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A New Error Bound for Shifted Surface Spline Interpolation

Lin-Tian Luh[1]

Abstract: Shifted surface spline is a frequently used radial function for scattered data interpolation. The most frequently used error bounds for this radial function are the one raised by Wu and Schaback in [17] and the one raised by Madych and Nelson in [14]. Both are $O(d^l)$ as $d \to 0$, where l is a positive integer and d is the well-known fill-distance which roughly speaking measures the spacing of the data points. Then RBF people found that there should be an error bound of the form $O(\omega^{\frac{1}{d}})$ because shifted surface spline is smooth and every smooth function shares this property. The only problem was that the value of the cucial constant ω was unknown. Recently Luh raised an exponential-type error bound with convergence rate $O(\omega^{\frac{1}{d}})$ as $d \to 0$ where $0 < \omega < 1$ is a fixed constant which can be accurately computed [11]. Although the exponential-type error bound converges much faster than the algebraic-type error bound, the constant ω is intensely influenced by the dimension n in the sense $\omega \to 1$ rapidly as $n \to \infty$. Here the variable x of both the interpolated and interpolating functions lies in R^n. In this paper we present an error bound which is $O(\sqrt{d}\omega'^{\frac{1}{d}})$ where $0 < \omega' < 1$ is a fixed constant for any fixed n, and is only mildly influenced by n. In other words, $\omega' \to 1$ very slowly as $n \to \infty$, and $\omega' << \omega$, especially for high dimensions. Moreover, ω' can be accurately computed without slight difficulty. This provides a good error estimate for high-dimensional problems which are of growing importance.

Key Words: Radial Basis Function; Shifted Surface Spline; Error Bound; High-Dimensional Approximation

1. INTRODUCTION

In the theory of radial basis functions, it's well known that any conditionally positive definite radial function can form an interpolant for any set of scattered data. We make a simple sketch of this process as follows.

Suppose h is a continuous function on R^n which is strictly conditionally positive definite of order m. For any set of data points (x_j, f_j), $j = 1, \ldots, N$, where $X = \{x_1, \ldots, x_N\}$ is a subset of R^n and the $f_j's$ are real or complex numbers, there is a unique function of the form

$$s(x) = p(x) + \sum_{j=1}^{N} c_j h(x - x_j), \tag{1}$$

[1]Department of Mathematics, Providence University, Shalu, Taichung, Taiwan.
*Corresponding author. E-mail address: ltluh@pu.edu.tw.
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where $p(x)$ is a polynomial in P^n_{m-1}, satisfying

$$\sum_{j=1}^{N} c_j q(x_j) = 0, \tag{2}$$

for all polynomials q in P^n_{m-1} and

$$p(x_i) + \sum_{j=1}^{N} c_j h(x_i - x_j) = f_i,\ i = 1, \ldots, N, \tag{3}$$

if X is a determining set for P^n_{m-1}.

A complete treatment of this topic can be seen in [13] and many other papers.

The function $s(x)$ is called the h-spline interpolant of the data points and is of central importance in the theory of radial basis functions. In this paper h always denotes a radial function in the sense that the value of $h(x)$ is completely determined by the norm $|x|$ of x. Here, P^n_{m-1} denotes the class of those n-variable polynomials of degree not more than $m-1$.

In this paper we are mainly interested in a radial function called shifted surface spline defined by

$$\begin{aligned} h(x) &:= (-1)^m(|x|^2 + c^2)^{\frac{\lambda}{2}} log(|x|^2 + c^2)^{\frac{1}{2}},\ \lambda \in Z_+,\ m = 1 + \frac{\lambda}{2},\ c > 0, \\ & \quad x \in R^n,\ \lambda, n\ even, \end{aligned} \tag{4}$$

where $|x|$ is the Euclidean norm of x, and λ, c are constants. In fact, the definition of shifted surface spline covers odd dimensions. For odd dimensions, it's of the form

$$\begin{aligned} h(x) &:= (-1)^{\lceil \lambda - \frac{n}{2} \rceil}(|x|^2 + c^2)^{\lambda - \frac{n}{2}},\ n\ odd,\ \lambda \in Z_+ = \{1, 2, 3, \ldots\} \\ & \quad and\ \lambda > \frac{n}{2}. \end{aligned} \tag{5}$$

However, this is just multiquadric and we treated it in another paper [12]. Therefore we will not discuss it. Instead, we will focus on (4) and even dimensions only.

1.1 Polynomials and Simplices

Let E denote an n-dimensional simplex [4] with vertices $v_1, \ldots, v_{n+1}$. If we adopt barycentric coordinates, then any point $x \in E$ can be written as a convex combination of the vertices:

$$x = \sum_{i=1}^{n+1} \lambda_i v_i,\ \sum_{i=1}^{n+1} \lambda_i = 1,\ \lambda_i \geq 0.$$

We define the "**evenly spaced**" points of degree k to be those points whose barycentric coordinates are of the form

$$(k_1/k, k_2/k, \ldots, k_{n+1}/k),\ k_i\ nonnegative\ integers\ and\ k_1 + \cdots + k_{n+1} = k.$$

It's easily seen that the number of such points is exactly $dimP^n_k$, i.e., the dimension of P^n_k. In this section we use N to denote $dimP^n_k$. What's noteworthy is that our notion of being evenly spaced is very different from the notion of grid. Moreover the shape of the simplex is determined by its vertices and is very flexible. Therefore the distribution of the evenly spaced points has a lot of freedom.

