFINITE NUMBER OF VARIABLES

Let $(P_k)_{k=1}^{\infty}$ be a fixed sequence of positive twice continuous differentiable probability weights, with respect to this sequence on $\mathfrak{R}^\infty = \mathfrak{R}^1 \times \mathfrak{R}^1 \times \dots$, of points $x = (x_k)_{k=1}^{\infty}$, $(x_k \in \mathfrak{R}^1)$ the (weighted) product measure

$d\rho(x)$ can be introduced in the following way

$$\begin{aligned} d\rho(x) &= (P_1(x_1)dx_1) \otimes (P_2(x_2)dx_2) \otimes \ldots \\ &= (d\rho_1(x_1)) \otimes (d\rho_2(x_2)) \otimes \ldots . \end{aligned}$$

The examples of construction of the measure $d\rho(x)$ are given in Ref. [10].

Below we consider $G \subset \Re^\infty$ with G be a bounded open domain with infinitely differentiable boundary Γ.

We denoted by $L^2(G, d\rho)$ the space of all square integrable functions on G which is a Hilbert space[11] for the scalar product

$$(\phi, \psi)_{L^2(G,d\rho)} = \int_G \phi(x)\, \psi(x)\, d\rho(x).$$

The Sobolev space of order ℓ (with infinite number of variables), $W^\ell(G, d\rho)$ is defined by

$$W^\ell(G, d\rho) := \{y |\, D^\alpha y \in L^2(G, d\rho), \quad \forall \alpha, |\, \alpha \,| \le \ell\},$$

such that

$$||y||_{W^\ell(G,d\rho)} := \left(\sum_{|\alpha| \le \ell} ||D^\alpha y||^2_{L^2(G,d\rho)} \right)^{\frac{1}{2}} < \infty,$$

where

$$D^\alpha = D_1^{\alpha_1}\; D_2^{\alpha_2} \ldots \quad, \alpha = (\alpha_i)_{i=1}^\infty \quad, |\, \alpha \,| = \sum_{i=1}^\infty \alpha_i,$$

and

$$D_k y(x) = \frac{1}{\sqrt{P_k(x_k)}} \frac{\partial}{\partial x_k} (\sqrt{P_k(x_k)} y(x)).$$

Next one can define the following space

$$W_0^\ell(G, d\rho) := \{y |\, y \in W^\ell(G, d\rho), \quad D^\alpha y = 0 \text{ on } \Gamma \quad, |\alpha| \le \ell - 1,\ \ell > 1\}.$$

For the spaces $W_0^\ell(G, d\rho)$ ($\ell = 1, 2, \ldots,$) one can construct their dual $W^{-\ell}(G, d\rho)$.
The duality between spaces $W_0^\ell(G, d\rho)$ and $W^{-\ell}(G, d\rho)$ is induced by the scalar product of the space $W^0(G, d\rho) = L^2(G, d\rho)$. For them we have the following chain

$$W_0^\ell(G, d\rho) \subseteq L^2(G, d\rho) \subseteq W^{-\ell}(G, d\rho).$$

3. PETROWSKY SYSTEMS WITH INFINITE VARIABLES

For $y = (y_i)_{i=1}^n$, $\phi = (\phi_i)_{i=1}^n$ and $t \in [0, T]$, let us define a family of continues bilinear forms

$$\pi(t; ., .) : (W_0^\ell(G, d\rho))^n \times (W_0^\ell(G, d\rho))^n \to \Re,$$

by

$$\pi(t; y, \phi) = \sum_{i=1}^n \int_G \left[\sum_{\alpha \le \ell} \sum_{k=1}^\infty \left(D_k^\alpha y_i\right)\left(D_k^\alpha \phi_i\right) + a_i(x, t) y_i \phi_i \right] d\rho(x) - \sum_{i,j=1}^n \int_G a_{ij}(x, t) y_j \phi_i d\rho(x), \tag{1}$$

where

$$\left.\begin{array}{l} a(x,t) \text{ and } a_{ij}(x,t) \text{ are positive functions in } L^{\infty}(S), \\ a_{ij} = a_{ji} \text{ (symmetric condition)} \\ a_{ij} = 0 \quad \text{when } i = j \\ a_i \geq \sum_{j=1}^{n} a_{ij}, \quad \forall i = 1,2,\dots n \end{array}\right\} \tag{2}$$

The bilinear form (1) can be but in the operator form:

$$\begin{aligned} \pi(t; y, \phi) &= \sum_{i=1}^{n} \int_G \left[\sum_{\alpha \leq \ell} \sum_{k=1}^{\infty} (-1)^{\alpha} \left(D_k^{2\alpha} y_i \right) + a_i(x,t) y_i \right] \phi_i d\rho(x) - \sum_{i,j=1}^{n} \int_G a_{ij}(x,t) y_j \phi_i d\rho(x) \\ &= \langle A(t)y, \phi \rangle_{(L^2(G,d\rho))^n} \end{aligned}$$

where $A(t)$ is $n \times n$ matrix operator which maps $\left(W^{\ell}(G, d\rho)\right)^n$ onto $\left(W^{-\ell}(G, d\rho)\right)^n$ and takes the form

$$A(t) = \begin{pmatrix} B_1(t) & a_{12} & \dots & a_{1n} \\ a_{21} & B_2(t) & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & B_n(t) \end{pmatrix}_{n \times n},$$

where $B_i(t) = \left[\sum_{\alpha \leq \ell} \sum_{k=1}^{\infty} (-1)^{\alpha} \left(D_k^{2\alpha} \right) + a_i(x,t) \right]$ are bounded self-adjoint elliptic partial differential operators of order 2ℓ with infinite number of variables.

