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Fractional Euler-Lagrange Equations of Order (α,β) for Lie Algebroids

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Abstract: The main purpose of this paper is to derive the fractional Euler-Lagrange equations which depend on the Riemann-Liouville derivatives of order $(\alpha,\beta), \alpha > 0, \beta > 0$ for Lie algebroids. The fractional Hamiltonian formalism was also discussed. Two examples in particular the fractional geodesics for Lie algebroids and the Wong's fractional equations which arise in the dynamics of a colored particle in Yang-Mills field and on the falling cat theorem were also derived.

Key Words: Fractional Action-like Variational Approach; Fractional Lagrangian and Hamiltonian Formalisms; Lie Algebroids

1. INTRODUCTION

The Fractional Calculus of Variations (FCV) based on fractional calculus [1-18] was proved recently to be a useful tool for description of weak dissipative and nonconservative dynamical systems with holonomic and nonholonomic constraints. The respective Euler-Lagrange type equations are a subject of current strong research and investigations [19-25]. An extension of Noether's symmetry theorem to the FCV has recently introduced by the author as the fractional action-like variational approach (FALVA) with many interesting applications and features [26-32]. In this work, our main aim is to derive the fractional Euler-Lagrange equation on Lie algebroids. However, the fractional Euler-Lagrange equations based on the FALVA with one parameter α defined on Lie algebroids were investigated more recently [33]. In this paper, we will enlarge the problem and describe the fractional Euler-Lagrange equations which depend on the Riemann-Liouville derivatives of order $(\alpha,\beta), \alpha > 0, \beta > 0$ for Lie algebroids. We will in addition explore the corresponding fractional Hamilton equations.

In fact, since a Lie algebroid is a concept which simply unifies tangent bundles and Lie algebras, naturally, one can expects their relation to classical mechanics. Further, it has proved to be a powerful tool in the investigating of many fundamental problems in applied mathematics in general and differential geometry in particular [34-38]. In the context of classical mechanics, a theory of Lagrangian and Hamiltonian systems on Lie algebroids has been explored in some details using the linear Poisson structure and the dual of the Lie algebroid and the Legendre transformation associated with the regular Lagrangian [34]. Thus, a powerful mathematical structure has emerged. Within FALVA framework, it was proved that the set of admissible curve on a Lie algebroid with fixed endpoints can be endowed with a structure of Banach manifold, that the fractional action integral in FALVA is continuously differentiable and that the

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equations for the critical points are precisely of the fractional Euler-Lagrange equations obtained in FALVA for the given Lagrange system [33].

The fundamental problem of the FCV with Riemann-Liouville fractional integral of order $(\alpha,\beta),\alpha>0,\beta>0$, as introduced by El-Nabulsi-Torres in [39] is the following: consider a smooth manifold M and let L be an admissible smooth Lagrangian function $L:\mathbb{C}^d\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}, d\geq 1$. For any piecewise smooth path $q:[t_0,t_1]\to M$ satisfying the boundary conditions $q(a)=q_a$ and $q(b)=q_b$, the fractional functional associated to L is defined by

$$S^{\alpha,\beta}_{\gamma,(a,b)}[q]=\frac{1}{\Gamma(\alpha)}\int_a^b L\left(D^{\alpha,\beta}_{\gamma}q(\tau),q(\tau),\tau\right)(t-\tau)^{\alpha-1}\,d\tau\,, \tag{1}$$

where the fractional derivative operator of order (α,β) is defined by

$$D^{\alpha,\beta}_{\gamma}=\frac{1}{2}\left[D^{\alpha}_{a_+}-D^{\beta}_{b_+}\right]+\frac{i\gamma}{2}\left[D^{\alpha}_{a_+}+D^{\beta}_{b_+}\right],\quad \gamma\in\mathbb{C}\,. \tag{2}$$

The critical points of $S^{\alpha,\beta}_{\gamma,(a,b)}[q]$ were proved to satisfy the following Euler-Lagrange equations:

$$\begin{aligned}\frac{\partial L}{\partial q}&\left(D^{\alpha,\beta}_{\gamma}q(\tau),q(\tau),\tau\right)-D^{\beta,\alpha}_{-\gamma;\tau}\frac{\partial L}{\partial\dot{q}}\left(D^{\alpha,\beta}_{\gamma}q(\tau),q(\tau),\tau\right)\\&=\frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial\dot{q}}\left(D^{\alpha,\beta}_{\gamma}q(\tau),q(\tau),\tau\right).\end{aligned} \tag{3}$$

where $D^{\beta,\alpha}_{-\gamma,\tau}$ represents the fractional time derivative with respect to τ.