The above-defined evenly spaced points can induce a polynomial interpolation process as follows. Let $x_1, \ldots, x_N$ be the evenly spaced points in E of degree k. The associated Lagrange polynomials l_i of degree

k are defined by the condition $l_i(x_j) = \delta_{ij},\ 1 \le i, j \le N$. For any continuous map $f \in C(E)$, $(\Pi_k f)(x) := \sum_{i=1}^N f(x_i)l_i(x)$ is its interpolating polynomial. If both spaces are equipped with the supremum norm, the mapping

$$\Pi_k : C(E) \to P_k^n$$

has a well-known norm

$$\|\Pi_k\| = \max_x \sum_{i=1}^N |l_i(x)|$$

which is the maximum value of the famous Lebesgue function. It's easily seen that for any $p \in P_k^n$,

$$\|p\|_\infty := \max_{x\in E} |p(x)| \le \|\Pi_k\| \max_{1\le i\le N} |p(x_i)|.$$

The next result is important in our construction of the error bound, and we cite it directly from [2].

Lemma 1.1 *For the above evenly spaced points* $\{x_1, \ldots, x_N\}$, $\|\Pi_k\| \le \begin{pmatrix} 2k-1 \\ k \end{pmatrix}$. *Moreover, as* $n \to \infty$, $\|\Pi_k\| \to \begin{pmatrix} 2k-1 \\ k \end{pmatrix}$.

Then we need another lemma which must be proven because it plays a crucial role in our development.

Lemma 1.2 *Let* $Q \subseteq R^n$ *be an* n *simplex in* R^n *and* Y *be the set of evenly spaced points of degree* k *in* Q. *Then, for any point* x *in* Q, *there is a measure* σ *supported on* Y *such that*

$$\int p(y)d\sigma(y) = p(x)$$

for all p *in* P_k^n, *and*

$$\int d|\sigma|(y) \le \begin{pmatrix} 2k-1 \\ k \end{pmatrix}.$$

Proof. Let $Y = \{y_1, \ldots, y_N\}$ be the set of evenly spaced points of degree k in Q. Denote P_k^n by V. For any $x \in Q$, let δ_x be the point-evaluation functional. Define $T : V \to T(V) \subseteq R^N$ by $T(v) = (\delta_{y_i}(v))_{y_i \in Y}$. Then T is injective. Define $\tilde{\psi}$ on $T(V)$ by $\tilde{\psi}(w) = \delta_x(T^{-1}w)$. By the Hahn-Banach theorem, $\tilde{\psi}$ has a norm-preserving extension $\tilde{\psi}_{ext}$ to R^N. By the Riesz representation theorem, each linear functional on R^N can be represented by the inner product with a fixed vector. Thus, there exists $z \in R^N$ with

$$\tilde{\psi}_{ext}(w) = \sum_{j=1}^N z_j w_j$$

and $\|z\|_{(R^N)^*} = \|\tilde{\psi}_{ext}\|$. If we adopt the l_∞-norm on R^N, the dual norm will be the l_1-norm. Thus $\|z\|_{(R^N))^*} = \|z\|_1 = \|\tilde{\psi}_{ext}\| = \|\tilde{\psi}\| = \|\delta_x T^{-1}\|$.

Now, for any $p \in V$, by setting $w = T(p)$, we have

$$\delta_x(p) = \delta_x(T^{-1}w) = \tilde{\psi}(w) = \tilde{\psi}_{ext}(w) = \sum_{j=1}^N z_j w_j = \sum_{j=1}^N z_j \delta_{y_j}(p).$$

This gives

$$p(x) = \sum_{j=1}^N z_j p(y_j), \tag{6}$$

where $|z_1| + \cdots + |z_N| = \|\delta_x T^{-1}\|$.

Note that

$$\begin{aligned}
\|\delta_x T^{-1}\| &= \sup_{\substack{w \in T(V) \\ w \neq 0}} \frac{\|\delta_x T^{-1}(w)\|}{\|w\|_{R^N}} \\
&= \sup_{\substack{w \in T(V) \\ w \neq 0}} \frac{|\delta_x p|}{\|T(p)\|_{R^N}} \\
&\leq \sup_{\substack{p \in V \\ p \neq 0}} \frac{|p(x)|}{\max_{j=1,\ldots,N} |p(y_j)|} \\
&\leq \sup_{\substack{p \in V \\ p \neq 0}} \frac{\|\Pi_k\| \max_{j=1,\ldots,N} |p(y_j)|}{\max_{j=1,\ldots,N} |p(y_j)|} \\
&= \|\Pi_k\| \\
&\leq \begin{pmatrix} 2k-1 \\ k \end{pmatrix}.
\end{aligned}$$

Therefore $|z_1| + \cdots + |z_N| \leq \begin{pmatrix} 2k-1 \\ k \end{pmatrix}$ and our lemma follows immediately from (6) by letting $\sigma(\{y_j\}) = z_j,\ j = 1, \ldots, N$.

