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On Nonlinear Sum-difference Inequality with Two Variables and Application to BVP

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Abstract: Sum-difference inequalities with n nonlinear terms in two variables which generalize some existing results and can be used more effectively in the analysis of certain boundary value problem for certain partial difference equation are discussed. Application example is also given to show the boundedness of solutions of a difference equation.

Key Words: Sum-difference inequality; Nonlinear; Boundedness

1. INTRODUCTION

The well-known and widely used Gronwall-Bellman inequality plays a fundamental role in the study of existence, uniqueness, boundedness and other qualitative properties of the solution of differential equation. An enormous amount of effort has been devoted to the discovery of new types of inequalities and their applications in various branches of ordinary differential equations, partial differential equations and integral equations (see [1-5], and the references given therein). In the recent past, more attention has been paid to the discrete versions of such integral inequalities (for example, [6-10]). These inequalities are applied to study the boundedness and uniqueness of the solutions of the following boundary value problem (BVP, for short) for the partial difference equation

$$\begin{aligned} &\Delta_1\Delta_2\psi(u(m,n)) = F(m,n,u(m,n)), \\ &u(m,n_0) = f(m), \quad u(m_0,n) = g(n), \end{aligned} \tag{1}$$

where ψ, F, f, g are given functions and u is the unknown function to be found.

Unfortunately, the discretization of some nonlinear integral inequalities can not directly be processed because there still exist some technical difficulties between a continuous form and a discontinuous form. In this paper, we investigate the sum-difference inequality in two variables with n nonlinear terms

$$\psi(u(m,n)) \le a(m,n) + \sum_{i=1}^{k} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f_i(m,n,s,t) w_i(u(s,t)), \tag{2}$$

which modifies the incorrect proof in [11] and can be used more effectively in the study of the qualitative properties of solutions of the difference equation. Moreover, we also present an example to show boundedness of solution of a partial difference equation.

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2. STATEMENT OF MAIN RESULT

Assumptions 2.1

(A1) $\psi \in C(R_+, R_+)$ is a strictly increasing function satisfying $\psi(\infty) = \infty$;

(A2) $a(m,n)$ is nonnegative for $m,n \in N_0$ and $a(m_0,n_0) > 0$ for $m_0,n_0 \in N_0$;

(A3) $f_i(m,n,s,t)$ $(i = 1,\cdots,k)$ is nonnegative for $m,n,s,t \in N_0$;

(A4) $w_i(u) \in C(R_+, R_+)$ $(i = 1,\cdots,k)$ is a nondecreasing function. They satisfy the relationship $w_1 \propto w_2 \propto \cdots \propto w_k$, where $w_i \propto w_{i+1}$ means that $\frac{w_{i+1}}{w_i}$ is nondecreasing on $(0,\infty)$.

Definitions 2.2

(D1) $\tilde{a}(m,n) = \max_{m_0 \le \tau \le m, n_0 \le \eta \le n, \tau,\eta \in N_0} a(\tau,\eta)$. Clearly, $\tilde{a}(m,n)$ is nonnegative and nondecreasing in m and n, and $\tilde{a}(m,n) \ge a(m,n)$.

(D2) $\tilde{f}_i(m,n,s,t) = \max_{m_0 \le \tau \le m, n_0 \le \eta \le n, \tau,\eta \in \mathbf{N}_0} f_i(\tau,\eta,s,t)$. Then, $\tilde{f}_i(m,n,s,t)$ is nonnegative and nondecreasing in m and n, and $\tilde{f}_i(m,n,s,t) \ge f_i(m,n,s,t)$.

(D3) $\Delta_1 u(m,n) = u(m+1,n) - u(m,n)$ and $\Delta_3 r(m,n,s,t) = r(m,n,s+1,t) - r(m,n,s,t)$.

(D4) For $u \ge u_i > 0$, $W_i(u) = \int_{u_i}^{u} \frac{dz}{w_i(\psi^{-1}(z))}$. From Assumption (A4), W_i is strictly increasing so its inverse W_i^{-1} is well defined, continuous and increasing in its corresponding domain.

Our main result is given in the following.