It is easy to see that i-th component $(A(t)y)_i$ takes the form

$$(A(t)y)_i = \left[\sum_{\alpha \leq \ell} \sum_{k=1}^{\infty} (-1)^{\alpha} \left(D_k^{2\alpha} \right) + a_i(x,t) \right] y_i - \sum_{j=1}^{n} a_{ij}(x,t) y_j.$$

Notation $\|y\|_m = \left(\int_G m(x,t) y^2 d\rho(x) \right)^{\frac{1}{2}}$.

Lemma 3.1

$$\sum_{i,j=1}^{n} \int_G a_{ij}(x,t) y_i y_j d\rho(x) \leq \frac{1}{2} \sum_{i=1}^{n} \|y_i\|^2_{\sum_{j=1}^{n} a_{ij}}.$$

Proof. Using Cauchy Schwarz inequality, we have

$$\begin{aligned} \sum_{i=j=1}^{n} \int_G a_{ij}(x,t) y_i y_j d\rho(x) &\leq \sum_{i,j=1}^{n} \|y_i\|_{a_{ij}} \|y_j\|_{a_{ij}} \\ &\leq \frac{1}{2} \sum_{i,j=1}^{n} \left[\|y_i\|^2_{a_{ij}} + \|y_j\|^2_{a_{ij}} \right] = \frac{1}{2} \sum_{i=1}^{n} \|y_i\|^2_{\sum_{j=1}^{n} a_{ij}}. \end{aligned}$$

□

Theorem 3.2 *Assume that* (2) *holds, then, there exist a unique weak solution*

$$y = (y_i)_{i=1}^n \in \left\{ y : y \in L^2(0,T; (W_0^{\ell}(G, d\rho))^n), \quad \frac{\partial y}{\partial t} \in (L^2(S))^n \right\},$$

satisfying the following $n \times n$ Petrowsky system involving operators with infinite number of variables:

$$\left.\begin{array}{lll} \dfrac{\partial^2 y_i}{\partial t^2} + (A(t)y)_i = u_i, & u_i \in L^2(S) & in\ S =]0,T[\times G, \\ y_i(x,0) = y_{i,0}(x), & y_{i,0}(x) \in W_0^\ell(G,d\rho) & in\ G, \\ y_i'(x,0) = y_{i,1}(x), & y_{i,1}(x) \in L^2(G,d\rho) & in\ G, \\ D^\omega y_i = 0, & \omega = 0,1,\ldots,\ell-1 & on\ \Sigma =]0,T[\times\Gamma. \end{array}\right\} \tag{3}$$

Proof. In fact

$$\pi(t;y,y) = \sum_{i=1}^{n}\int_G \left[\sum_{\alpha\le\ell}\sum_{k=1}^{\infty}\left|(D_k^\alpha y_i\right|^2\right]d\rho(x) + \sum_{i=1}^{n}\|y_i\|_{a_i}^2 - \sum_{i,j=1}^{n}\int_G a_{ij}(x,t)y_iy_j d\rho(x).$$

By using Lemma 3.1, we obtain

$$\pi(t;y,y) \ge \sum_{i=1}^{n}\int_G \left[\sum_{\alpha\le\ell}\sum_{k=1}^{\infty}\left|(D_k^\alpha y_i\right|^2\right]d\rho(x) + \sum_{i=1}^{n}\left[\|y_i\|^2_{a_i - \frac{1}{2}\sum_{j=1}^{n} a_{ij}}\right].$$

Hence

$$\pi(t;y,y) + c_0\|y\|^2_{\left(L^2(G,d\rho)\right)^n} \ge c_1\|y\|^2_{\left(W_0^\ell(G,d\rho)\right)^n}, \quad c_0, c_1 > 0.$$

Now, since $\pi(t;y,\phi)$ is symmetric ,we can apply Theorem 1.1 chapter4 of Lions[4] with

$$V = W_0^\ell(G,d\rho), \quad H = L^2(G,d\rho), \quad V' = W^{-\ell}(G,d\rho),$$

to obtain the result. □

4. CONTROL PROBLEM

Let us consider the following optimization problem

$$T \to \min, \tag{4}$$

under the constraints

$$\left.\begin{array}{ll} \dfrac{\partial^2 y_i}{\partial t^2} + (A(t)y)_i = u_i & \text{in } S, \\ y_i(x,0) = y_{i,0}(x) & \text{in } G, \\ y_i'(x,0) = y_{i,1}(x) & \text{in } G, \\ D^\omega y_i = 0, \quad \omega = 0,1,\ldots,\ell-1 & \text{on } \Sigma, \end{array}\right\} \tag{5}$$

$$z(x,T) = D\left(y(x,T), y'(x,T)\right) \in K, \tag{6}$$

$$u \in U_{ad}. \tag{7}$$

Let us denote by $U = (L^2(S))^n$, the space of controls, by $Y := L^2(0,T;(W_0^\ell(G,d\rho))^n) \times (L^2(S))^n$ the space of states, by a Hilbert space $\mathcal{H}$ the space of observations.

We assume that

$$\left.\begin{array}{l} U_{ad} \ \text{ is a closed , convex subset of } U, \\ K \ \text{ be a closed convex subset of } \mathcal{H} \text{ with a non -empty interior,} \\ D \ \text{ is a linear operator from } (W_0^{\ell}(G,d\rho))^n \times (L^2(G,d\rho))^n \to \mathcal{H}. \end{array}\right\} \tag{8}$$

Notation 1 *We will call the problem* (4)-(7) *under assumptions* (8)*, problem* **I**

The optimization problem **I** can be replaced by another equivalent one with a fixed time T. To show that we need tow auxiliary lemmas.