1.2 Radial Functions and Borel Measures

Our theory is based on a fundamental fact that any continuous conditionally positive definite radial function corresponds to a unique positive Borel measure. Before discussing this property in detail, we first clarify some symbols and definitions. In this paper $\mathcal{D}$ denotes the space of all compactly supported and infinitely differentiable complex-valued functions on R^n. For each function ϕ in $\mathcal{D}$, its Fourier transform is

$$\hat{\phi}(\xi) = \int e^{-i<x,\xi>} \phi(x) dx.$$

Then we have the following lemma which was introduced in [8] but modified by Madych and Nelson in [14].

Lemma 1.3 *For any continuous conditionally positive definite function h on R^n of order m, there are a unique positive Borel measure μ on $R^n \sim \{0\}$ and constants $a_r,\ |r| = 2m$ such that for all $\psi \in \mathcal{D}$,*

$$\begin{aligned}
\int h(x)\psi(x)dx &= \int \{\hat{\psi}(\xi) - \hat{\chi}(\xi) \sum_{|r|<2m} D^r \hat{\psi}(0) \frac{\xi^r}{r!}\} d\mu(\xi) \\
&\quad + \sum_{|r|\leq 2m} D^r \hat{\psi}(0) \frac{a_r}{r!}
\end{aligned} \tag{7}$$

, where for every choice of complex numbers $c_\alpha,\ |\alpha| = m$,

$$\sum_{|\alpha|=m} \sum_{|\beta|=m} a_{\alpha+\beta} c_\alpha \overline{c_\beta} \geq 0.$$

Here χ is a function in $\mathcal{D}$ such that $1-\hat{\chi}(\xi)$ has a zero of order $2m+1$ at $\xi = 0$; both of the integrals

$$\int_{0<|\xi|<1} |\xi|^{2m} d\mu(\xi), \int_{|\xi|\geq 1} d\mu(\xi)$$

are finite. The choice of χ affects the value of the coefficients a_r for $|r| < 2m$.

2. MAIN RESULT

In order to show our main result, we need some lemmas, including the famous Stirling's formula.

Stirling's Formula: $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$.

The approximation is very reliable even for small n. For example, when $n = 10$, the relative error is only 0.83%. The larger n is, the better the approximation is. For further details, we refer the reader to [5, 6].

Lemma 2.1 *For any positive integer k,*

$$\frac{\sqrt{(2k)!}}{k!} \leq 2^k.$$

Proof. This inequality holds for $k = 1$ obviously. We proceed by induction.

$$\begin{aligned}
\frac{\sqrt{[2(k+1)]!}}{(k+1)!} &= \frac{\sqrt{(2k+2)!}}{k!(k+1)} = \frac{\sqrt{(2k)!}}{k!} \cdot \frac{\sqrt{(2k+2)(2k+1)}}{k+1} \\
&\leq \frac{\sqrt{(2k)!}}{k!} \cdot \frac{\sqrt{(2k+2)^2}}{k+1} \leq 2^k \cdot \frac{(2k+2)}{k+1} = 2^{k+1}.
\end{aligned}$$

Now recall that the function h defined in (4) is conditionally positive definite of order $m = 1 + \frac{\lambda}{2}$. This can be found in [3] and many relevant papers. Its Fourier transform [7] is

$$\hat{h}(\theta) = l(\lambda, n)|\theta|^{-\lambda-n}\tilde{\mathcal{K}}_{\frac{n+\lambda}{2}}(c|\theta|), \tag{8}$$

where $l(\lambda, n) > 0$ is a constant depending on λ and n, and $\tilde{\mathcal{K}}_\nu(t) = t^\nu \mathcal{K}_\nu(t)$, $\mathcal{K}_\nu(t)$ being the modified Bessel function of the second kind [1]. Then we have the following lemma.

Lemma 2.2 *Let h be as in (4) and m be its order of conditional positive definiteness. There exists a positive constant ρ such that*

$$\int_{R^n} |\xi|^k d\mu(\xi) \leq l(\lambda, n) \cdot \sqrt{2\pi} \cdot n \cdot \alpha_n \cdot c^{\lambda-k} \cdot \Delta_0 \cdot \rho^k \cdot k!, \tag{9}$$

for all integer $k \geq 2m+2$ where μ was defined in Lemma1.3, α_n denotes the volume of the unit ball in R^n, c is as in (4), and Δ_0 is a positive constant.

