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# Positive Solutions for a Class of Quasilinear Elliptic Equations with a Dirichlet Problem

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**Abstract:** In this paper, we study the following problem

$$-\triangle_p u = h(x)u^q + f(u), \quad u \in W_0^{1,p}(\Omega), \ u > 0 \text{ in } \Omega,$$

where $\Omega$ is a bounded smooth domain in $\mathbf{R}^N (N \geq 3)$, $0 < q < 1$. By using Mountain Pass Theorem, we prove that there exists at least two positive solutions under suitable assumptions on the nonlinearity.

**Key Words:** Quasilinear elliptic equation; Positive solution; Dirichlet problem; Mountain Pass Theorem

## 1. INTRODUCTION

In this paper, we study the following Dirichlet problem:

$$\begin{cases} -\Delta_p u = h(x)u^q + f(u), \\ u \in W_0^{1,p}(\Omega), \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1}$$

where $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, $1 < p < \infty$, and $\Omega$ is a bounded smooth domain in $\mathbf{R}^N$, $p^* = \frac{Np}{N-p}$ if $N > p$, $p^* = \infty$ if $N \leq p$, $0 < q < p-1$ and $h(x)$, $f(s)$ satisfy the following conditions:

(h1) $h(x) \in L^\infty(\Omega)$, $h(x) \geq 0$ and $h(x) \not\equiv 0$,

(f1) $f(s) \in C(R)$; $f(0) = 0$; $f(s) \geq (\not\equiv) 0$ if $s \geq 0$ and $f(s) \equiv 0$ if $s \leq 0$,

(f2) $\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0$; $\lim_{s\to +\infty} \frac{f(s)}{s^{p-1}} = l > 0$,

(f3) $\frac{f(s)}{s^{p-1}}$ is non-decreasing in $s > 0$,

(f4) There exists a constant $M > 0$ such that $f(s)s - pF(s) \leq M$ for all $s \geq 0$.

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For $p = 2$, the following problem

$$\begin{cases} -\Delta u = h(x)u^q + f(u), \\ u \in W_0^{1,p}(\Omega), \\ u > 0 \text{ in } \Omega, \end{cases} \tag{2}$$

has been studied extensively under various conditions, for example, if $f(u) = 0$ and $0 < q < 1$ (i.e. the sublinear case, in which $I(u)$ is coercive ), see e.g., the papers [3, 4]. If $h(x) \equiv \lambda_1$ and $q = 1, l = 0$, problem (2) is the so-called resonant problem; the related results can be found in [5, 6], etc. If $q \geq 1$ and $f(s)$ is superlinear in $s$, the typical results were given in [1, 7–9]. A similar problem to (2) with $h(x) \equiv 0$ and $f(x)$ being asymptotically linear at infinity was studied by [10–13], etc. For the case of $0 < q < 1$ and $f(x)$ being superliner in $s$, some existence and multiplicity results to problem (2) on a bounded domain were given in [14].

For $p > 1$, the existence and uniqueness of the positive solutions for the quasilinear elliptic equation with eigenvalue problems

$$\begin{cases} \triangle_p u + \lambda f(u) = 0 \quad \text{in } \ \Omega, \\ u(x) = 0 \quad \text{on } \ \partial\Omega, \end{cases} \tag{3}$$

with $\lambda > 0, p > 1, \Omega \subset \mathbf{R^N}, N \geq 2$ have been studied by many authors, see [21–28] and the references therein. When $f$ is strictly increasing on $\mathbf{R}^+, f(0) = 0, \lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^\mu, 0 < \mu < p-1, \alpha_1, \alpha_2 > 0$, it was shown in [23] that there exists at least two positive solutions for Eqs. (3) when $\lambda$ is sufficiently large. If $\lim_{s\to 0^+} \inf f(s)/s^{p-1} > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved in [24] that the problem (3) has a unique positive solution when $\lambda$ is sufficiently large. Moreover, it was also shown in [25] that problem (3) has a unique positive large solution and at least one positive small solution when $\lambda$ is large if $f$ is nondecreasing; there exists $\alpha_1, \alpha_2 > 0$ such that $f(s) \leq \alpha_1 + \alpha_2 s^\beta, 0 < \beta < p-1; \lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0$, and there exists $T, Y > 0$ with $Y \geq T$ such that