Theorem 2.3 Under Assumptions (A1)-(A4), if $\Delta_1 \tilde{a}(m,n)$ is nondecreasing in n and $u(m,n)$ is a nonnegative function for $m,n \in N_0$ satisfying (2), then

$$
\begin{aligned}
u(m,n) \le \psi^{-1}\{W_k^{-1}[W_k(\tilde{a}(m_0,n)) &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde{f}_k(m,n,s,t) \\
&+ \sum_{s=m_0}^{m-1} \frac{\Delta_3 r_k(m,n,s,n)}{\varphi_k(W_{k-1}^{-1}(r_k(m_0,n_0,s,n_0)))}]\},
\end{aligned} \tag{3}
$$

for $m_0 \le m \le M_1, n_0 \le n \le N_1$, where $r_k(m,n,s,t)$ is determined recursively by

$$
\begin{aligned}
r_1(m,n,s,t) &= \tilde{a}(s,t), \\
r_{i+1}(m,n,s,t) &= W_i(r_1(m,n,m_0,t)) + \sum_{\tau=m_0}^{s-1}\sum_{\eta=n_0}^{t-1} \tilde{f}_i(m,n,\tau,\eta) \\
&\quad + \sum_{\tau=m_0}^{s-1} \frac{\Delta_3 r_i(m,n,\tau,t)}{\varphi_i(W_{i-1}^{-1}(r_i(m_0,n_0,\tau,n_0)))}, i = 1,\cdots,k-1,
\end{aligned} \tag{4}
$$

$\varphi_i(u) = \frac{w_i(\psi^{-1}(u))}{w_{i-1}(\psi^{-1}(u))}$, $\varphi_1(u) = w_1(\psi^{-1}(u))$, $W_0 = I$ (Identity), and M_1 and N_1 are positive integers satisfying

$$
\begin{aligned}
W_i(\tilde{a}(m_0,N_1)) + \sum_{s=m_0}^{M_1-1}\sum_{t=n_0}^{N_1-1} \tilde{f}_i(M_1,N_1,s,t) + \sum_{s=m_0}^{M_1-1} \frac{\Delta_3 r_i(M_1,N_1,s,N_1)}{\varphi_i(W_{i-1}^{-1}(r_i(m_0,n_0,s,n_0)))} \\
\le \int_{u_i}^{\infty} \frac{dz}{w_i(\psi^{-1}(z))}, \quad i = 1,\cdots,k.
\end{aligned} \tag{5}
$$

Remarks: If $w_i\ (i=1,\cdots,k)$ satisfies $\int_{u_i}^{\infty}\frac{dz}{w_i(\psi^{-1}(z))}=\infty$, then we may choose $N_1=\infty$ and $M_1=\infty$. As explained in [3], different choices of u_i in W_i do not affect our results.

Before we prove the theorem, the following lemma should be introduced. For the proof, see [13].

Lemma 2.4 For $i=1,\cdots,k$, $\Delta_3 r_i(m,n,s,t)$ is nonnegative and nondecreasing in m, n and t, and $r_i(m,n,s,t)$ is nonnegative and nondecreasing in its arguments.

3. PROOF OF THEOREM 2.3

Proof. Take any arbitrary positive integers M and N where $M\le M_1,\ N\le N_1$. By the definition of functions $\tilde{a}$, $\tilde{f}_i$ and r_1, we have an auxiliary inequality from (2),

$$\psi(u(m,n))\le r_1(M,N,m,n)+\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_i(M,N,s,t)w_i(u(s,t)), \tag{6}$$

for $m_0\le m\le M, n_0\le n\le N$, where $r_1(M,N,m,n)=\tilde{a}(m,n)$. Assume that $u(m,n)$ in (6) satisfies

$$\begin{aligned}u(m,n)\le \psi^{-1}\{W_k^{-1}[W_k(r_1(M,N,m_0,n))&+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_k(M,N,s,t)\\ &+\sum_{s=m_0}^{m-1}\frac{\Delta_3 r_k(M,N,s,n)}{\varphi_k(W_{k-1}^{-1}(r_k(m_0,n_0,s,n_0)))}]\},\end{aligned} \tag{7}$$

for $m_0\le m\le \min\{M,M_2\}$ and $n_0\le n\le \min\{N,N_2\}$, where M_2 and N_2 are positive integers satisfying

$$\begin{aligned}W_i(r_1(M,N,m_0,N_2))+\sum_{s=m_0}^{M_2-1}\sum_{t=n_0}^{N_2-1}\tilde{f}_i(M,N,s,t)+\sum_{s=m_0}^{M_2-1}\frac{\Delta_3 r_i(M,N,s,N_2)}{\varphi_i(W_{i-1}^{-1}(r_i(m_0,n_0,s,n_0)))}\\ \le\int_{u_i}^{\infty}\frac{dz}{w_i(\psi^{-1}(z))}, i=1,\cdots,k.\end{aligned} \tag{8}$$