Lemma 4.1 *Let $T^0 > 0$ be the optimal time for the problem* **I**. *If $int\, K \neq \emptyset$ then*

$$z(x,T^0) = D\left(y(x,T), y'(x,T)\right) \in \partial K \quad (\textit{ boundary of } K), \tag{9}$$

for any set y satisfying (5)-(6).

Proof. Any solution of (5) is continuous with respect to t. If (9) is not true, then there exists an admissible state y such that the observation $z(x,T^0) \in int\, K$. Thus a $\hat{T} < T^0$ exists so that $z(x,\hat{T}) \in \partial K$. This contradicts the optimality of T^0 and hence (9) must be fulfilled. □

Lemma 4.2 *Let $T^0 > 0$ be the optimal time for the problem* **I**, *let u^0 and y^0 be an optimal control and corresponding state, respectively. Then there exist a non-trivial vector $g(x) \in \mathcal{H}'$ so that the pair (u^0, y^0) is the optimal for the following control problem with the fixed time T^0:*

$$I(y,u) := \left\langle g(x), z(x,T^0)\right\rangle \to \min, \tag{10}$$

subject to the constraints (5)-(7)*, where $\langle\rangle$ denotes the duality between $\mathcal{H}$, $\mathcal{H}'$.*

Proof. The linearity of the equations (5) and the linearity of D implies that the endpoints $z(x,T^0)$ of all admissible states y form a convex set $\mathcal{H}_{T^0}$. From Lemma 4.1 we have

$$\mathcal{H}_{T^0} \cap int\, K = \emptyset \ \text{ and } \ z(x,T^0) \in \partial K.$$

Since $int\, K \neq \emptyset$ thus there exists a closed hyperplane separating $\mathcal{H}_{T^0}$ and K containing $z(x,T^0)$, i.e. there is a nonzero vector $g \in \mathcal{H}'$ such as[12]

$$\sup_{z \in \mathcal{H}_{T^0}} \left\langle g(x), z(x,T^0)\right\rangle \leq \left\langle g(x), z^0(x,T^0)\right\rangle \leq \inf_{z\in K} \langle g(x), z\rangle .$$

This completes the proof. □

Remarks: The method fails if $int\, K \neq \emptyset$, e.g., in the case when K consists of a single point.

Remarks: If the set K has a special form i.e

$$K = \{z \in \mathcal{H}; \|z - z_d\|_{\mathcal{H}} \leq \epsilon\},$$

where $\epsilon > 0$ and $z_d \in \mathcal{H}$ are given, then g is Known explicitly and is expressed by

$$g(x) = z^0(x,T^0) - z_d.$$

According to Lemma 4.2, problem **I** is equivalent to the one with the fixed time T^0 and the performance index in the form (10).

Let us denote by Q_1, Q_2, Q_3 the sets in the space $E = Y \times U$ as follows

$$Q_1 := \left\{ (y,u) \in E; \quad \begin{array}{lr} \dfrac{\partial^2 y_i}{\partial t^2} + (A(t)y)_i = u_i & \text{in } S, \\ y_i(x,0) = y_{i,0}(x) & \text{in } G, \\ y_i'(x,0) = y_{i,1}(x) & \text{in } G, \\ D^\omega y_i = 0, \quad \omega = 0,1,\ldots,\ell-1 & \text{on } \Sigma, \end{array} \right\} \tag{11}$$

$$Q_2 := \Big\{(y,u) \in E; \qquad y \in Y, \quad u \in U_{ad}\Big\}, \tag{12}$$

$$Q_3 := \Big\{(y,u) \in E; \qquad z(x,T^0) \in K, \quad u \in U_{ad}\Big\}. \tag{13}$$

Thus the optimization problem **I** may be formulated in such a form

$$I(y,u) \to \min \quad \text{subject to} \quad (y,u) \in Q_1 \cap Q_2 \cap Q_3. \tag{14}$$

We approximate the sets Q_1 and Q_2 by the regular tangent cones (RTC),Q_3 by the regular admissible cone (RAC)and the performance functional by the regular cone of decrease (RFC).

The tangent cone to the set Q_1 at (y^0,u^0) has the form

$$TC(Q_1,(y^0,u^0)) = \Big\{(\bar{y},\bar{u}) \in E : P'(z^0,u^0)(\bar{y},\bar{u}) = 0\Big\}$$

$$= \left\{ (\bar{y},\bar{u}) \in E; \quad \begin{array}{lr} \dfrac{\partial^2 y_i}{\partial t^2} + (A(t)y)_i = u_i & \text{in } S, \\ y_i(x,0) = y_{i,0}(x) & \text{in } G, \\ y_i'(x,0) = y_{i,1}(x) & \text{in } G, \\ D^\omega y_i = 0, \quad \omega = 0,1,\ldots,\ell-1 & \text{on } \Sigma, \end{array} \right\} \tag{15}$$

where $P'(y^0,u^0)(\bar{y},\bar{u})$ is the Fréchet differential of the operator

$$P\left((y,y'),u\right) := \left(\frac{\partial^2 y}{\partial t^2} + A(t)y - u, y(x,0) - y_0(x), y'(x,0) - y_1(x)\right),$$

mapping from the space E in to the space F where

$$F = L^2(0,T;(W_0^{-\ell}(G,d\rho))^n) \times (W_0^{\ell}(G,d\rho))^n \times (L^2(G,d\rho))^n.$$

According to Theorem3.2 on the existence of solution to the equation (5) it is easy to prove that $P'(y^0,u^0)$ is the mapping from the space Eon to the space F as required in the Lusternik Theorem (Theorem 9.1 in [13]).