2. FRACTIONAL EULER-LAGRANGE EQUATIONS (α,β) FOR LIE ALGEBROIDS

A Lie algebroid structure on a vector bundle $\pi:E\to M$ is given by a vector bundle morphism $\rho:E\to TM$ over the identity in M, called the anchor map, together with a Lie algebra structure on the space $C^{\infty}(M)$ -module of sections of E determined by the Lie bracket which induces a Lie algebra homomorphism $\bar{\rho}$ from $\mathrm{Sec}(E)\to\chi(M)$ by the anchor map $\rho:E\to TM$ given by $\sigma\in\mathrm{Sec}(E)\to\bar{\rho}(\sigma)(x)\in\chi(M)$, where $\bar{\rho}(\sigma)(x)=\rho(s(x)),\forall x\in M$, satisfying the Leibniz compatibility identity:

$$\left[\sigma,f\eta\right]_E=(\rho(\sigma)f)\eta+f\left[\sigma,\eta\right]_E,\ \forall f\in C^{\infty}(M),\sigma,\eta\in\mathrm{Sec}(E)\,. \tag{4}$$

A vector bundle (E,ξ,M) endowed with a Lie algebroid structure $([\cdot,\cdot]_E,\rho)$ is called Lie algebroid over M and is denoted by $(E,[\cdot,\cdot]_E,\rho)$.

A local coordinate system (x^i) in the base manifold M and a local basis $\{e_\gamma\}$ of section of E, determine the local coordinate system (x^i,y^η) on E. The anchor and the bracket are locally determined by the local structure functions ρ_i^γ and C^i_{jk} structure functions $\rho_i^\gamma,C^k_{ij}\in C^{\infty}(M)$ of $(E,[\cdot,\cdot]_E,\rho)$, which satisfies the following relations which results from the Leibniz identity and the Jacobi identity:

$$\rho_i^\eta \frac{\partial \rho_j^\sigma}{\partial x^\eta} - \rho_j^\eta \frac{\partial \rho_i^\sigma}{\partial x^\eta} = \rho_k^\sigma C_{ij}^k, \eta = \overline{1,n}, \ i = \overline{1,m}, \tag{5}$$

$$\sum_{cyclic(i,j,k)} \left(\rho_i^\eta \frac{\partial C_{jk}^l}{\partial x^\eta} + C_{jk}^h C_{ih}^l \right) = 0, \tag{6}$$

where $\bar{\rho}(e_i) = \rho_i^\eta \partial_\eta$, $\partial_\eta \doteq \partial / \partial x^\eta$ and $\left[e_i, e_j\right] = C_{ij}^k e_k$.

We consider now the space of E-paths on the Lie algebroid denoted by $\mathcal{P}(J,E), J = [a,b]$, which is a differentiable Banach manifold [33].

Theorem 2.1 *Let* $L(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau)$ *be a Lagrangian on the Lie algebroid* E *with admissible curve* q *in* E *and with two fixed endpoints*

$$A, B \in M \subset \mathcal{P}(J,E)_A^B = \left\{ \mathcal{P}(J,E) \middle| \pi(q(a)) = A \text{ and } \pi(q(b)) = B \right\}.$$