$$(f(s)/s^{p-1})' > 0 \ \text{ for } \ s \in (0, T),$$

and

$$(f(s)/s^{p-1})' < 0 \ \text{ for } \ s > Y.$$

Recently, Hai[26] considered the case when $\Omega$ is an annular domain, and obtained the existence of positive large solutions for the problem (3) when $\lambda$ sufficiently small. Xuan & Chen proved in [27] the singular problem (3) has a unique positive radial solution if $f$ is a continuous function and positive on $\overline{\Omega} = B_R$ (here $B_R$ is a ball).

Moreover, it was also shown in [28] that problem

$$\triangle_p u + q(x)u^{-\gamma} = 0, \quad x \in \mathbf{R}^N,$$

has a positive entire solution if $q \in C(\mathbf{R}^+), 0 \leq \gamma < p-1$, for any $0 < \varepsilon < (N-p)(p-1-|\gamma|)/(p-1)$,

$$\int_1^\infty r^{p+\varepsilon-1+[(N-p)|\gamma|/(p-1)]} q(r)dr < \infty,$$

and for $r \in (0, 1), \delta < 1, q(r) = O(r^{-\delta})$.

Still in [29], the authors studied the existence of nontrivial solutions for the problem

$$-\triangle_p u + |u|^{p-2}u = 0,$$

in a bounded smooth domain $\Omega \subset R^N$ with a nonlinear boundary condition by variational and topological argument, and the authors in [30] obtained ground and bound state solution of quasilinear equation

$$-\triangle_p u + V(x)|u|^{p-2}u = f(x,u),$$

with unbounded or decaying radial potential.

To the author's knowledge, it seems that there are few results for problem (1). Motivated by the results of the above cited papers, we study the existence of two positive solutions of problem (1) under the condition $(f1)-(f4)$, the results of the semilinear equations are extended to the quasilinear ones. We modify the methods developed in [1, 17, 20] and extend the results of [20] to a quasilinear elliptic equation (1).

**Definition 1.1** *$u \in W_0^{1,p}$ is a positive weak solution to problem (1) if $u > 0, a.e.$ on $\Omega$ and satisfies*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx = \int_\Omega h(x)u^q \varphi dx + \int_\Omega f(u)\varphi dx, \qquad \textit{for all } \varphi \in W_0^{1,p}(\Omega).$$

Throughout this paper, we denote by $\lambda_1 > 0$ the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, that is:

$$\lambda_1 = \inf_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}, \tag{3}$$

Denote the norm of $u \in W_0^{1,p}(\Omega)(L^p(\Omega), p \geq 1$, respectively) by

$$\|u\| = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}, \qquad (|u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}, \qquad \text{respectively })$$

Define

$$F(t) = \int_0^t f(s)dx \qquad \text{for } t \geq 0.$$

By (f1)-(f3), it is easy to see that $\frac{F(t)}{t^p} \to 0$ as $t \to 0$ and $\frac{F(t)}{t^{p^*}} \to 0$ as $t \to \infty$ where $p^* = \frac{Np}{N-p}$. Then, for any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ such that

$$F(s) \leq \varepsilon s^p + C_\varepsilon s^{p^*}, \qquad \text{for all } s \geq 0. \tag{4}$$

It is well known that to seek a nontrivial weak solution for (1) is equivalent to finding a nonzero critical point of the following variational functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q+1}\int_\Omega h(x)(u^+)^{q+1}dx - \int_\Omega F(u^+)dx, \tag{5}$$

for $u \in W_0^{1,p}(\Omega)$, where $u^+ = \max\{0,u\}, u^- = \min\{0,u\}$. By (f1)-(f3), $I(u)$ is well defined and in $C^1(W_0^{1,p}(\Omega))$. Moreover, by the maximum principle for weak solution[2], we know that a nonzero critical point of $I(u)$ is in fact a positive weak solution to problem (1).