Proof. We first transform the integral of the left-hand side of the inequality into a simpler form.

$$\begin{aligned}
& \int_{R^n} |\xi|^k d\mu(\xi) \\
= & \int_{R^n} |\xi|^k l(\lambda, n)\tilde{\mathcal{K}}_{\frac{n+\lambda}{2}}(c|\xi|)|\xi|^{-\lambda-n} d\xi \text{ (by (8) of this paper and Theorem5.2 of [14])} \\
= & \ l(\lambda, n)c^{\frac{n+\lambda}{2}} \int_{R^n} |\xi|^{k-\frac{n+\lambda}{2}} \cdot \mathcal{K}_{\frac{n+\lambda}{2}}(c|\xi|) d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad l(\lambda,n)c^{\frac{n+\lambda}{2}}\sqrt{2\pi}\int_{R^n}|\xi|^{k-\frac{n+\lambda}{2}}\cdot\frac{1}{\sqrt{c|\xi|}\cdot e^{c|\xi|}}d\xi \text{ (See the explanation after the deduction.)}\\
&= \quad l(\lambda,n)c^{\frac{n+\lambda}{2}}\cdot\sqrt{2\pi}\cdot n\cdot\alpha_n\int_0^\infty r^{k-\frac{n+\lambda}{2}}\cdot\frac{r^{n-1}}{\sqrt{cr}\cdot e^{cr}}dr\\
&= \quad l(\lambda,n)c^{\frac{n+\lambda}{2}}\sqrt{2\pi}\cdot n\cdot\alpha_n\cdot\frac{1}{\sqrt{c}}\int_0^\infty\frac{r^{k+\frac{n-\lambda-3}{2}}}{e^{cr}}dr\\
&= \quad l(\lambda,n)c^{\frac{n+\lambda}{2}}\sqrt{2\pi}\cdot n\cdot\alpha_n\cdot\frac{1}{\sqrt{c}}\cdot\frac{1}{c^{k+\frac{n-\lambda-1}{2}}}\int_0^\infty\frac{r^{k+\frac{n-\lambda-3}{2}}}{e^r}dr\\
&= \quad l(\lambda,n)\sqrt{2\pi}\cdot n\cdot\alpha_n\cdot c^{\lambda-k}\int_0^\infty\frac{r^{k'}}{e^r}dr \textit{ where } k'=k+\frac{n-\lambda-3}{2}.
\end{aligned}
$$

Note that $k\geq 2m+2=4+\lambda$ implies $k'\geq\frac{n+\lambda+5}{2}>0$. In the preceding deduction we used Lemma 5.13 and 5.14 of [16] to build the inequality in the fourth line. The two lemmas were used for $c|\xi|\geq 1$ and $0<c|\xi|\leq 1$, respectively. For $c|\xi|\geq 1$, the inequality is obviously acceptable, with a very small gap which can be ignored. For $0<c|\xi|\leq 1$, it's also suitable because k is a very large number, making the integral nearly zero when c is not close to zero. The only trouble for $0<c|\xi|\leq 1$ is that if c is extremely small, the gap may be larger. However, if c is very small, the function h in (4) is essentially the famous radial function thin plate spline which is beyond the scope of this paper. Moreover, k is a fixed number making the influence of very small c quite limited and can be ignored. In the forthcoming Theorem2.3 and Corollary2.4 it will be clear that $k\to\infty$ as the essential fill-distance $\delta\to 0$. In almost all cases $k>>0$.

Now we divide the proof into three cases. Let $k''=\lceil k'\rceil$ which is the smallest integer greater than or equal to k'.

case 1. Assume $k''>k$. Let $k''=k+s$. We first note that $\int_0^1\frac{r^{k'}}{e^r}dr\sim 0$ because $k'\geq 4.5$. The larger k' is, the more accurate it is. Also, by the technique of integration by parts, we find

$$\int_0^\infty\frac{r^{k+1}}{e^r}dr=(k+1)\int_0^\infty\frac{r^k}{e^r}dr.$$

Therefore

$$\int_0^\infty\frac{r^{k'}}{e^r}dr\leq\int_0^\infty\frac{r^{k''}}{e^r}dr=k''!=(k+s)(k+s-1)\cdots(k+1)k!$$

and

$$\int_0^\infty\frac{r^{k'+1}}{e^r}dr\leq\int_0^\infty\frac{r^{k''+1}}{e^r}dr=(k''+1)!=(k+s+1)(k+s)\cdots(k+2)(k+1)k!.$$