The critical points of the fractional action integral $S_{\gamma,(a,b)}^{\alpha,\beta} : \mathcal{P}(J,E) \to \mathbb{R}$ *defined by:*

$$S_{\gamma,(a,b)}^{\alpha,\beta}[q] = \frac{1}{\Gamma(\alpha)} \int_a^b L\left(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau\right) (t-\tau)^{\alpha-1} d\tau, (\alpha,\beta) \in (0,1],$$

on the Banach manifold $\mathcal{P}(J,E)_A^B$ *are exactly those elements of that space which satisfy the following fractional Euler-Lagrange equations of order* (α,β) *:*

$$\delta^r L(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau) = 0,$$

where

$$\begin{aligned} < \delta^r L(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau) > &= < dL(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau), \Sigma_q(\sigma) > \\ &- D_{-\gamma;\tau}^{\beta,\alpha} < \tau_{L(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau)} \circ q, \sigma > \\ &- \frac{1-\alpha}{t-\tau} < \tau_{L(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau)} \circ q, \sigma > . \end{aligned} \tag{7}$$

Here $\tau_L = (\partial L / \partial y^i) dy^i$ *in local coordinates.*

Proof. The tangent space on $\mathcal{P}(J,E)_A^B$ at $q \in \mathcal{P}(J,E)_A^B$ is nothing than the set of vector fields along q are of the form $\Xi_q(\sigma) : \sigma \in \mathrm{Sec}(E) / \sigma(a) = \sigma(b) = 0$. The fractional action is smooth, then:

$$\begin{aligned} 0 = < dS_{\gamma,(a,b)}^{\alpha,\beta}, \Xi_q(f\sigma) > &= \frac{1}{\Gamma(\alpha)} \int_a^b < dL(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau), \Xi_q(f\sigma) > (t-\tau)^{\alpha-1} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \left(\int_a^b \Big[f(\tau) < dL(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau), \Xi_q(\sigma) > \right. \\ &- D_\gamma^{\alpha,\beta} < \tau_{L(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau)} \circ q, \sigma > (t-\tau)^{\alpha-1} \\ &+ (\alpha-1) < \tau_{L(D_\gamma^{\alpha,\beta} q(\tau), q(\tau), \tau)} \circ q, \sigma > (t-\tau)^{\alpha-2} \Big] d\tau \end{aligned}$$

$$+f(\tau) < \tau_{L(D_{\gamma}^{\alpha,\beta}q(\tau),q(\tau),\tau)} \circ q, \sigma > \left(t-\tau\right)^{\alpha-1}\Big|_{a}^{b}\Bigg)$$

$$= \frac{1}{\Gamma(\alpha)}\int_{a}^{b} f(\tau) < \delta^{r} L(D_{\gamma}^{\alpha,\beta}q(\tau),q(\tau),\tau),\sigma(\tau) > d\tau,$$

where we have used the fact that $\Xi_q(f\sigma) = f\sum_q(\sigma) + D_{\gamma}^{\alpha,\beta}\sigma_q^{\mathrm{v}}$. $v=\rho(q)$ is the actual velocity and $D_{\gamma}^{\alpha,\beta}\sigma_q^{\mathrm{v}}$ is the Riemann-Liouville fractional derivative of the canonical vertical lift of σ. This equation is satisfied for every function $f \in C^{\infty}(\mathbb{R})$ and every section $\sigma \in \mathrm{Sec}(E)$. Thus, the critical points are satisfied by $\delta^{r} L(D_{\gamma}^{\alpha,\beta}q(\tau),q(\tau),\tau) = 0$. □

Definition 2.2 *In local coordinates* $\tau_L = (\partial L/\partial y^i)dy^i$, *the fractional Euler-Lagrange equations of order* (α,β) *is:*

$$\rho_i^{\eta}\frac{\partial L}{\partial x^{\eta}}\left(D_{\gamma}^{\alpha,\beta}x^{\eta},x^{\eta},\tau\right) - D_{-\gamma;\tau}^{\beta,\alpha}\frac{\partial L}{\partial y^{i}}\left(D_{\gamma}^{\alpha,\beta}x^{\eta},x^{\eta},\tau\right)$$
$$-\frac{\partial L}{\partial y^{k}}C_{ij}^{k}y^{j} + \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial y^{i}}\left(D_{\gamma}^{\alpha,\beta}x^{\eta},x^{\eta},\tau\right) = 0, \tag{8}$$

where $D_{\gamma}^{\alpha,\beta}x^{\eta} = \rho_i^{\eta}y^{i}$.