We end this section by giving our main theorem. Before this, we give a further condition on $h(x)$.

For any fixed $\tau \in (0, \frac{1}{p})$ (as small as we want). Let $\varepsilon = \lambda_1 \tau$ in (4). Then there is a fixed constant $C_\tau(\lambda_1) > 0$ such that

$$F(s) \leq \lambda_1 \tau s^p + C_\tau s^{p^*}, \qquad \text{for all } s \geq 0. \tag{6}$$

For the above $\tau$ and $C_\tau$, we suppose that $|h|_\infty$ satisfies:

$$(h2) \qquad |h|_\infty^{\frac{p^*-p}{\nu}} < \frac{(\frac{1}{p}-\tau)\beta\alpha^{\frac{p-q-1}{\nu}}}{1+(q+1)\alpha C_\tau},$$

where

$$\nu = p^* - q - 1, \qquad \alpha = \frac{(p-q-1)(N-p)}{p^2(1+q)C_\tau}, \quad \beta = \frac{(q+1)S^p}{|\Omega|^{\frac{p}{N}}}.$$

where $|\Omega|$ denotes the measure of the domain $\Omega, S$ is the best Sobolev constant, that is,

$$S^p = \inf_{0\neq u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{(\int_\Omega |u|^{p^*} dx)^{\frac{p}{p^*}}}. \tag{7}$$

We modify the methods developed in [20], which give the following theorems

**Theorem 1.2** *If (h1), (h2), (f1)-(f4) hold and $l > \lambda_1$, then problem (1) has two positive weak solutions $u_1$ and $u_2$ in $W_0^{1,p}(\Omega)$ such that*

$$I(u_1) < 0 < I(u_2).$$

*Remarks:* By the proof of Theorem 3.1 (see Section 3), we know that if (h1), (h2) and (f1), (f2) hold, then for all $l > 0$ (even if $l \le \lambda_1$), problem (1) has also a positive weak solution $u_1 \in W_0^{1,p}(\Omega)$ with $I(u_1) < 0$.

## 2. PRELIMINARIES

Let's first recall the Mountain Pass Theorem and Ekeland' variational principle which will be used to prove Theorem 1.2.

**Proposition 2.1** *(Mountain Pass Theorem[18]) Let $E$ be a real Banach space with its dual space $E^*$ and suppose that $I \in C(E,R)$ satisfies the condition*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho} I(u),$$

*for some $\mu < \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characteriaed by*

$$c = \inf_{\gamma\in\Gamma} \max_{0\le\tau\le 1} I(\gamma(\tau)),$$

*where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e. Then, there exists a sequence $\{u_n\} \subset E$ such that*

$$I(u_n) \to c \ge \eta \qquad and \qquad (1+\|u_n\|)\|I'(u_n)\|_{E^*} \to 0 (n\to\infty).$$

**Proposition 2.2** *(Ekeland' variational principle, see [16]) Let $V$ be a complete metric space and $F : V \to R \cup \{+\infty\}$ be lower semi-continuous, bounded from below. For any $\varepsilon > 0$, there is some point $\upsilon \in V$ with*

$$F(\upsilon) \le \inf_V + \varepsilon \qquad and \qquad \forall\omega \in V, F(\omega) \ge F(\upsilon) - \varepsilon d(\upsilon,\omega).$$

**Lemma 2.3** *For $I$ defined by (6), if (f1), (f3), (f4) hold and there exists $\{u_n\} \subset W_0^{1,p}(\Omega)$ satisfying*

$$\langle I'(u_n), u_n\rangle \to 0 (n \to \infty),$$

*then, for any $t > 0$, by extracting a suitable subsequence, we have*

$$I(tu_n) \leq \frac{t^p}{pn} + [\frac{t^p}{p} - \frac{t^{q+1}}{1+q}] \int_\Omega h(x)(u_n^+)^{1+q} dx + \frac{1}{p} M|\Omega|.$$