Note that

$$\frac{(k+s+1)(k+s)\cdots(k+2)}{(k+s)(k+s-1)\cdots(k+1)}=\frac{k+s+1}{k+1}.$$

The condition $k\geq 2m+2$ implies that

$$\frac{k+s+1}{k+1}\leq\frac{2m+3+s}{2m+3}=1+\frac{s}{2m+3}.$$

Let $\rho=1+\frac{s}{2m+3}$. Then the ratio of $\int_0^\infty\frac{r^{k''+1}}{e^r}dr$ and $\int_0^\infty\frac{r^{k''}}{e^r}dr$ is less than or equal to $\rho\cdot(k+1)$ and

$$\int_0^\infty\frac{r^{k''+1}}{e^r}dr\leq\Delta_0\cdot\rho^{k+1}\cdot(k+1)!$$

if $\int_0^\infty\frac{r^{k''}}{e^r}dr\leq\Delta_0\cdot\rho^k\cdot k!$. The smallest k'' is $k_0''=2m+2+s$ when $k_0=2m+2$. Now,

$$\int_0^\infty\frac{r^{k_0''}}{e^r}dr \quad = \quad k_0''!=(2m+2+s)(2m+1+s)\cdots(2m+3)(2m+2)!$$

$$\begin{aligned}
&= \frac{(2m+2+s)(2m+1+s)\cdots(2m+3)}{\rho^{2m+2}}\cdot\rho^{2m+2}\cdot(2m+2)! \\
&= \Delta_0\cdot\rho^{2m+2}\cdot(2m+2)! \\
&\quad \textit{where } \Delta_0 = \frac{(2m+2+s)(2m+1+s)\cdots(2m+3)}{\rho^{2m+2}}. \\
&= \Delta_0\cdot\rho^{k_0}\cdot k_0!
\end{aligned}$$

It follows that $\int_0^\infty \frac{r^{k'}}{e^r}dr \le \Delta_0\cdot\rho^k\cdot k!$ for all $k\ge 2m+2$.

case 2. Assume $k''<k$. Let $k''=k-s$ where $s>0$. Then

$$\int_0^\infty \frac{r^{k'}}{e^r}dr \le \int_0^\infty \frac{r^{k''}}{e^r}dr = k''! = (k-s)! = \frac{1}{k(k-1)\cdots(k-s+1)}\cdot k!$$

and

$$\begin{aligned}
\int_0^\infty \frac{r^{k'+1}}{e^r}dr &\le \int_0^\infty \frac{r^{k''+1}}{e^r}dr \\
&= (k''+1)! = (k-s+1)! = \frac{1}{(k+1)k\cdots(k-s+2)}\cdot(k+1)!.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left\{\frac{1}{(k+1)k\cdots(k-s+2)}\Big/\frac{1}{k(k-1)\cdots(k-s+1)}\right\} \\
&= \frac{k(k-1)\cdots(k-s+1)}{(k+1)k\cdots(k-s+2)} \\
&= \frac{(k-s+1)}{k+1} \\
&\le 1.
\end{aligned}$$

Let $\rho=1$. Then

$$\int_0^\infty \frac{r^{k''+1}}{e^r}dr \le \Delta_0\cdot\rho^{k+1}\cdot(k+1)!$$

if $\int_0^\infty \frac{r^{k''}}{e^r}dr \le \Delta_0\cdot\rho^k\cdot k!$. The smallest k is $k_0=2m+2$. Hence the smallest k'' is $k_0''=k_0-s=2m+2-s$. Now,

$$\begin{aligned}
\int_0^\infty \frac{r^{k_0''}}{e^r}dr &= k_0''! = (2m+2-s)! = (k_0-s)! \\
&= \frac{1}{k_0(k_0-1)\cdots(k_0-s+1)}\cdot(k_0!) \\
&= \Delta_0\cdot\rho^{k_0}\cdot k_0! \textit{ where } \Delta_0 = \frac{1}{(2m+2)(2m+1)\cdots(2m-s+3)}.
\end{aligned}$$

It follows that $\int_0^\infty \frac{r^{k'}}{e^r}dr \le \Delta_0\cdot\rho^k\cdot k!$ for all $k\ge 2m+2$.

case 3. Assume $k''=k$. Then

$$\int_0^\infty \frac{r^{k'}}{e^r}dr \le \int_0^\infty \frac{r^{k''}}{e^r}dr = k! \ \textit{ and } \ \int_0^\infty \frac{r^{k'+1}}{e^r}dr \le (k+1)!.$$

Let $\rho=1$. Then $\int_0^\infty \frac{r^{k'}}{e^r}dr \le \Delta_0\cdot\rho^k\cdot k!$ for all k where $\Delta_0=1$.

The lemma is now an immediate result of the three cases.

Remark 2.1: For the convenience of the reader we should express the constants Δ_0 and ρ in a clear form. It's easily shown that
(a)$k'' > k$ if and only if $n-\lambda > 3$,
(b)$k'' < k$ if and only if $n-\lambda \le 1$, and
(c)$k'' = k$ if and only if $1 < n-\lambda \le 3$,
where k'' and k are as in the proof of the lemma. The proof of (a) is as follows. Note that $k'' = \lceil k' \rceil$ where $k' = k + \frac{n-\lambda-3}{2}$. Therefore $k'' > k$ iff $\frac{n-\lambda-3}{2} > 0$. This gives that $k'' > k$ iff $n-\lambda > 3$. The proofs for (b) and (c) are similar and we omit them.We thus have the following situations.
(a)$n-\lambda > 3$. Let $s = \lceil \frac{n-\lambda-3}{2} \rceil$. Then

$$\rho = 1 + \frac{s}{2m+3} \;\; and \;\; \Delta_0 = \frac{(2m+2+s)(2m+1+s)\cdots(2m+3)}{\rho^{2m+2}}.$$