Definition 2.3 *The term*

$$< F_{\gamma,\tau}^{\alpha,\beta} \circ q, \sigma > = \frac{1-\alpha}{t-\tau} < \tau_{L(D_{\gamma}^{\alpha,\beta}q(\tau),q(\tau),\tau)} \circ q, \sigma >, \tag{9}$$

is called the fractional decaying friction force for Lie algebroids.

Remark 2.1 When $\beta = 1$, equation (7) is the same as the one obtained in [33].

3. THE FRACTIONAL HAMILTONIAN FORMALISM FOR LIE ALGEBROIDS

We now consider the following general fractional variational problem in local coordinates:

$$S_{\gamma,(a,b)}^{\alpha,\beta}\left[x,u\right] = \frac{1}{\Gamma(\alpha)}\int_{a}^{b} L\left(y(\tau),x(\tau),\tau\right)\left(t-\tau\right)^{\alpha-1}d\tau \to \min \tag{10}$$

$$D_{\gamma}^{\alpha,\beta}x(\tau) = \varphi\left(u(\tau),x(\tau),\tau\right). \tag{11}$$

A necessary optimality conditions to the following problem for Lie algebroids may be obtained if we introduce the augmented fractional action integral [39]:

$$S_{\gamma,(a,b)}^{\alpha,\beta}\left[x,u,p^{\alpha,\beta}\right] = \frac{1}{\Gamma(\alpha)}\int_{a}^{b}\Big[\mathcal{H}^{\alpha,\beta}\left(u(\tau),x(\tau),p^{\alpha,\beta}(\tau),\tau\right)$$
$$-p^{\alpha,\beta}(\tau)D_{\gamma}^{\alpha,\beta}x(\tau)\Big]d\tau, \tag{12}$$

where $p^{\alpha,\beta}$ is the fractional Lagrange multiplier and

$$\mathcal{H}^{\alpha,\beta}\left(u(\tau),x(\tau),p^{\alpha,\beta}(\tau),\tau\right)=L\left(u(\tau),x(\tau),\tau\right)(t-\tau)^{\alpha-1}$$

$$+p^{\alpha,\beta}(\tau)\varphi\left(u(\tau),x(\tau),\tau\right), \tag{13}$$

is the fractional Hamiltonian.

Theorem 3.1: *Let* $L\left(u(\tau),x(\tau),\tau\right)$ *be a hyperregular Lagrangian on the Lie algebroid* E *with admissible curve* q *in* E *and with two fixed endpoints*

$$A,B\in M\subset\mathcal{P}(J,E)_A^B=\left\{\mathcal{P}(J,E)\middle|\pi(q(a))=A \text{ and } \pi(q(b))=B\right\}.$$

If (x,u) *is a minimizer of problem (9), then there exists a co-vector function* $p^{\alpha,\beta}$ *such that the following conditions hold:*

- *The fractional Hamiltonian system for Lie algebroids:*

$$D_{-\gamma;\tau}^{\beta,\alpha}\dot{x}^{\alpha}(\tau)=\rho_i^{\eta}\frac{\partial\mathcal{H}^{\alpha,\beta}}{\partial p_i^{\alpha,\beta}}\left(u(\tau),x(\tau),p^{\alpha,\beta}(\tau),\tau\right), \tag{14}$$

$$D_{-\gamma;\tau}^{\beta,\alpha}p_i^{\alpha,\beta}(\tau)=-\rho_i^{\eta}\frac{\partial\mathcal{H}^{\alpha,\beta}}{\partial x^{\eta}}\left(u(\tau),x(\tau),p^{\alpha,\beta}(\tau),\tau\right)$$

$$-C_{ij}^{k}p_k^{\alpha,\beta}(\tau)\frac{\partial\mathcal{H}^{\alpha,\beta}}{\partial p_j^{\alpha,\beta}}\left(u(\tau),x(\tau),p^{\alpha,\beta}(\tau),\tau\right), \tag{15}$$

- *The fractional stationary condition for Lie algebroids:*

$$\frac{\partial\mathcal{H}^{\alpha,\beta}}{\partial u}\left(u(\tau),x(\tau),p^{\alpha,\beta}(\tau),\tau\right)=0. \tag{16}$$