Notice that we may choose $M_1\le M_2$ and $N_1\le N_2$. In fact, by Lemma 2.4 and Definition (D2), $r_i(M,N,m,n)$, $\Delta_3 r_i(M,N,m,n)$ and $\tilde{f}_i(M,N,m,n)$ are nondecreasing in M and N. Thus, M_2 and N_2 satisfying (8) get smaller as M and N are chosen larger. Moreover, M_2 and N_2 satisfy the same (5) as M_1 and N_1 for $M=M_1$ and $N=N_1$.

Next, we use the mathematical induction to prove our result.

(I) $k=1$

Let $z(m,n)=\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_1(M,N,s,t)w_1(u(s,t))$. Obviously, $z(m_0,n)=0$ and $z(m,n)$ is nonnegative and nondecreasing in each variable. It follows from (6) that

$$\psi(u(m,n))\le r_1(M,N,m,n)+z(m,n),$$

for $m_0\le m\le M, n_0\le n\le N$. By Definition (D3), we get

$$\Delta_1 z(m,n)=z(m+1,n)-z(m,n)=\sum_{t=n_0}^{n-1}\tilde{f}_1(M,N,m,t)w_1(u(m,t))$$

$$\le \sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,m,t) w_1(\psi^{-1}(r_1(M,N,m,t)+z(m,t)))$$
$$\le w_1(\psi^{-1}(r_1(M,N,m,n)+z(m,n))) \sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,m,t),$$

where we apply the fact that w_1 and ψ^{-1} are nondecreasing. Since $r_1(M,N,m,n)=\tilde{a}(m,n)>0$, we have

$$\frac{\Delta_1 z(m,n)+\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(z(m,n)+r_1(M,N,m,n)))}$$
$$\le \sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,m,t)+\frac{\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(z(m,n)+r_1(M,N,m,n)))}$$
$$\le \sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,m,t)+\frac{\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,m,n_0)))}. \tag{9}$$

Then

$$\int_{z(m,n)+r_1(M,N,m,n)}^{z(m+1,n)+r_1(M,N,m+1,n)} \frac{d\tau}{w_1(\psi^{-1}(\tau))}$$
$$\le \int_{z(m,n)+r_1(M,N,m,n)}^{z(m+1,n)+r_1(M,N,m+1,n)} \frac{d\tau}{w_1(\psi^{-1}(z(m,n)+r_1(M,N,m,n)))}$$
$$\le \frac{\Delta_1 z(m,n)+\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(z(m,n)+r_1(M,N,m,n)))}$$

$$\le \sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,m,t)+\frac{\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,m,n_0)))}.$$

So

$$\int_{z(m_0,n)+r_1(M,N,m_0,n)}^{z(m,n)+r_1(M,N,m,n)} \frac{d\tau}{w_1(\psi^{-1}(\tau))} = \sum_{s=m_0}^{m-1} \int_{z(s,n)+r_1(M,N,s,n)}^{z(s+1,n)+r_1(M,N,s+1,n)} \frac{d\tau}{w_1(\psi^{-1}(\tau))}$$
$$\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,s,t)+\sum_{s=m_0}^{m-1} \frac{\Delta_3 r_1(M,N,s,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,s,n_0)))}.$$

The definition of W_1 and $z(m_0,n)=0$ shows that

$$W_1(z(m,n)+r_1(M,N,m,n)) \le W_1(r_1(M,N,m_0,n))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1} \frac{\Delta_3 r_1(M,N,s,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,s,n_0)))}, \quad m_0 \le m \le M, n_0 \le n \le N. \tag{10}$$

It is easy to check from (8) that the right side of (10) is in the domain of W_1^{-1} for all $m_0 \le m \le M$ and $n_0 \le n \le N$. Thus by the monotonicity of W_1^{-1}, we obtain

$$u(m,n) \le \psi^{-1}(z(m,n)+r_1(M,N,m,n))$$
$$\le \psi^{-1}\{W_1^{-1}[W_1(r_1(M,N,m_0,n))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1} \frac{\Delta_3 r_1(M,N,s,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,s,n_0)))}]\}, \tag{11}$$

for $m_0 \le m \le M$ and $n_0 \le n \le N$, that is, (7) is true for $k=1$.