(b)$n-\lambda \le 1$. Let $s = -\lceil \frac{n-\lambda-3}{2} \rceil$. Then

$$\rho = 1 \;\; and \;\; \Delta_0 = \frac{1}{(2m+2)(2m+1)\cdots(2m-s+3)}.$$

(c)$1 < n-\lambda \le 3$. We have

$$\rho = 1 \;\; and \;\; \Delta_0 = 1.$$

Before introducing our main theorem, we must introduce a function space called **native space**, denoted by $\mathbf{C_{h,m}}$, for each conditionally positive definite radial function h of order m. If

$$\mathcal{D}_m = \{\phi \in \mathcal{D} : \int x^\alpha \phi(x)dx = 0 \;\; for\ all\ |\alpha| < m\},\ \mathcal{D} = C_0^\infty$$

, then $C_{h.m}$ is the class of those continuous functions f which satisfy

$$\left| \int f(x)\phi(x)dx \right| \le c(f)\left\{ \int h(x-y)\phi(x)\overline{\phi(y)}dxdy \right\}^{1/2}, \tag{10}$$

for some constant $c(f)$ and all ϕ in $\mathcal{D}_m$. The definition of $\mathcal{D}$ can be seen in the beginning of 1.2. If $f \in C_{h,m}$, let $\|f\|_h$ denote the smallest constant $c(f)$ for which (10) is true. Then $\|\cdot\|_h$ is a semi-norm and $C_{h,m}$ is a semi-Hilbert space; in the case $m = 0$ it is a norm and a Hilbert space respectively. For further details, we refer the reader to [13, 14]. This definition of native space was introduced by Madych and Nelson, and characterized by Luh in [9, 10]. Although there is an equivalent definition [16] which is easier to handle, we still adopt Madych and Nelson's definition to show the author's respect for them.

Now we have come to the main theorem of this paper.

Theorem 2.3 *Let h be as in (4). For any positive number b_0, there exist positive constants δ_0, c_1, C, ω', $0 < \omega' < 1$, completely determined by h and b_0, such that for any n-dimensional simplex Q_0 of diameter b_0, any $f \in C_{h,m}$, and any $0 < \delta \le \delta_0$, there is a number r satisfying the property that $\frac{1}{3C} \le r \le b_0$ and for any n-dimensional simplex Q of diameter r, $Q \subseteq Q_0$, there is an interpolating function $s(\cdot)$ as defined in (1) such that*

$$|f(x) - s(x)| \le c_1 \sqrt{\delta}(\omega')^{\frac{1}{\delta}} \cdot \|f\|_h, \tag{11}$$

for all x in Q, where C is defined by

$$C := \max\left\{8\rho',\ \frac{2}{3b_0}\right\},\ \rho' = \frac{\rho}{c}$$

where ρ and c appear in Lemma2.2 and (4) respectively. The function $s(\cdot)$ interpolates f at $x_1, \cdots, x_N$ which are evenly spaced points of degree $k-1$ on Q as defined in the beginning of 1.1, with $k = \frac{r}{\delta}$. Here $\|f\|_h$ is the h-norm of f in the native space.

The numbers δ_0, c_1 and ω' are given by $\delta_0 := \frac{1}{3C(m+1)}$ where m was introduced in (4); $c_1 := \sqrt{l(\lambda, n)} \cdot (2\pi)^{1/4} \cdot \sqrt{n\alpha_n} \cdot c^{\lambda/2} \cdot \sqrt{\Delta_0}\sqrt{3C} \cdot \sqrt{(16\pi)^{-1}}$ where λ is as in (4), $l(\lambda, n)$ was introduced in (8), α_n is the volume of the unit ball in R^n, and Δ_0, together with the computation of ρ, was defined in Lemma2.2 and the remark following its proof; $\omega' := (\frac{2}{3})^{1/3C}$.

Proof. Let δ_0, and C be as in the statement of the theorem. For any $0 < \delta \le \delta_0$, we have $0 < \delta \le \frac{1}{3C(m+1)}$ and $0 < 3C\delta \le \frac{1}{m+1}$. Since $\frac{1}{m+1} < 1$, there exists an integer k such that

$$1 \le 3C\delta k \le 2.$$

For such k, $\delta k \le \frac{2}{3C} \le b_0$, $\frac{1}{3C\delta} \le k \le \frac{b_0}{\delta}$, and $8\rho'\delta k \le \frac{2}{3}$ where $\rho' = \frac{\rho}{c}$.