According to (12) the tangent cone $RTC(Q_2,(y^0,u^0))$ to the set Q_2 at $(y^0,u^0)))$ has the form

$$RTC(Q_2,(y^0,u^0)) = Y \times RTC(U_{ad},u^0), \tag{16}$$

where $RTC(U_{ad},u^0)$ is the tangent cone to the set U_{ad} at the point u^0. From [14] it is known that tangent cones are closed.

Applying the same arguments as in Section 2.2 from [1] we can show that

$$RTC(Q_1 \cap Q_2, (z^0, u^0)) = RTC(Q_1, (z^0, u^0)) \cap RTC(Q_2, (z^0, u^0)).$$

We have to use Theorem 1.2 (Theorem 3.3 from [2]),to show that $\left[RTC(Q_1(y^0, u^0))\right]^*$ and $[RTC(Q_2, (y^0, u^0))\right]^*$ are of the same sense. Note that we do not need to determine the explicit form of $[RTC(Q_1, (y^0, u^0))]^*$ in order to derive this conclusion. It is enough to use the Theorem 3.2 about the existence and uniqueness of the solution for hyperbolic system (5)which determine $RTC(Q_1, (y^0, u^0))$ in (15). According to this theorem the solution of such a system depends continuously on the right side ; i.e., in our case on $\bar{u}$ so we can rewrite the cone given by (15) in the form

$$RTC(Q_1, (y^0, u^0)) = \{(\bar{y}, \bar{u}) \in Y \times U : \bar{y} = M\bar{u}\}, \tag{17}$$

where $M : U \to Y$ is a linear and continuous operator. Then, applying Theorem 1.2 (Theorem 3.3 from [2]) to the cones given by (16) and (17), we get the assumption (iv) of Theorem 1.1 is satisfied.

The admissible cone $RAC(Q_3, (y^0, u^0))$ to the set Q_3 at (y^0, u^0) has the form

$$RAC(Q_3, (y^0, u^0)) = RAC(K, y^0(T^0)) \times U, \tag{18}$$

where, $RAC(K, y^0(T^0))$, is the admissible cone to the set K at the point $y^0(x, T^0)$.

Using Theorem 7.5 in Ref. [13], the regular cone of decrease for the performance functional I is given by

$$RFC(I, (y^0, u^0)) = \left\{(\bar{y}, \bar{u}) \in E; I'(y^0, u^0)(\bar{y}, \bar{u}) < 0\right\}, \tag{19}$$

where $I'(y^0, u^0)(\bar{y}, \bar{u})$ is the fréchet differential of the performance functional$I$.

If $RFC(I, (y^0, u^0)) \neq \emptyset$ then the adjoint cone consists of the elements of the form (Theorem 10.2 in [13])

$$f_0((y^0, u^0)) = -\lambda_0 I'(y^0, u^0)(\bar{y}, \bar{u}), \quad \text{where} \quad \lambda_0 \geq 0,$$

The functionals belonging to $[RTC(Q_1, (y^0, u^0))]^*$ have the form (Theorem 10.1 in [13])

$$f_2(\bar{y}, \bar{u}) = 0 \quad \forall (\bar{y}, \bar{u}) \in RTC(Q_1, (y^0, u^0)).$$

The functionals

$$f_3(\bar{y}, \bar{u}) \in [RTC(Q_2, (y^0, u^0)))]^* \quad \text{and} \quad f_1(\bar{y}, \bar{u}) \in [RAC(Q_3, (y^0, u^0)))]^*,$$

can be expressed as follows

$$f_3(\bar{y}, \bar{u}) = f_3^1(\bar{y}) + f_3^2(\bar{u}), \quad f_1(\bar{y}, \bar{u}) = f_1^1(\bar{y}) + f_1^2(\bar{u}),$$

where $f_3^1(\bar{y}) = 0 \quad \forall \bar{y} \in Y$ and $f_1^2(\bar{u}) = 0 \quad \forall \bar{u} \in U$ (Theorem 10.1 in [13]), $f_3^2(\bar{u})$ is the support functional to the set U_{ad} at the point u^0 and, $f_1^1(\bar{y})$ is the support functional to the set K at the point $y^0(x, T^0)$ (Theorem 10.5 in [13]).

Since all assumptions of Theorem 1.1 (the generalized Dubovitskii-Milyutin Theorem, Theorem 1.8.1 in [1]) are satisfied and we know suitable adjoint cones then we ready to write down the Euler-Lagrange Equation in the following form.

$$f_3^2(\bar{u}) + f_1^1(\bar{y}) = \lambda_0 I'(y^0, u^0)(\bar{y}, \bar{u}) \quad \forall (\bar{y}, \bar{u}) \in RTC(Q_1, (y^0, u^0)). \tag{20}$$

Since I depended on the observation z which depended on the operator D, we shall interpret (20) after choosing the observation$z$,the observation space $\mathcal{H}$ and the target set K in a less form fashions.

5. OBSERVATION ON THE POSITION Y

Let

$$\left.\begin{array}{l} z(x,T) = y(x,T), \quad \mathcal{H} = (L^2(G,d\rho))^n \\ K = \left\{ z = (z_i)_{i=1}^n \in (L^2(G,d\rho))^n : \|z_i - z_{id}\|_{L^2(G,d\rho)} \leq \epsilon \right\}, \end{array}\right\} \tag{21}$$

where z_{id} and ϵ are given such that $z_{id} \in L^2(G,d\rho),\ i = 1,2,\ldots n\ \epsilon > 0$.

Notation 2 *We will call the problem* **I** *with* z, $\mathcal{H}$ *and* K *are given by* (21)*, problem* $\mathbf{I_1}$.