(II) For $k+1$

Assume that (7) is true for k. Consider

$$\psi(u(m,n)) \le r_1(M,N,m,n) + \sum_{i=1}^{k+1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde{f}_i(M,N,s,t)w_i(u(s,t)), \tag{12}$$

for $m_0 \le m \le M$ and $n_0 \le n \le N$. Let

$$z(m,n) = \sum_{i=1}^{k+1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde{f}_i(M,N,s,t)w_i(u(s,t)).$$

Obviously, $z(m_0,n)=0$ holds and $z(m,n)$ is nonnegative and nondecreasing in m and n. (12) can be rewritten as $\psi(u(m,n)) \le r_1(M,N,m,n) + z(m,n)$ for $m_0 \le m \le M$, $n_0 \le n \le N$. Moreover, we have

$$\begin{aligned}\Delta_1 z(m,n) &= \sum_{i=1}^{k+1}\sum_{t=n_0}^{n-1} \tilde{f}_i(M,N,m,t)w_i(u(m,t)) \\ &\le \sum_{i=1}^{k+1}\sum_{t=n_0}^{n-1} \tilde{f}_i(M,N,m,t)w_i(\psi^{-1}(r_1(M,N,m,t)+z(m,t))).\end{aligned}$$

Since w_i is nondecreasing and $r_1(M,N,m,n)>0$, by Lemma 2.4, we have

$$\begin{aligned}&\frac{\Delta_1 z(m,n)+\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(z(m,n)+r_1(M,N,m,n)))} \\ &\le \frac{\sum_{i=1}^{k+1}\sum_{t=n_0}^{n-1} \tilde{f}_i(M,N,m,t)w_i(\psi^{-1}(z(m,t)+r_1(M,N,m,t)))}{w_1(\psi^{-1}(z(m,n)+r_1(M,N,m,n)))} + \frac{\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(r_1(M,N,m,n)))} \\ &\le \sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,m,t) + \frac{\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,m,n_0)))} \\ &\qquad + \sum_{i=2}^{k+1}\sum_{t=n_0}^{n-1} \tilde{f}_i(M,N,m,t)\frac{w_i(\psi^{-1}(z(m,t)+r_1(M,N,m,t)))}{w_1(\psi^{-1}(z(m,t)+r_1(M,N,m,t)))} \\ &\le \sum_{t=n_0}^{n-1} \tilde{f}_1(M,N,m,t) + \frac{\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,m,n_0)))} \\ &\qquad + \sum_{i=1}^{k}\sum_{t=n_0}^{n-1} \tilde{f}_{i+1}(M,N,m,t)\upsilon_{i+1}(z(m,t)+r_1(M,N,m,t)),\end{aligned}$$

for $m_0 \le m \le M$, $n_0 \le n \le N$, where

$$\upsilon_{i+1}(u) = \frac{w_{i+1}(\psi^{-1}(u))}{w_1(\psi^{-1}(u))},$$

for $i=1,\cdots,k$. Notice that

$$\begin{aligned}&\int_{z(m,n)+r_1(M,N,m,n)}^{z(m+1,n)+r_1(M,N,m+1,n)} \frac{d\tau}{w_1(\psi^{-1}(\tau))} \\ &\le \int_{z(m,n)+r_1(M,N,m,n)}^{z(m+1,n)+r_1(M,N,m+1,n)} \frac{d\tau}{w_1(\psi^{-1}(z(m,n)+r_1(M,N,m,n)))} \\ &\le \frac{\Delta_1 z(m,n)+\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(r_1(M,N,m,n)+z(m,n)))}\end{aligned}$$

$$\le \sum_{t=n_0}^{n-1} \tilde f_1(M,N,m,t) + \frac{\Delta_3 r_1(M,N,m,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,m,n_0)))}$$
$$+\sum_{i=1}^{k}\sum_{t=n_0}^{n-1} \tilde f_{i+1}(M,N,m,t)\upsilon_{i+1}(z(m,t)+r_1(M,N,m,t)).$$