Let $r = \delta k$. Then $\frac{1}{3C} \le r \le b_0$. For any n simplex Q of diameter r with vertices $v_0, v_1, \cdots, v_n$ such that $Q \subseteq Q_0$, let $X := \{x_1, \cdots, x_N\}$ be the set of evenly spaced points of degree $k-1$ on Q where $N = dimP^n_{k-1}$. By (9) and Lemma2.1,

For any $x \in \Omega$, choose arbitrarily an n simplex Q of diameter $diamQ = \delta k$ with vertices $v_0, v_1, \ldots, v_n$ such that $x \in Q \subseteq \Omega$. Let $x_1, \ldots, x_N$ be evenly spaced points of degree $k-1$ on Q where $N = dimP^n_{k-1}$. By (9) and Lemma2.1, whenever $k > m$,

$$\begin{aligned} c_k &:= \left\{\int_{R^n} \frac{|\xi|^{2k}}{(k!)^2} d\mu(\xi)\right\}^{1/2} \\ &\le \sqrt{l(\lambda, n)} \cdot (2\pi)^{1/4} \cdot \sqrt{n\alpha_n} \cdot c^{\lambda/2} \cdot c^{-k} \cdot \sqrt{\Delta_0} \cdot (2\rho)^k. \end{aligned} \tag{12}$$

Theorem4.2 of [14] implies that

$$|f(x) - s(x)| \le c_k \|f\|_h \cdot \int_{R^n} |y - x|^k d|\sigma|(y), \tag{13}$$

whenever $k > m$, and σ is any measure supported on X such that

$$\int_{R^n} p(y) d\sigma(y) = p(x), \tag{14}$$

for all polynomials p in P^n_{k-1}. The fact $s(\cdot) \in C_{h,m}$ can be seen in [9, 10, 13, 14].

Let σ be the measure supported on $\{x_1, \ldots, x_N\}$ as mentioned in Lemma1.2. We essentially need to bound the quantity

$$I = c_k \int_{R^n} |y - x|^k d|\sigma|(y)$$

only.

By Stirling's Formula,

$$\begin{aligned} \begin{pmatrix} 2(k-1)-1 \\ k-1 \end{pmatrix} &= \frac{(2k-3)!}{(k-1)!(k-2)!} \\ &\sim \frac{\sqrt{2\pi(2k-3)} \cdot (2k-3)^{2k-3}}{\sqrt{2\pi(k-1)} \cdot (k-1)^{k-1} \cdot \sqrt{2\pi(k-2)} \cdot (k-2)^{k-2}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2k-3}}{\sqrt{k-1}\sqrt{k-2}} \frac{(2k-3)^{2k-3}}{(k-1)^{k-1}(k-2)^{k-2}} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{k-1}}\frac{\sqrt{2k-3}}{\sqrt{2k-4}}\frac{(2k-2)^{2k-3}}{(k-1)^{k-1}(k-2)^{k-2}} \\
&\leq \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{k-1}}\cdot 1\cdot 2^{2k-3}\cdot\frac{(k-1)^{k-2}}{(k-2)^{k-2}} \\
&\leq \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{k-1}}2^{2k-2}
\end{aligned}
$$

.Thus for k mentioned as above and Δ_0 defined in Lemma2.2, by Lemma1.2,

$$
\begin{aligned}
I &\leq \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}c^{-k}\sqrt{\Delta_0}(2\rho)^k(\delta k)^k\begin{pmatrix}2(k-1)-1\\ k-1\end{pmatrix} \\
&\leq \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}c^{-k}\sqrt{\Delta_0}(2\rho)^k(\delta k)^k\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{k-1}}2^{2(k-1)} \\
&\sim \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}c^{-k}\sqrt{\Delta_0}(2\rho)^k(\delta k)^k\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{k}}4^{k-1} \text{ when k is large}
\end{aligned}
$$

(In fact it holds for $k \geq 2$ because we already added a multiplying factor 2 in the last expression.)

$$
\begin{aligned}
&= \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{k}}\left(\frac{2\rho\delta k}{c}\right)^k\frac{4^k}{4} \\
&= \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{1}{\sqrt{16\pi}}\frac{1}{\sqrt{k}}\left(\frac{8\rho\delta k}{c}\right)^k \\
&\leq \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{1}{\sqrt{16\pi}}\frac{1}{\sqrt{k}}\left(\frac{2}{3}\right)^k \\
&\leq \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{1}{\sqrt{16\pi}}\frac{1}{\sqrt{k}}\left(\frac{2}{3}\right)^{\frac{1}{3C\delta}} \\
&= \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{1}{\sqrt{16\pi}}\frac{1}{\sqrt{k}}\left[\left(\frac{2}{3}\right)^{\frac{1}{3C}}\right]^{\frac{1}{\delta}} \\
&= \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{1}{\sqrt{16\pi}}\frac{1}{\sqrt{k}}[\omega']^{\frac{1}{\delta}} \text{ where } \omega' = \left(\frac{2}{3}\right)^{\frac{1}{3C}} \\
&\leq \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{1}{\sqrt{16\pi}}\sqrt{3C\delta}[\omega']^{\frac{1}{\delta}} \\
&= \sqrt{l(\lambda,n)}(2\pi)^{1/4}\sqrt{n\alpha_n}c^{\lambda/2}\sqrt{\Delta_0}\frac{\sqrt{3C}}{\sqrt{16\pi}}\sqrt{\delta}[\omega']^{\frac{1}{\delta}}
\end{aligned}
$$

Our theorem thus follows from (13).