In the present case, according to Remark 4.,

$$I'(y^0,u^0)(\bar{y},\bar{u}) = \sum_{i=1}^{n} \int_G (y_i^0(x,T^0) - z_{id})\bar{y}_i(x,T^0)d\rho. \tag{22}$$

Introducing the adjoint variable p by the solution of the following systems

$$\left.\begin{array}{lll} \dfrac{\partial^2 p_i}{\partial t^2} + (A(t)p)_i = 0 & , \quad x \in G, & t \in]0,T^0[, \\ p_i(x,T^0) = 0, & x \in G, & \\ p_i'(x,T^0) = -(y_i^0(x,T^0) - z_{id}), & x \in G, & \\ D^\omega p_i = 0, \quad \omega = 0,1,\ldots,\ell-1 & x \in \Gamma, & t \in]0,T^0[. \end{array}\right\} \tag{23}$$

The existence of a unique solution for the equation (23) can be proved using Theorem3.2 with an obvious change of variables.

Taking into account that $\bar{y}$ is the solution of $P'((y^0,u^0))(\bar{y},\bar{u}) = 0$ for any fixed $\bar{u}$,we obtain.

$$\begin{aligned} 0 &= \int_0^{T^0} \int_G \left[\frac{\partial^2 p_i}{\partial t^2} + (A(t)p)_i \right] \bar{y}_i d\rho dt \\ &= -\int_G \frac{\partial p_i}{\partial t}\bar{y}_i \Big|_0^{T^0} d\rho + \int_G p_i.\frac{\partial \bar{y}_i}{\partial t}\Big|_0^{T^0} d\rho - \int_0^{T^0}\int_G p_i.\left[\frac{\partial^2 \bar{y}_i}{\partial t^2} + (A(t)\bar{y})_i\right] d\rho dt \\ &= -\int_G \frac{\partial p_i}{\partial t}(x,T^0).\bar{y}_i(x,T^0)d\rho - \int_0^{T^0}\int_G p_i.\,\bar{u}_i d\rho dt. \end{aligned}$$

Hence

$$\int_G (y_i(x,T^0) - z_{id}).\bar{y}_i(x,T^0)d\rho = \int_0^{T^0}\int_G p_i\,\bar{u}_i d\rho dt. \tag{24}$$

So, the Euler-Lagrange Equation (20) takes the form:

$$f_3^2(\bar{u}) + f_1^1(\bar{y}) = \frac{1}{2}\lambda_0 \sum_{i=1}^{n}\int_0^{T^0}\int_G p_i\,\bar{u}_i d\rho dt + \frac{1}{2}\lambda_0 \sum_{i=1}^{n}\int_G (y_i^0(x,T^0) - z_{id})\bar{y}_i(x,T^0)d\rho. \tag{25}$$

A number λ_0 cannot be equal to 0 because in such a case all functionals in the Euler-Lagrange Equation would be zero which is impossible according to the DM Theorem. Using the definition of the support functional and dividing both members of the obtained inequalities by λ_0 from (25) we obtain the maximum conditions:

$$\sum_{i=1}^{n}\int_0^{T^0}\int_G p_i(u_i - u_i{}^0)d\rho\,dt \geq 0 \quad \forall u \in U_{ad}, \tag{26}$$

$$\sum_{i=1}^{n}\int_G (y_i^0(x,T^0)-z_{id}).(y_i-{y_i}^0)d\rho \geq 0 \quad \forall y\in K. \tag{27}$$

.

If $RFC(I_1,(y^0,\mathbf{u}^0)) = \emptyset$ then the optimality conditions are fulfilled with equality in the maximum conditions (26)-(27).

We have thus proved:

Theorem 5.1 *Assuming that $T^0 > 0$ is the optimal time for the problem $\mathbf{I_1}$, u^0 and y^0 are the optimal control and corresponding state respectively. Then, their exists the adjoint state p, $p = (p_i)_{i=1}^n \in \left\{p : p \in L^2(0,T;(W_0^\ell(G,d\rho))^n),\quad \frac{\partial p}{\partial t} \in (L^2(S))^n\right\}$ so that the following system of equations and inequalities must be satisfied:*

State equations

$$\left.\begin{array}{rll}
\dfrac{\partial^2 {y_i}^0}{\partial t^2} + \left(A(t)y^0\right)_i = {u_i}^0, & x\in G, & t\in]0,T^0[,\\
{y_i}^0(x,0) = y_{i,0}(x), & x\in G, & \\
{y'_i}^0(x,0) = y_{i,1}(x), & x\in G, & \\
D^\omega y_i^0 = 0,\quad \omega = 0,1,\ldots,\ell-1 & x\in\Gamma, & t\in]0,T^0[\\
y(x,T)\in K. & &
\end{array}\right\} \tag{28}$$

Adjoint equations

$$\left.\begin{array}{rll}
\dfrac{\partial^2 p_i}{\partial t^2} + (A(t)p)_i = 0 \qquad , & x\in G, & t\in]0,T^0[,\\
p_i(x,T^0) = 0, & x\in G, & \\
p'_i(x,T^0) = -(y_i^0(x,T^0)-z_{id}), & x\in G, & \\
D^\omega p_i = 0,\quad \omega = 0,1,\ldots,\ell-1 & x\in\Gamma, & t\in]0,T^0[.
\end{array}\right\} \tag{29}$$

Maximum conditions

$$\sum_{i=1}^{n}\int_0^{T^0}\int_G p_i(u_i-{u_i}^0)d\rho\, dt \geq 0 \quad \forall u\in U_{ad}, \tag{30}$$

$$\sum_{i=1}^{n}\int_G (y_i^0(x,T^0)-z_{id}).(y_i-{y_i}^0)d\rho \geq 0 \quad \forall y\in K. \tag{31}$$

6. OBSERVATION ON THE VELOCITY Y'

Let

$$\left.\begin{array}{l}
z(x,T) = y'(x,T),\quad \mathcal{H} = (L^2(G,d\rho))^n\\
K = \left\{z = (z_i)_{i=1}^n \in (L^2(G,d\rho))^n : \|z_i - z_{id}\|_{L^2(G,d\rho)} \leq \epsilon\right\},
\end{array}\right\} \tag{32}$$

where z_{id} and ϵ are given such that $z_{id}\in L^2(G,d\rho)$, $i = 1,2,\ldots n$ $\epsilon > 0$.