Therefore,

$$\int_{z(m_0,n)+r_1(M,N,m_0,n)}^{z(m,n)+r_1(M,N,m,n)} \frac{d\tau}{w_1(\psi^{-1}(\tau))} \le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde f_1(M,N,s,t) + \sum_{s=m_0}^{m-1} \frac{\Delta_3 r_1(M,N,s,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,s,n_0)))}$$
$$+\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde f_{i+1}(M,N,s,t)\upsilon_{i+1}(z(s,t)+r_1(M,N,s,t)),$$

that is,

$$W_1(z(m,n)+r_1(M,N,m,n)) \le W_1(r_1(M,N,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde f_1(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1} \frac{\Delta_3 r_1(M,N,s,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,s,n_0)))}$$
$$+\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde f_{i+1}(M,N,s,t)\upsilon_{i+1}(z(s,t)+r_1(M,N,s,t)),$$

or equivalently

$$\psi(\Xi(m,n)) \le \theta_1(M,N,m,n) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde f_{i+1}(M,N,s,t)\upsilon_{i+1}(W_1^{-1}(\psi(\Xi(s,t)))),$$

for $m_0 \le m \le M$ and $n_0 \le n \le N$ the same as (6) for k, where

$$\psi(\Xi(m,n)) = W_1(z(m,n)+r_1(M,N,m,n))$$
$$\theta_1(M,N,m,n) = W_1(r_1(M,N,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde f_1(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1} \frac{\Delta_3 r_1(M,N,s,n)}{w_1(\psi^{-1}(r_1(m_0,n_0,s,n_0)))}.$$

From the Assumptions (A1) and (A4), each $\upsilon_{i+1}(W_1^{-1}(\psi))$, $i=1,\cdots,k$, is continuous and nondecreasing on $[0,\infty)$ and is positive on $(0,\infty)$ since W_1^{-1} is continuous and nondecreasing on $[0,\infty)$. Moreover, $\upsilon_2(W_1^{-1}(\psi)) \propto \upsilon_3(W_1^{-1}(\psi)) \propto \cdots \propto \upsilon_{k+1}(W_1^{-1}(\psi))$. By the inductive assumption, we have

$$\Xi(m,n) \le \psi^{-1}\{\Upsilon_{k+1}^{-1}[\Upsilon_{k+1}(\theta_1(M,N,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \tilde f_{k+1}(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1} \frac{\Delta_3 c_k(M,N,s,n)}{\phi_{k+1}(\Upsilon_k^{-1}(\theta_k(m_0,n_0,s,n_0)))}]\}, \tag{13}$$

for $m_0 \le m \le \min\{M,M_3\}$ and $n_0 \le n \le \min\{N,N_3\}$, where

$$\Upsilon_{i+1}(u) = \int_{\tilde u_{i+1}}^{u} \frac{dz}{\upsilon_{i+1}(W_1^{-1}(\psi(z)))},$$

$u>0$, $\Phi_1 = I$ (Identity), $\tilde{u}_{i+1} = W_1(\psi(u_{i+1}))$, Υ_{i+1}^{-1} is the inverse of Υ_{i+1},

$$\phi_{i+1}(u) = \frac{\upsilon_{i+1}(W_1^{-1}(\psi(u)))}{\upsilon_i(W_1^{-1}(\psi(u)))} = \frac{w_{i+1}(\psi^{-1}(W_1^{-1}(\psi(u))))}{w_i(\psi^{-1}(W_1^{-1}(\psi(u))))}, \qquad i=1,\cdots,k,$$

$$\theta_{i+1}(M,N,m,n) = \Upsilon_{i+1}(\theta_1(M,N,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{i+1}(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1}\frac{\Delta_3\theta_i(M,N,s,n)}{\phi_{i+1}(\Upsilon_i^{-1}(\theta_i(m_0,n_0,s,n_0)))}, \quad i=1,\cdots,k-1,$$

and M_3 and N_3 are positive integers satisfying

$$\Upsilon_{i+1}(\theta_1(M,N,m_0,N_3)) + \sum_{s=m_0}^{M_3-1}\sum_{t=n_0}^{N_3-1}\tilde{f}_{i+1}(M,N,s,t) + \sum_{s=m_0}^{M_3-1}\frac{\Delta_3\theta_i(M,N,s,N_3)}{\phi_{i+1}(\Upsilon_i^{-1}(\theta_i(m_0,n_0,s,n_0)))}$$
$$\le \int_{\tilde{u}_{i+1}}^{W_1(\infty)}\frac{dz}{\upsilon_{i+1}(W_1^{-1}(\psi(z)))}, \quad i=1,\cdots,k. \tag{14}$$