**Proof.** The main idea of proving this lemma is essentially due to [18, 20], so we omit it here.

**Lemma 2.4** *If (h1), (h2) and (f1), (f2) hold, then*

*(i) There exists $\eta > 0, \rho > 0$ such that $I(u) \geq \eta > 0$, for all $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho$.*

*(ii) There exists $e \in W_0^{1,p}(\Omega)$ with $\|e\| > \rho$ such that $I(e) < 0$.*

**Proof.** (i) By the definition of $I$ given in (6), applying (7) and (4) as well as Sobolev inequality (8), we have

$$\begin{aligned} I(u) &\geq \frac{1}{p}\|u_n\|^p - \frac{|h|_\infty}{1+q}\int_\Omega (u_n^+)^{1+q} dx - \tau\lambda_1 \int_\Omega (u_n^+)^p dx - C_\tau \int_\Omega (u_n^+)^{p^*} dx \\ &\geq (\frac{1}{p} - \tau)\|u_n\|^p - \frac{|h|_\infty}{1+q}(\int_\Omega (u_n^+)^{p^*} dx)^{\frac{1+q}{p^*}} |\Omega|^{\frac{\nu}{p^*}} - C_\tau S^{-p^*}\|u_n\|^{p^*} \\ &\geq [(\frac{1}{p} - \tau) - \frac{|h|_\infty |\Omega|^{\frac{\nu}{p^*}}}{(1+q)S^{1+q}}\|u_n\|^{q+1-p} - C_\tau S^{-p^*}\|u_n\|^{p^*-p}]\|u_n\|^p, \end{aligned}$$

where $\nu$ is given by (h2). Motivated by [15], we let

$$Q(t) = \frac{|h|_\infty |\Omega|^{\frac{\nu}{p^*}}}{(1+q)S^{1+q}} t^{q+1-p} + C_\tau S^{-p^*} t^{p^*-p}.$$

So, to prove (i) it suffices to show that there exists $t_0 > 0$ such that

$$Q(t_0) < \frac{1}{p} - \tau. \tag{8}$$

In fact,

$$Q'(t) = \frac{(q+1-p)|h|_\infty |\Omega|^{\frac{\nu}{p^*}}}{(1+q)S^{1+q}} t^{q-p} + C_\tau S^{-p^*}(p^* - p) t^{p^*-p-1},$$

then

$$Q'(t_0) = \frac{(q+1-p)|h|_\infty |\Omega|^{\frac{\nu}{p^*}}}{(1+q)S^{1+q}} t_0^{q-p} + C_\tau S^{-p^*}(p^* - p) t_0^{p^*-p-1} = 0.$$

We have

$$t_0^{p^*-p-1} = \frac{(p-q-1)|h|_\infty |\Omega|^{\frac{\nu}{p^*}} S^{p^*}}{(1+q)S^{1+q}(p^*-p)C_\tau}.$$

Then

$$t_0 = [\frac{(p-q-1)|h|_\infty S^{p^*-p-1}}{(1+q)(p^*-p)C_\tau}]^{\frac{1}{\nu}} |\Omega|^{\frac{1}{p^*}} \qquad (\nu = p^* - p - 1)$$

$$
\begin{aligned}
&= [\frac{(p-q-1)|h|_\infty}{(1+q)(p^*-p)C_\tau}]^{\frac{1}{\nu}} S|\Omega|^{\frac{1}{p^*}} \\
&= [\frac{(p-q-1)(N-p)|h|_\infty}{(1+q)p^2C_\tau}]^{\frac{1}{\nu}} S|\Omega|^{\frac{1}{p^*}} \qquad (p^* = \tfrac{Np}{N-p}) \\
&= (\alpha|h|_\infty)^{\frac{1}{\nu}} S|\Omega|^{\frac{1}{p^*}} \qquad (\alpha = \tfrac{(p-q-1)(N-p)}{p^2(1+q)C_\tau}) .
\end{aligned}
$$