Notation 3 *We will call the problem* **I** *with z, $\mathcal{H}$ and K are given by* (32)*, problem* $\mathbf{I_2}$.

In the present case, according to Remark 4.,

$$I'(y^0,u^0)(\bar{y},\bar{u}) = \sum_{i=1}^{n} \int_G (y_i'^0(x,T^0) - z_{id})\bar{y_i'}(x,T^0)d\rho. \tag{33}$$

Introducing the adjoint variable p by the solution of the following systems

$$\left.\begin{aligned} &\frac{\partial^2 p_i}{\partial t^2} + (A(t)p)_i = 0, && x \in G, \quad t \in]0,T^0[, \\ &p_i(x,T^0) = (y_i'^0(x,T^0) - z_{id}), && x \in G, \\ &p_i'(x,T^0) = 0 && x \in G, \\ &D^\omega p_i = 0, \quad \omega = 0,1,\dots,\ell-1 && x \in \Gamma, \quad t \in]0,T^0[. \end{aligned}\right\} \tag{34}$$

Since $(y_i'(\tau^0;u^0) - z_{id}) \notin W_0^\ell(G,d\rho)$,the existence of a unique solution for the problem (34) can not be proved using Theorem3.2.

The existence of a unique solution $p(u)$ for the problem (34) can be proved by applying Transposition Theorem (Theorem 3.1, chapter 4 of Lions[4]) with

$$V = \left(W_0^\ell(G,d\rho)\right)^n, \quad H = L^2(G,d\rho), \quad V' = \left(W_0^{-\ell}(G,d\rho)\right)^n.$$

Then we have the following lemma

Lemma 6.1 *The solution of* (34), $p(u)$ *is defined as the unique element of* $(L^2(S))^2$ *such that*

$$\left.\begin{aligned} &\int_0^{T^0}\int_G p_i(u)\left[\frac{\partial^2\phi_i}{\partial t^2} + (A(t)\phi)_i\right]d\rho(x)dt = \int_G (y_i'(\tau^0;u^0) - z_{id})\phi_i' d\rho(x) \\ &\forall\phi = (\phi_i)_{i=1}^n \text{ such that } \left[\frac{\partial^2\phi_i}{\partial t^2} + (A(t)\phi)_i\right] \in L^2(S), \quad \phi_i(0) = 0, \quad \phi_i'(0) = 0. \end{aligned}\right\} \tag{35}$$

Therefore, in (35) we can take $\phi_i = \bar{y}_i$ and taking into account that $\bar{y}$ is the solution of $P'((y^0,u^0))(\bar{y},\bar{u}) = 0$ for any fixed $\bar{u}$,we obtain.

$$\int_G (y_i'(x,T^0) - z_{id}).\bar{y_i'}(x,T^0)d\rho = \int_0^{T^0}\int_G p_i\,\bar{u}_i d\rho dt. \tag{36}$$

Hence, as the above section, we obtain the following theorem:

Theorem 6.2 *Assuming that* $T^0 > 0$ *is the optimal time for the problem* $\mathbf{I_2}$, u^0 *and* y^0 *are the optimal control and corresponding state respectively. Then, their exists the adjoint state* $p = (p_i)_{i=1}^n \in (L^2(S))$ *so that the following system of equations and inequalities must be satisfied:*

State equations

$$\left.\begin{aligned} &\frac{\partial^2 y_i{}^0}{\partial t^2} + \left(A(t)y^0\right)_i = u_i{}^0, && x \in G, \quad t \in]0,T^0[, \\ &y_i{}^0(x,0) = y_{i,0}(x), && x \in G, \\ &y_i'^0(x,0) = y_{i,1}(x), && x \in G, \\ &D^\omega y_i^0 = 0, \quad \omega = 0,1,\dots,\ell-1 && x \in \Gamma, \quad t \in]0,T^0[\\ &y(x,T) \in K. \end{aligned}\right\} \tag{37}$$

Adjoint equations

$$\left.\begin{array}{ll}\dfrac{\partial^2 p_i}{\partial t^2} + (A(t)p)_i = 0, & x \in G, \quad t \in]0, T^0[, \\ p_i(x, T^0) = (y_i'^0(x, T^0) - z_{id}), & x \in G, \\ p_i'(x, T^0) = 0 & x \in G, \\ D^\omega p_i = 0, \quad \omega = 0, 1, \ldots, \ell - 1 & x \in \Gamma, \quad t \in]0, T^0[. \end{array}\right\} \tag{38}$$

Maximum conditions

$$\sum_{i=1}^{n} \int_0^{T^0} \int_G p_i(u_i - u_i{}^0) d\rho \, dt \geq 0 \quad \forall u \in U_{ad}, \tag{39}$$

$$\sum_{i=1}^{n} \int_G ((y_i^0)'(x, T^0) - z_{id}).(y_i' - (y_i{}^0)')d\rho \geq 0 \quad \forall y \in K. \tag{40}$$

7. OBSERVATION ON THE POSITION-VELOCITY (Y, Y')

We now consider the optimal control problem with position-velocity observation. Let

$$\left.\begin{array}{l} z(t) = (y(t), y'(t)), \quad \mathcal{H} = \left(W_0^\ell(G, d\rho)\right)^n \times (L^2(G, d\rho))^n \\ K = \left\{(z, z') \in \mathcal{H} : \displaystyle\sum_{\alpha \leq \ell} \|D^\alpha z_i - z_{id}\|_{L^2(G, d\rho)} + \|z_i' - z_{id}\|_{L^2(G, d\rho)} \leq \epsilon \right\}, \end{array}\right\} \tag{41}$$

where z_{id} and ϵ are given such that $z_{id} \in L^2(G, d\rho)$, $i = 1, 2, \ldots n$, $\epsilon > 0$.