Note that

$$\Upsilon_i(u) = \int_{\tilde{u}_i}^{u}\frac{dz}{\upsilon_i(W_1^{-1}(\psi(z)))} = \int_{W_1(\psi(u_i))}^{u}\frac{w_1(\psi^{-1}(W_1^{-1}(\psi(z))))dz}{w_i(\psi^{-1}(W_1^{-1}(\psi(z))))}$$
$$= \int_{u_i}^{W_1^{-1}(\psi(u))}\frac{dz}{w_i(\psi^{-1}(z))} = W_i\circ W_1^{-1}\circ\psi(u), \qquad i=2,\cdots,k+1,$$

and

$$\phi_{i+1}(\Upsilon_i^{-1}(u)) = \frac{w_{i+1}(\psi^{-1}(W_1^{-1}(\psi(\Upsilon_i^{-1}(u)))))}{w_i(\psi^{-1}(W_1^{-1}(\psi(\Upsilon_i^{-1}(u)))))}$$
$$= \frac{w_{i+1}(\psi^{-1}(W_1^{-1}(\psi(\psi^{-1}(W_1(W_i^{-1}(u)))))))}{w_i(\psi^{-1}(W_1^{-1}(\psi(\psi^{-1}(W_1(W_i^{-1}(u)))))))}$$
$$= \frac{w_{i+1}(\psi^{-1}(W_i^{-1}(u)))}{w_i(\psi^{-1}(W_i^{-1}(u)))} = \varphi_{i+1}(W_i^{-1}(u)), \qquad i=1,\cdots,k.$$

Thus, using the fact that $\theta_1(M,N,m_0,n) = W_1(r_1(M,N,m_0,n))$, we have from (13) that

$$u(m,n) \le \psi^{-1}(r_1(M,N,m,n)+z(m,n)) = \psi^{-1}(W_1^{-1}(\Xi(m,n)))$$
$$\le \psi^{-1}\{W_{k+1}^{-1}[W_{k+1}(W_1^{-1}(\theta_1(M,N,m_0,n))) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{k+1}(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1}\frac{\Delta_3\theta_k(M,N,s,n)}{\varphi_{k+1}(W_k^{-1}(\theta_k(m_0,n_0,s,n_0)))}]\}, \tag{15}$$

for $m_0 \le m \le \min\{M, M_3\}$ and $n_0 \le n \le \min\{N, N_3\}$.

In the following, we prove that $\theta_i(M,N,m,n) = r_{i+1}(M,N,m,n)$ by induction again. It is clear that $\theta_1(M,N,m,n) = r_2(M,N,m,n)$ for $i=1$. Suppose that

$$\theta_l(M,N,m,n) = r_{l+1}(M,N,m,n),$$

for $i=l$. We have

$$\theta_{l+1}(M,N,m,n) = \Upsilon_{l+1}(\theta_1(M,N,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{l+1}(M,N,s,t)$$
$$+\sum_{s=m_0}^{n-1}\frac{\Delta_3\theta_l(M,N,s,n)}{\phi_{l+1}(\Upsilon_l^{-1}(\theta_l(m_0,n_0,s,n_0)))}$$
$$= W_{l+1}(r_1(M,N,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{l+1}(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1}\frac{\Delta_3 r_{l+1}(M,N,s,n)}{\varphi_{l+1}(W_l^{-1}(r_{l+1}(m_0,n_0,s,n_0)))}$$
$$= r_{l+2}(M,N,m,n),$$

where $\theta_1(M,N,m_0,n) = W_1(r_1(M,N,m_0,n))$ is applied. It implies that it is true for $i = l+1$. Thus, $\theta_i(M,N,m,n) = r_{i+1}(M,N,m,n)$ for $i = 1,\cdots,k$. (14) becomes