So

$$
\frac{t_0}{S} = (\alpha|h|_\infty)^{\frac{1}{\nu}}|\Omega|^{\frac{1}{p^*}}.
$$

Then

$$
\begin{aligned}
Q(t_0) &= \frac{|h|_\infty|\Omega|^{\frac{\nu}{p^*}} t_0^{q+1-p}}{(1+q)S^{1+q}} + \frac{(p-q-1)(N-p)|h|_\infty|\Omega|^{\frac{\nu}{p^*}} t_0^{q+1-p}}{(1+q)p^2S^{1+q}} \\
&= \frac{|h|_\infty|\Omega|^{\frac{\nu}{p^*}} t_0^{q+1-p}}{(1+q)S^pS^{q+1-p}} + \frac{(1+q)\alpha C_\tau|h|_\infty|\Omega|^{\frac{\nu}{p^*}} t_0^{q+1-p}}{(1+q)S^pS^{q+1-p}} \\
&= \frac{|h|_\infty|\Omega|^{\frac{\nu}{p^*}} t_0^{q+1-p}}{\beta|\Omega|^{\frac{p}{N}}S^{q+1-p}} + \frac{(1+q)\alpha C_\tau|h|_\infty|\Omega|^{\frac{\nu}{p^*}} t_0^{q+1-p}}{\beta|\Omega|^{\frac{p}{N}}S^{q+1-p}} \qquad (\beta = \tfrac{(q+1)S^p}{|\Omega|^{\frac{p}{N}}}) \\
&= \frac{|h|_\infty|\Omega|^{\frac{p-q-1}{p^*}}}{\beta}[1+(1+q)\alpha C_\tau](\frac{t_0}{S})^{q+1-p} \\
&= \frac{|h|_\infty|\Omega|^{\frac{p-q-1}{p^*}}}{\beta}[1+(1+q)\alpha C_\tau](\alpha|h|_\infty)^{\frac{q+1-p}{\nu}}|\Omega|^{\frac{q+1-p}{p^*}} \\
&= \frac{|h|_\infty^{\frac{p^*-p}{\nu}}}{\beta}[1+(1+q)\alpha C_\tau]\alpha^{\frac{q+1-p}{\nu}} \\
&< \frac{(\frac{1}{p}-\tau)\alpha^{\frac{p-q-1}{\nu}}}{1+(1+q)\alpha C_\tau}[1+(1+q)\alpha C_\tau]\alpha^{\frac{q+1-p}{\nu}} \qquad \text{(by (h2))} \\
&= \frac{1}{p}-\tau.
\end{aligned}
$$

So (10) holds. But $Q(t) \to \infty$ whenever $t \to 0^+$ or $t \to +\infty$, which means $Q(t)$ has a minimum at $t = t_0$. Substituting $t_0$ in $Q(t)$ and noticing the condition (h2), we see that (10) holds. Taking $\rho = t_0$, then there exists $\eta > 0$ such that (i) holds.

(ii) Let $\varphi_1$ be the $\lambda_1-$eigenfunction, that is, $\varphi_1$ achieves the infimum of (11). For $t > 0$,

$$
\begin{aligned}
\lim_{t\to\infty}\frac{I(t\varphi_1)}{t^p} &= \frac{1}{p}||\varphi_1||^p - \lim_{t\to\infty}\frac{t^{q+1-p}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}dx - \lim_{t\to\infty}\int_\Omega\frac{F(t\varphi_1)}{t^p}dx \\
&\le \frac{1}{p}||\varphi_1||^p - \int_\Omega \liminf_{t\to\infty}\frac{F(t\varphi_1)}{t^p\varphi_1^p}\varphi_1^p dx \qquad \text{(by Fatou's lemma)} \\
&\le \frac{1}{p}||\varphi_1||^p - \frac{l}{p}\int_\Omega\varphi_1^p dx \qquad \text{(by (f2))} \\
&= \frac{1}{p}(1-\frac{l}{\lambda_1})||\varphi_1||^p \\
&< 0 \qquad (\text{by } l > \lambda_1) .
\end{aligned}
$$

So, for $t_0 > 0$ large enough, choosing $e = t_0\varphi_1$, then (ii) is proved.