Remark 3.1 From equation (15), equations (13) and (14) yield:

$$D_{-\gamma;\tau}^{\beta,\alpha}\left[\frac{\partial L}{\partial u}\left(D_{\gamma}^{\alpha,\beta}x^{\eta},x^{\eta},\tau\right)(t-\tau)^{\alpha-1}\right]$$

$$-\rho_i^{\eta}\frac{\partial L}{\partial x^{\eta}}\left(D_{\gamma}^{\alpha,\beta}x^{\eta},x^{\eta},\tau\right)(t-\tau)^{\alpha-1}+\frac{\partial L}{\partial y^k}C_{ij}^{k}y^{j}=0, \tag{17}$$

where $u(\tau)=D_{\gamma}^{\alpha,\beta}x(\tau)$. Equation (17) is equivalent to equation (9). Thus, theorem 3.1 is a generalization of theorem 2.1 to the fractional optimal control problem for Lie algebroids.

Remark 3.2 Equations (14), (15) and (16) are obtained easily if we apply the fractional Euler-Lagrange equations (9) of order (α,β) to the augmented fractional action integral with respect to $p^{\alpha,\beta}$, x and u respectively.

Remark 3.3 The fractional Hamilton dynamics for equation (9) on the dual bundle E^{*} is represented by the fractional vector field:

$$\mathcal{D}_{\alpha,\beta}^{r}(x,\xi)=\rho_i^{\eta}\frac{\partial\mathcal{H}^{\alpha,\beta}}{\partial p_i^{\alpha,\beta}}\frac{\partial}{\partial x^{\eta}}$$

$$-\left(\rho_i^\eta \frac{\partial \mathcal{H}^{\alpha,\beta}}{\partial x^\eta} + C_{ij}^k p_k^{\alpha,\beta}(\tau) \frac{\partial \mathcal{H}^{\alpha,\beta}}{\partial p_j^{\alpha,\beta}}\right) \frac{\partial}{\partial \xi_i} . \tag{18}$$

Remark 3.4 It is well-know that Noether's theorem is a consequence of the existence of a variational description of the dynamical problem. In the standard case when $\alpha = \beta = 1$, the Noether energy is conserved. This statement does not hold for the case where $(\alpha,\beta) \in (0,1]$ over Lie algebroids. To solve the problem, one may use the following new notion of fractional constant of notion C: $D_{-\gamma;\tau}^{\beta,\alpha} C = 0$.

Example 3.1 As a simple example, we will discuss the geodesics for Lie algebroids. In local coordinates, the Lagrangian is $L = (1/2) g_{ij}(x) y^i y^j$ where the metric $g = g_{ij}(x) e^i \otimes e^j$ is expected to induce an isomorphism of the vector bundles $\tilde{g} : E \to E^*$. The fractional Euler-Lagrange equations read:

$$\begin{aligned} D_{-\gamma;\tau}^{\beta,\alpha}\left[g_{ik} y^i\right](D_\gamma^{\alpha,\beta} x^\eta, x^\eta, \tau) = \\ \frac{1-\alpha}{t-\tau} g_{ik} y^i (D_\gamma^{\alpha,\beta} x^\eta, x^\eta, \tau) \\ +\left(C_{ik}^j g_{sj} + \frac{1}{2}\rho_k^\eta \frac{\partial g_{ij}}{\partial x^\eta}\right) y^i (D_\gamma^{\alpha,\beta} x^\eta, x^\eta, \tau) y^j (D_\gamma^{\alpha,\beta} x^\eta, x^\eta, \tau), \end{aligned} \tag{19}$$

which can be rewritten in the form

$$D_{-\gamma;\tau}^{\beta,\alpha} y^i + \Gamma_{ij}^l y^i y^j - \frac{1-\alpha}{t-\tau} y^l = 0, k = \overline{1,n} , \tag{20}$$

where

$$\begin{aligned} \Gamma_{ij}^l = \frac{1}{2} g^{kl}\left(\rho_j^\eta \frac{\partial g_{ik}}{\partial x^\eta} + \rho_i^\eta \frac{\partial g_{jk}}{\partial x^\eta} - \rho_k^\eta \frac{\partial g_{ij}}{\partial x^\eta}\right) \\ - C_{ik}^s g_{sj} - C_{jk}^s g_{si} , \end{aligned} \tag{21}$$

are the Christoffel symbols for Lie algebroids. Equation (20) with $D_\gamma^{\alpha,\beta} x^\eta = \rho_i^\eta y^i$ are the fractional r – geodesics equations of order (α,β) for $\rho : E \to TM$.