$$W_{i+1}(r_1(M,N,m_0,N_3)) + \sum_{s=m_0}^{M_3-1}\sum_{t=n_0}^{N_3-1}\tilde{f}_{i+1}(M,N,s,t) + \sum_{s=m_0}^{M_3-1}\frac{\Delta_3 r_{i+1}(M,N,s,N_3)}{\varphi_{i+1}(W_i^{-1}(r_{i+1}(m_0,n_0,s,n_0)))}$$
$$\le \int_{\tilde{u}_{i+1}}^{W_1(\infty)}\frac{dz}{\upsilon_{i+1}(W_1^{-1}(\psi(z)))} = \int_{\tilde{u}_{i+1}}^{W_1(\infty)}\frac{w_1(W_1^{-1}(\psi(z)))}{w_{i+1}(W_1^{-1}(\psi(z)))}dz = \int_{u_{i+1}}^{\infty}\frac{dz}{w_{i+1}(\psi^{-1}(z))}, \tag{16}$$

for $i = 1,\cdots,k$. It implies that we may choose $w_1(u) = \sqrt{u},\ w_2(u) = u$ and $N_3 = N_2$. Thus, (15) becomes

$$u(m,n) \le \psi^{-1}\{W_{k+1}^{-1}[W_{k+1}(r_1(M,N,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{k+1}(M,N,s,t)$$
$$+\sum_{s=m_0}^{m-1}\frac{\Delta_3 r_{k+1}(M,N,s,n)}{\varphi_{k+1}(W_k^{-1}(r_{k+1}(m_0,n_0,s,n_0)))}]\},$$

for $m_0 \le m \le M$ and $n_0 \le n \le N$. It shows that (7) is true for $k+1$. Thus, the claim is proved.

Now we prove (3). Replacing m and n by M and N in (7) respectively, we have

$$u(M,N) \le \psi^{-1}\{W_k^{-1}[W_k(r_1(M,N,m_0,N)) + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\tilde{f}_k(M,N,s,t)$$
$$+\sum_{s=m_0}^{M-1}\frac{\Delta_3 r_k(M,N,s,N)}{\varphi_k(W_{k-1}^{-1}(r_k(m_0,n_0,s,n_0)))}]\}.$$

Since (7) is true for any $M \le M_1$ and $N \le N_1$, we replace M and N by m and n and get

$$u(m,n) \le \psi^{-1}\{W_k^{-1}[W_k(r_1(m,n,m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_k(m,n,s,t)$$
$$+\sum_{s=m_0}^{m-1}\frac{\Delta_3 r_k(m,n,s,n)}{\varphi_k(W_{k-1}^{-1}(r_k(m_0,n_0,s,n_0)))}]\}.$$

This is exactly (3) since $r_1(m,n,m_0,t) = \tilde{a}(m_0,t)$. This proves Theorem 2.3.

Remarks: If we exchange the order of the two sum symbols for s and t, (2) also holds. Therefore, with a suitable modification in the proof, another form of our result can be obtained as follows

$$u(m,n) \le \psi^{-1}\{W_k^{-1}[W_k(\tilde{a}(m,n_0)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_k(m,n,s,t)$$

$$+\sum_{t=n_0}^{n-1}\frac{\Delta_4 r_k(m,n,m,t)}{\varphi_k(W_{k-1}^{-1}(r_k(m_0,n_0,m_0,t)))}]\},\quad m_0 \le m \le \tilde{M}_1, n_0 \le n \le \tilde{N}_1,$$

where $r_k(m,n,s,t)$ is determined by

$$\begin{aligned} r_1(m,n,s,t) &= \tilde{a}(s,t),\\ r_{i+1}(m,n,s,t) &= W_i(r_1(m,n,s,n_0)) + \sum_{\tau=m_0}^{s-1}\sum_{\eta=n_0}^{t-1}\tilde{f}_i(m,n,\tau,\eta)\\ &\quad +\sum_{\eta=n_0}^{t-1}\frac{\Delta_4 r_i(m,n,s,\eta)}{\varphi_i(W_{i-1}^{-1}(r_i(m_0,n_0,m_0,\eta)))},\quad i=1,\cdots,k-1,\\ \Delta_4 r_i(m,n,s,t) &= r_i(m,n,s,t+1) - r_i(m,n,s,t), i=1,\cdots,k, \end{aligned}$$