**Lemma 2.5** *If* $\{u_n\} \subset W_0^{1,p}(\Omega)$ *is bounded and*

$$I(u_n) \to c, \qquad I'(u_n) \to 0, \qquad in\ W_0^{-1,p^*}(\Omega),$$

*then, there exists a subsequence, still denoted by* $\{u_n\}$*, such that for some* $u \in W_0^{1,p}(\Omega), u_n \to u$ *strongly in* $W_0^{1,p}(\Omega)$ *and* $I(u) = c$*.*

The proof of the Lemma 2.5 is similar to the Lemma 2.5 of [20], so we omit it here.

# 3. EXISTENCE OF THE FIRST SOLUTION

For $\rho$ given in Lemma 2.4, we set

$$\overline{B}_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\| \le \rho\} \qquad \text{and} \qquad \partial B_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\| = \rho\},$$

and$\overline{B}_\rho$ is a complete metric space with the distance

$$\text{dist}(u, v) = \|u - v\|, \qquad \text{foy any } u, v \in \overline{B}_\rho.$$

By Proposition 2.2, we have

$$I(u)\mid_{\partial B_\rho} \ge \eta > 0. \tag{9}$$

Moreover, it is easy to see that $I \in C^1(\overline{B}_\rho, R)$, hence $I$ is lower semi-continuous and bounded from below on $\overline{B}_\rho$. Let

$$c_1 = \inf\{I(u) : u \in \overline{B}_\rho\}. \tag{10}$$

Taking $v \in C_0^\infty(\infty), v \ge 0$ with $\int_\Omega h(x)v(x)^{q+1}dx > 0$, and for $t > 0$, we have

$$\begin{aligned} I(tv) &= \frac{t^p}{p}\int_\Omega |\nabla v|^p dx - \frac{t^{q+1}}{1+q}\int_\Omega h(x)v^{1+q}dx - \int_\Omega F(tv)dx \\ &\le \frac{t^p}{p}\int_\Omega |\nabla v|^p dx - \frac{t^{q+1}}{1+q}\int_\Omega h(x)v^{1+q}dx \\ &< 0 \qquad \text{for } t > 0 \text{ small enough} \end{aligned}$$

Therefore, $c_1 < 0$.

**Theorem 3.1** *If (h1), (h2) and (f1), (f2) hold, then there exists* $u_1 \in W_0^{1,p}(\Omega)$ *which is a weak solution of problem (1) and* $I(u_1) = c_1 < 0, c_1$ *being given by (12).*

**Proof.** By Proposition 2.2, for any $k \ge 1$ there is a $u_k$ with

$$c_1 \le I(u_k) \le c_1 + \frac{1}{k}, \tag{11}$$

$$I(w) \ge I(u_k) - \frac{1}{k}\|u_k - w\|. \tag{12}$$

Then, $\|u_k\| < \rho$ for $k \geq 1$ large enough. Otherwise, if $\|u_k\| = \rho$ for infinitely many $k$, without loss of generality, we may assume that $\|u_k\| = \rho$ for all $k \geq 1$, and it follows from (11) that

$$I(u_k) \geq \eta > 0.$$

This and (13) imply that $0 > c_1 \geq \eta > 0$ by letting $k \to \infty$, a contradiction.