Example 3.2 Another example concerns the Wong's equations which arise in the dynamics of a colored particle in Yang-Mills field and on the falling cat theorem [36-38]. The Lagrangian and the Hamiltonian of the theory on the Lie algebroid E are given by:

$$L(x^\eta, D_\gamma^{\alpha,\beta} x^\eta, \overline{\mathrm{v}}^{\mathrm{i}}) = \frac{1}{2}\left(h_{ij} \overline{\mathrm{v}}^i \overline{\mathrm{v}}^j + g_{\eta\sigma} u^\eta u^\sigma\right), , \tag{22}$$

$$\mathcal{H}(x^\eta, p_\eta, \overline{p}_i) = \frac{1}{2}\left(h^{ij} \overline{p}_i \overline{p}_j + g^{\eta\sigma} p_\eta p_\sigma\right), \tag{23}$$

$$u(\tau) = D_\gamma^{\alpha,\beta} x(\tau) , \tag{24}$$

where $(x^\eta, D_\gamma^{\alpha,\beta} x^\eta, \overline{\mathrm{v}}^{\mathrm{i}})$ is the corresponding dual fibered coordinates on TQ/G and $(x^\eta, p_\eta, \overline{p}_i)$ is the dual coordinates on T^*Q/G , G being a compact Lie group, $E = TM \times \mathcal{A}$ with $\dim TM = m$ and $\dim \mathcal{A} = n$, $\mathcal{A}$ being an arbitrary Lie $\mathbb{R}$ -algebra of dimension n . h is a metric on $\mathcal{A}$ and g is assumed to be a Riemannian metric on M . The corresponding fractional Euler-Lagrange equations of order (α,β) are given by

$$D_{-\gamma;\tau}^{\beta,\alpha}\overline{\nabla}^{k}=C_{ji}^{l}h^{ik}h_{ls}\overline{\nabla}^{j}\overline{\nabla}^{s}+\frac{1-\alpha}{t-\tau}\overline{\nabla}^{k},k=\overline{1,n}, \tag{25}$$

$$D_{-\gamma;\tau}^{\beta,\alpha}\left(D_{-\gamma;\tau}^{\beta,\alpha}x^{\sigma}\right)=\Gamma_{\lambda\varepsilon}^{\sigma}D_{-\gamma;\tau}^{\beta,\alpha}x^{\lambda}D_{-\gamma;\tau}^{\beta,\alpha}x^{\varepsilon}+\frac{1-\alpha}{t-\tau}D_{-\gamma;\tau}^{\beta,\alpha}x^{\sigma}, \tag{26}$$

which are the fractional Wong's equations of order (α,β) and they are exactly the fractional Euler-Lagrange equations of order (α,β) .

Remark 3.5 Being that symmetry plays a capital role on classical and modern physics, one can write the fractional Euler-Lagrange equations of order (α,β) for systems with symmetry based on the Atiyah algebroid. This will be explored in a future work. See [40-43] for other interesting applications.

4. CONCLUSIONS

Using the fractional action-like variational approach together with Riemann-Liouville fractional derivatives of order (α,β) , we generalize previous results of the FCV for Lie algebroids. The generalized fractional Euler-Lagrange equations, the generalized Hamilton equations and the generalized geodesics equation for Lie algebroids are derived. It was also proved that the fractional Wong's equations of order (α,β) are exactly the fractional Euler-Lagrange equations of same order. It would be of interest in the future to explore the fractional Lagrange-d'Alembert-Poincaré equations on Atiyah algebroids and problems with symmetry. Another interesting problem concerns the gauge invariant Lagrangians in terms of groupoids fractional action of order (α,β) on complex plane [44].

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