Corollary 2.4 *Let h be as in (4), and b_0 be any positive number. Let Ω be a subset of R^n satisfying the property that for any x in Ω, there exists an n-dimensional simplex Q_0 of diameter b_0 such that $x \in Q_0 \subseteq \Omega$. Then there exist positive constants $\delta_0,\ 0 < \omega' < 1,\ c_1$ and C, completely determined by h and b_0, such that for any $x \in \Omega$, any $f \in C_{h,m}$, and any $0 < \delta \leq \delta_0$, there exists an interpolating function $s(\cdot)$ satisfying*

$$|f(x) - s(x)| \leq c_1\sqrt{\delta}(\omega')^{1/\delta}\|f\|_h, \tag{15}$$

where $\|f\|_h$ is the h-norm of f in the native space $C_{h,m}$. The function $s(\cdot)$ is defined as in (1) and interpolates f at $X = \{x_1, \cdots, x_N\}$, a set of evenly spaced points of degree $k-1$ on an n simplex Q of diameter $k\delta$, $\frac{1}{3C} \leq k\delta \leq b_0$ and $x \in Q \subset \Omega$. The values of $\delta_0,\ \omega',\ c_1$ and C are the same as Theorem2.3.

Remark. Note that (15) holds not only for one point x. Once the simplex Q is chosen, the error bound is suitable for all points in Q. Also, $k \to \infty$ as $\delta \to 0$ to keep $\frac{1}{3C} \le k\delta \le b_0$. It means that the number of data points will increase if δ decreases.

In the preceding theorem and corollary we didn't mention the well-known fill-distance. In fact δ is in spirit equivalent to the fill-distance $d(Q, X) := \sup_{y \in Q} \min_{1 \le i \le N} \|y - x_i\|$ where $X = \{x_1, \cdots, x_N\}$ as mentioned in the text. Note that $\delta \to 0$ if and only if $d(Q, X) \to 0$. However we avoid using fill-distance because in our approach the data points are not purely scattered. In this paper we adopt neither grid-data interpolating, nor scattered data interpolating. Instead, the data points are located in a special way, called evenly spaced, as defined in the beginning of 1.1. This to some extent seems to be a drawback. However the shape of the simplex is very flexible. Consequently the centers $x_1, \cdots, x_N$ are nearly scattered in the domain Ω. This requirement does not pose any trouble for us both theoretically and practically. The evenly spaced centers $x_1, \cdots, x_N$ in the simplex Q are friendly and easily tractable.

As a last comment on Corollary2.4, it should be mentioned that the domain Ω is very flexible. It can be bounded or unbounded, closed or open or neither, and with a smooth or unsmooth boundary.

3. COMPARISON

The exponential-type error bound for (4) presented by Luh in [11] is of the form

$$|f(x) - s(x)| \le c_1 \omega^{\frac{1}{\delta}} \|f\|_h, \tag{16}$$

where $c_1 = \sqrt{l(\lambda, n)}(\pi/2)^{1/4} \sqrt{n\alpha_n} c^{\lambda/2} \sqrt{\Delta_0}$, δ is equivalent to the fill-distance and

$$\omega = \left(\frac{2}{3}\right)^{\frac{1}{3C\gamma_n}}$$

where

$$C = \max\left\{2\rho' \sqrt{n} e^{2n\gamma_n},\ \frac{2}{3b_0}\right\},\ \rho' = \frac{\rho}{c}$$

, ρ and c being the same as this paper, b_0 be the side length of a cube, and γ_n being defined recursively by

$$\gamma_1 = 2,\ \gamma_n = 2n(1 + \gamma_{n-1})\ if\ n > 1.$$

The constant c_1 is almost the same as the c_1 in (11). The number b_0 plays the same role as the b_0 of Theorem2.3. However, $\gamma_n \to \infty$ rapidly as $n \to \infty$. This can be seen by

$$\gamma_1 = 2,\ \gamma_2 = 12,\ \gamma_3 = 78,\ \gamma_4 = 632,\ \gamma_5 = 6330, \cdots$$

The fast growth of γ_n forces $e^{2n\gamma_n}$ and hence C to grow rapidly as dimension $n \to \infty$. This means that the crucial constant ω in (16) tends to 1 rapidly as $n \to \infty$, making the error bound (16) meaningless for high dimensions. Note that even for $n = 2$ or 3, ω is still too large.

The advantages of our new approach are:first, there is $\sqrt{\delta}$ in (11) which contributes to the convergence rate of the error bound as $\delta \to 0$, even for low dimensions; second, the crucial constant ω' in (11) is only mildly dependent of dimension n. Although ω' dependends on ρ which in turn depends on n, the situation is much better than before. In fact, ω' can be made completely independent of n by changing λ in (4) to keep $n - \lambda \le 3$. This can be seen in the remark following Lemma2.2. In other words, we have significantly improved the error bound (16), even for low dimensions.

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