We prove now that $I'(u_k) \to 0$ in $W_0^{-1,p^*}(\Omega)$. In fact, for any $u \in W_0^{1,p}(\Omega)$ with $\|u\| = 1$, let $w_k = u_k + tu$ and for a fixed $k \geq 1$, we have $\|w_k\| \leq \|u_k\| + t < \rho$ if $t > 0$ small enough. So, it follows from (14) that

$$I(u_k + tu) \geq I(u_k) - \frac{t}{k}\|u\|,$$

that is

$$\frac{I(u_k + tu) - I(u_k)}{t} \geq -\frac{1}{k}\|u\| = -\frac{1}{k}.$$

Letting $t \to 0$, we see that $\langle I'(u_k, u\rangle \geq -\frac{1}{k}$, and this gives

$$|\langle I'(u_k), u\rangle| < \frac{1}{k}, \qquad \text{for any } u \in W_0^{1,p}(\Omega) \text{ with } \|u\| = 1.$$

So, $I'(u_k) \to 0$ in $W_0^{-1,p^*}(\Omega)$ and by (13), $I(u_k) \to c_1 < 0$. Hence, it follows from Lemma 2.5 that there exists $u_1 \in W_0^{1,p}(\Omega)$ such that $I'(u_1) = 0$, that is, $u_1$ is a weak solution of problem (1) and $I(u_1) = c_1 < 0$. Moreover, the maximum principle implies that $u_1 > 0$ a.e. in $\Omega$.

# 4. PROOF OF THEOREM 1.2

By Theorem 3.1, to prove Theorem 1.2, we need only to show that there exists another nonzero critical point of $I$ in $W_0^{1,p}(\Omega)$. For this purpose, we use Proposition 2.1.

For $\eta, \rho$ and $e$ given in Lemma 2.4, by applying Proposition 2.1 with $\mu = 0, E = W_0^{1,p}(\Omega)$ and for $c$ defined as in Proposition 2.1, there exists a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$$I(u_n) \to c > 0; \qquad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0.$$

This implies that

$$I(u_n) = \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{1}{q+1}\int_\Omega h(x)(u_n^+)^{q+1} dx - \int_\Omega F(u_n^+)dx = c + o(1), \tag{13}$$

$$\langle I'(u_n), \varphi\rangle = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx - \int_\Omega h(x)(u_n^+)^q \varphi dx - \int_\Omega f(u_n^+)\varphi dx = o(1), \tag{14}$$

$$\langle I'(u_n), u_n\rangle = \int_\Omega |\nabla u_n|^p dx - \int_\Omega h(x)(u_n^+)^{q+1} dx - \int_\Omega f(u_n^+)u_n.dx = o(1). \tag{15}$$

**Proof.** By Lemma 2.5, if $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, then we can find some $u_2 \in W_0^{1,p}(\Omega)$ such that $I'(u_2) = 0$ and $I(u_2) = c > 0$ and $u_2$ is also a solution of problem (1), which is positive (by the maximum principle[2]) and different from the solution $u_1$ obtained in Theorem 3.1 since $I(u_1) = c_1 < 0$.

Therefore, to prove Theorem 1.2, it is enough to show that $\{u_n\}$ satisfying (15)-(17) is bounded in $W_0^{1,p}(\Omega)$

Indeed, if $\|u_n\| \to \infty$, for $M > 0$ given by (f4), we set

$$t_n = \frac{\sqrt[p]{pM|\Omega|}}{\|u_n\|}; \qquad w_n = t_n u_n = \frac{\sqrt[p]{pM|\Omega|}}{\|u_n\|} u_n. \tag{16}$$

Obviously, $w_n$ is bounded in $W_0^{1,p}(\Omega)$. By extracting a subsequence and we may suppose that

$$w_n \to w \qquad \text{weakly in } W_0^{1,p}(\Omega), \tag{17}$$

$$w_n \to w \qquad \text{a.e. in } \Omega, \tag{18}$$

$$w_n \to w \qquad \text{strongly in } L^r(\Omega), \qquad 1 \le r < p^* = \frac{Np}{N-p}. \tag{19}$$

We claim that

$$w \neq 0.$$

In fact, if $w \equiv 0$, then $w_n \to 0$ in $L^{p^*}(\Omega)$ and in $L^{q+1}(\Omega)$ by (21). Hence,

$$\lim_{n\to\infty} \int_\Omega h(x)(w_n^+)^{q+1} dx = 0; \qquad \lim_{n\to\infty} \int_\Omega F(w_n^+) dx = 0,$$

and consequently,

$$I(w_n) = \frac{1}{p}\|w_n\|^p - o(1) = M|\Omega| - o(1) \qquad \text{(by (4.4))}\,. \tag{20}$$