Notation 4 *We will call the problem* **I** *with* z, $\mathcal{H}$ *and* K *are given by* (41)*, problem* $\mathbf{I_3}$.

From the definition of the usual norm on $(W_0^\ell(G, d\rho))^n \times (L^2(G, d\rho))^n$, definition of K in (41), and Theorem 2 in [8], $I'(y^0, u^0)(\bar{y}, \bar{u})$ can be given by the following equation:

$$\begin{aligned} I'(y^0, u^0)(\bar{y}, \bar{u}) = & \sum_{i=1}^{n} \int_G -\left(\sum_{\alpha \leq \ell} \sum_{k=1}^{\infty} (-1)^\ell D_k^{2\ell}\right)(y_i(\tau^0; u^0) - z_{id})\bar{y}_i(x, T^0) d\rho \\ & + \sum_{i=1}^{n} \int_G (y_i'^0(x, T^0) - z_{id})\bar{y_i'}(x, T^0) d\rho, \end{aligned} \tag{42}$$

which can be interpreted as the above application to obtaining the following theorem

Theorem 7.1 *Assuming that* $T^0 > 0$ *is the optimal time for the problem* $\mathbf{I_3}$*,* u^0 *and* y^0 *are the optimal control and corresponding state respectively. Then, their exists the adjoint state* $p = (p_i)_{i=1}^n \in (L^2(S))$ *so that the following system of equations and inequalities must be satisfied:*

State equations

$$\left.\begin{array}{ll}\dfrac{\partial^2 y_i{}^0}{\partial t^2} + \left(A(t)y^0\right)_i = u_i{}^0, & x \in G, \quad t \in]0, T^0[, \\ y_i{}^0(x, 0) = y_{i,0}(x), & x \in G, \\ y_i'^0(x, 0) = y_{i,1}(x), & x \in G, \\ D^\omega y_i^0 = 0, \quad \omega = 0, 1, \ldots, \ell - 1 & x \in \Gamma, \quad t \in]0, T^0[\\ y(x, T) \in K. & \end{array}\right\} \tag{43}$$

Adjoint equations

$$\left.\begin{array}{ll} \dfrac{\partial^2 p_i}{\partial t^2} + (A(t)p)_i = 0, & x \in G, \quad t \in]0, T^0[, \\ p_i(x, T^0) = (y_i'^0(x, T^0) - z_{id}), & x \in G, \\ p_i'(x, T^0) = -\left(\sum_{\alpha \le \ell} \sum_{k=1}^{\infty} (-1)^\ell D_k^{2\ell}\right)(y_i(\tau^0; u^0) - z_{id}) & x \in G, \\ D^\omega p_i = 0, \quad \omega = 0, 1, \ldots, \ell - 1 & x \in \Gamma, \quad t \in]0, T^0[. \end{array}\right\} \tag{44}$$

Maximum conditions

$$\sum_{i=1}^{n} \int_0^{T^0} \int_G p_i(u_i - u_i{}^0) d\rho\, dt \ge 0 \quad \forall u \in U_{ad}, \tag{45}$$

$$\sum_{i=1}^{n} \int_G \left(\sum_{\alpha \le \ell} \sum_{k=1}^{\infty} (-1)^\ell D_{k\rho}^{2\ell}\right)((y_i^0)'(x, T^0) - z_{id})(y_i' - (y_i{}^0)')d\rho \ge 0 \quad \forall y \in K. \tag{46}$$

.

8. SCALER CASE

Here, we take the case where $n = 2$, in the case of section 5. the time optimal control is characterized b y

State equations

$$\left.\begin{array}{ll} \dfrac{\partial^2 y_1^0}{\partial t^2} + \left[\sum_{\alpha \le \ell} \sum_{k=1}^{\infty} (-1)^\alpha D_k^{2\alpha}\right] y_1^0 + a_1(x,t) y_1^0 - a_{12}(x,t) y_2^0 = u_1^0 & x \in G, \quad t \in]0, T^0[, \\ \dfrac{\partial^2 y_2^0}{\partial t^2} + \left[\sum_{\alpha \le \ell} \sum_{k=1}^{\infty} (-1)^\alpha D_k^{2\alpha}\right] y_2^0 - a_{21}(x,t) y_1^0 + a_2(x,t) y_2^0 = u_2^0 & x \in G, \quad t \in]0, T^0[, \\ y_1^0(x, 0) = y_{1,0}(x), \quad y_2^0(x, 0) = y_{2,0}(x) & x \in G, \\ y_1'^0(x, 0) = y_{1,1}(x), \quad y_1'^0(x, 0) = y_{2,1}(x) & x \in G, \\ D^\omega y_1^0 = D^\omega y_2^0 = 0, \quad \omega = 0, 1, \ldots, \ell - 1 & x \in \Gamma, \quad t \in]0, T^0[, \\ \|y_1^0(T^0, u^0) - z_{1d}\| \le \epsilon, \quad \|y_2^0(T^0, u^0) - z_{2d}\| \le \epsilon & \end{array}\right\} \tag{47}$$