$\tilde{M}_1$ and $\tilde{N}_1$ are positive integers satisfying

$$W_i(\tilde{a}(\tilde{M}_1,n_0)) + \sum_{s=m_0}^{\tilde{M}_1-1}\sum_{t=n_0}^{\tilde{N}_1-1}\tilde{f}_i(\tilde{M}_1,\tilde{N}_1,s,t) + \sum_{t=n_0}^{\tilde{N}_1-1}\frac{\Delta_4 r_i(\tilde{M}_1,\tilde{N}_1,\tilde{M}_1,t)}{\varphi_i(W_{i-1}^{-1}(r_i(m_0,n_0,m_0,t)))}$$

$$\le \int_{u_i}^{\infty}\frac{dz}{w_i(\psi^{-1}(z))},\quad i=1,\cdots,k.$$

and other functions are defined in Theorem 2.3. Here the condition in Theorem 2.3 that $\Delta_1\tilde{a}(m,n)$ is nondecreasing in n is replaced by the condition that $\Delta_2\tilde{a}(m,n) = \tilde{a}(m,n+1) - \tilde{a}(m,n)$ is nondecreasing in m.

4. APPLICATION TO A DIFFERENCE EQUATION

In this section, we apply our theorem to study the boundedness of solutions of a nonlinear difference equation

$$\psi(u(m,n)) = a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f_1(m,n,s,t)\sqrt{u(s,t)} + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f_2(m,n,s,t)u(s,t), \tag{17}$$

for $m,n \in N_0$, where ψ is a known function satisfying condition (A1) and $u(m,n) > 0$ is an unknown function for $m,n \in N_0$.

Let

$$w_1(u) = \sqrt{u},\qquad w_2(u) = u.$$

Clearly, $w_2(u)w_1(u) = \sqrt{u}$ is nondecreasing for $u > 0$, that is, $w_1 \propto w_2$. Hence, for $u_1,u_2 > 0$, we have

$$W_1(u) = \int_{u_1}^{u}\frac{dz}{w_1(\psi^{-1}(z))},\qquad W_2(u) = \int_{u_2}^{u}\frac{dz}{w_2(\psi^{-1}(z))},$$

Applying Theorem 2.3, we can compute

$$\begin{aligned} &r_1(m,n,s,t) = \tilde{a}(s,t) > 0,\quad r_1(m,n,0,t) = \tilde{a}(0,t),\\ &\Delta_3 r_1(m,n,\tau,t) = \Delta_1\tilde{a}(\tau,t),\\ &\varphi_1(W_0^{-1}(r_1(0,0,\tau,0))) = w_1(\psi^{-1}(\tilde{a}(\tau,0))), \end{aligned}$$

$$r_2(m,n,s,t) = W_1(\tilde{a}(0,t)) + \sum_{\tau=0}^{s-1}\sum_{\eta=0}^{t-1} \tilde{f}_1(m,n,\tau,\eta) + \sum_{\tau=0}^{s-1} \frac{\Delta_1 \tilde{a}(\tau,t)}{w_1(\psi^{-1}(\tilde{a}(\tau,0)))},$$

$$\Delta_3 r_2(m,n,s,t) = \sum_{\eta=0}^{t-1} \tilde{f}_1(m,n,s,\eta) + \frac{\Delta_1 \tilde{a}(s,t)}{w_1(\psi^{-1}(\tilde{a}(s,0)))},$$

$$r_2(0,0,s,0) = W_1(\tilde{a}(0,0)) + \sum_{\tau=0}^{s-1} \frac{\Delta_1 \tilde{a}(\tau,0)}{w_1(\psi^{-1}(\tilde{a}(\tau,0)))},$$

$$\varphi_2(u) = \frac{w_2(\psi^{-1}(u))}{w_1(\psi^{-1}(u))} = \sqrt{\psi^{-1}(u)}.$$

So

$$u(m,n) \le \psi^{-1}\{W_2^{-1}[W_2(\tilde{a}(0,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \tilde{f}_2(m,n,s,t) + \sum_{s=0}^{m-1} \frac{\sum_{\eta=0}^{t-1} \tilde{f}_1(m,n,s,\eta) + \frac{\Delta_1 \tilde{a}(s,n)}{w_1(\psi^{-1}(\tilde{a}(s,0)))}}{\sqrt{\psi^{-1}(W_1^{-1}(r_2(0,0,s,0)))}}]\}, \tag{18}$$

which implies the boundedness of solutions of the difference equation (17).

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