Since $\|u_n\| \to \infty$, noticing (18) we observe that $t_n \to 0$, then it follows from (18) and Lemma 2.3 that

$$I(w_n) \le \frac{t_n^p}{pn} + [\frac{t_n^p}{p} - \frac{t_n^{q+1}}{1+q}] \int_\Omega h(x)(u_n^+)^{1+q} dx + \frac{1}{p} M|\Omega| \to \frac{1}{p} M|\Omega|,$$

which contradicts (22). Hence, $w \neq 0$.

Now, we turn to showing that $w$ satisfies the following identity:

$$\int_\Omega |\nabla w|^{p-2} \nabla w \nabla \varphi dx = l \int_\Omega |w|^{p-2} w \varphi dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \tag{21}$$

In fact, by (16), (18) and noticing $t_n \to 0$, we have

$$\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla \varphi dx - \int_\Omega p_n(x)|w_n|^{p-2} w_n \varphi dx = o(1) \quad \text{for all } \varphi \in W_0^{1,p}(\Omega), \tag{22}$$

where

$$p_n(x) = \begin{cases} \frac{f(u_n^+(x))}{u_n^{p-1}(x)}, & \text{for } x \in \Omega \text{ with } u_n(x) \ge 0, \\ 0 & \text{for } x \in \Omega \text{ with } u_n(x) \le 0. \end{cases}$$

By condition (f1)-(f3), we know that

$$0 \le p_n(x) \le l, \qquad \text{for a.e. } x \in \Omega.$$

Passing to a suitable subsequence we may suppose that there is a function $g(x) \in L^{p^*}(\Omega)$ such that

$$\begin{aligned} &p_n(x) \rightharpoonup g(x) \qquad \text{weakly in } L^{p^*}(\Omega),\\ &0 \le g(x) \le l \qquad \text{a.e. in } \Omega. \end{aligned}$$

Since $\|u_n\| \to \infty$ and $w_n \to w$ a.e. in $\Omega$, it follows from (18) that $u_n \to \infty$ a.e. in $\Omega$ if $w(x) > 0$ a.e. on $\Omega$; then (f2) implies that

$$g(x) \equiv l \qquad \text{if } w(x) > 0 \text{ for a.e. } x \in \Omega. \tag{23}$$

For any $\varphi \in L^{p^*}(\Omega)$, it follows from $w_n \to w$ strongly in $L^{p^*}(\Omega)$ that

$$\begin{aligned} \int_\Omega p_n(x)|w_n(x)|^{p-2}w_n(x)\varphi dx &= \int_\Omega p_n(x)|w_n^+(x)|^{p-2}w_n^+(x)\varphi dx,\\ &\to \int_\Omega g(x)|w^+(x)|^{p-2}w^+(x)\varphi dx. \end{aligned}$$

Then, by (19), (24), we have

$$\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi dx - \int_\Omega g(x)|w^+|^{p-2}w^+\varphi dx = 0 \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

Taking $\varphi = w^-$, this yields $\|w^-\| = 0$ and so we have that $w \equiv w^+ \ge 0$ on $\Omega$; then the maximum principle implies that $w(x) > 0$ on $\Omega$. Thus, by (25) we have $g(x) \equiv l$ and (23) holds. So if we take $\varphi = \varphi_1$ (the $\lambda_1$-eigenfunction) in (23), we have

$$\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi_1 dx = l\int_\Omega |w|^{p-2}w\varphi_1 dx.$$

On the other hand, by $-\Delta_p\varphi_1 = \lambda_1\varphi_1$, we must have

$$\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi_1 dx = \lambda_1\int_\Omega |w|^{p-2}w\varphi_1 dx,$$

which is impossible since $l > \lambda_1$.

Thus, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, and the proof is completed.

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