Adjoint equations

$$\left.\begin{array}{ll} \dfrac{\partial^2 p_1}{\partial t^2} + \left[\sum_{\alpha \le \ell} \sum_{k=1}^{\infty} (-1)^\alpha D_{\rho k}^{2\alpha}\right] p_1 + a_1(x,t) p_1 - a_{21}(x,t) p_2 = 0 & x \in G, \quad t \in]0, T^0[, \\ \dfrac{\partial^2 p_2}{\partial t^2} + \left[\sum_{\alpha \le \ell} \sum_{k=1}^{\infty} (-1)^\alpha D_{\rho k}^{2\alpha}\right] p_2 - a_{12}(x,t) p_1 + a_2(x,t) p_2 = 0 & x \in G, \quad t \in]0, T^0[, \\ p_1(x, T^0) = 0, \quad p_2(x, T^0) = 0 & x \in G, \\ \dfrac{\partial p_1}{\partial t}(T^0; u^0) = -(y_1(T^0; u^0) - z_{1d}) & x \in G, \\ \dfrac{\partial p_2}{\partial t}(T^0; u^0) = -(y_2(T^0; u^0) - z_{2d}) & x \in G, \\ D^\omega p_1 = D^\omega p_2 = 0, \quad \omega = 0, 1, \ldots, \ell - 1 & x \in \Gamma, \quad t \in]0, T^0[. \end{array}\right\} \tag{48}$$

Maximum conditions

$$\sum_{i=1}^{2}\int_{0}^{T^0}\int_{G} p_i(u_i - {u_i}^0)d\rho\, dt \geq 0 \quad \forall u \in U_{ad}, \tag{49}$$

$$\sum_{i=1}^{2}\int_{G}(y_i^0(x,T^0) - z_{id})(y_i - {y_i}^0)d\rho \geq 0 \quad \forall y_i : \|y_i - z_{id}\| \leq \epsilon. \tag{50}$$

9. COMMENTS

- We would like to note that, if $\ell = 1$ and the sequence of weights $(P_k)_{k=1}^{\infty}$ is given by

$$(P_k)_{k=1}^{\infty} = (\underbrace{1,1,\ldots,1}_{N-\text{time}},0,0,\ldots),$$

then

$$A(t) = \begin{pmatrix} -\Delta + a_1 & -a_{12} \\ -a_{21} & -\Delta + a_2 \end{pmatrix},$$

where Δ is the Laplace operator: $\Delta = \sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2}$.

The results in this case are similar to the results in [8].

- As a final remark,we note that if there is no constraint in the states i.e for example if we take

$$Y = \left\{ y = (y_i)_{i=1}^n \in L^2(0,T;(W_0^{\ell}(G,d\rho))^n) \times (L^2(S))^n : (y_i(x,T) - z_{id}) \leq \epsilon \right\},$$

in the case of Section 5. the problem is equivalent to the fixed-time problem

$$\text{minimize } \sum_{i=1}^{n}\int_{G}|y_i(x,T) - z_{id}(x)|^2 d\rho, \quad T \text{ fixed },$$

subject to (5)-(7) [except in the trivial case where $z_d = y(x,T)$] Then we can prove in an analogous manner, that is the necessary conditions for optimality for this problem coincide with the results in [8].

REFERENCES

[1] Kotarski, W. (1997). *Some problems of optimal and pareto optimal control for distributed parameter systems.* Katowice, Poland: Reports of Silesian University, 1668.

[2] Walczak, S. (1984). On some properties of cones in normed spaces and their application to investing extremal problems. *Journal of Optimization Theory and Applications, 42*(2), 561-582.

[3] Fattorini, H. O. (1998). *Infinite dimensional optimization theory and opotimal control.* Cambridge: Cambridge University Press.

[4] Lions, J. L. (1971). *Optimal control of systems governed by partial differential equations.* Band: Springer-verlag.

[5] Wang, P. K. C. (1975). Optimal control of parabolic systems with boundary conditions involving time delays. *SIAM Journal of Control and Optimization, 13*(2), 274-293.

[6] Wong, K. H. (1987). Optimal control computation for parabolic systems with boundary conditions involving time delays. *Journal of Optimization Theory and Applications, 53*(3), 475-507.

[7] Dubinskii, Ju. A. (1986). Sobolev spaces of infinite order and differential equations. D. Rei

[8] El-Saify, H. A., Serag. H. M., & Shehata, M. A. (2009). Time-optimal control for co-operative hyperbolic systems involving Laplace operator. *Journal of Dynamical and Control systems, 15*(3), 405-423.

[9] Kotarski, W., El-Saify, H. A., & Shehata, M. A. (2007). Time optimal control of parabolic lag system with infinite variables. *Journal of the Egyptian mathematical society, 15*(1), 21-34.

[10] Berezanskii, Ju. M. (1986). *Self-adjoint operators in spaces of functions of infinitely many variables* (Translation of Mathematical Monographs, Volume 63). New York: American Mathematical Society.

[11] Lions, J. L., & Magenes, E. (1972). *Nonhomogeneous boundary value problem and applications.* New York: Spring-Verlage.

[12] Luenberger, D. G. (1969). *Optimization by vector space methods*. New York: Wiley.

[13] Girsanov, I. V. (1972). *Lectures on mathematical theory of extremum problems*. New York: Spring-Verlag.

[14] Lasiecka, I. (1974). *Conical approximations of sets with application to optimization*. Technical University, Warsaw, Poland, Ph.D. Thesisis.