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Classification of Positive Solutions for a Quasilinear Dynamic Equations on Time Scales

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Abstract
In this paper, A classification of nonoscillatory solution of the quasilinear dynamic equation on time scales are considered, by Schauder, Knaster's fixed-point theorem, Some necessary and sufficient conditions for nonoscillation of the dynamic equations on $\mathbb{T}$ are established.Our results as special case when $\mathbb{T} = R$ and $\mathbb{T} = N$, involve and improve some known results.

Key words
Positive Solution; Fixed-point theorem; Time Scales

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1. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D.Thesis in 1988 in order to unify continuous and discrete analysis[1]. A time scale $\mathbb{T}$, is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications[3].

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations . We refer the reader to the papers[2−5,7] and the reference cited therein.

In this paper, we consider a quasilinear dynamic equation

$$(\phi(y^{\Delta}(t)))^{\Delta} + f(t, y(g(t))) = 0. \tag{1}$$

Where $t \in [t_0, \infty) = T_0 \subseteq T$. the following conditions is always satisfied

(H_1) : $\phi \in C_{rd}(T, R)$ is a strictly increasing and odd function, and ϕ is submultiplicative, i.e. $\phi(xy) \leq \phi(x)\phi(y),\ x, y \geq 0$.

(H_2) : $g \in C_{rd}(T_0, R^+)$, and $\lim_{t\to\infty} g(t) = \infty$.

(H_3) : $f \in C_{rd}(T_0 \times R, R)$, for each fixed $t \geq t_0 > 0,\ f(t, y)$ is increasing in y, and $yf(t, y) > 0, y \neq 0$.

The behaviors of the solution for the equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)x(\sigma(t)) = 0.$$

have been studied by many authors, such as A.D. Medico et al.[4]. It is clear that the equation is a special case of Eq.(1).

2. NONOSCILLATION THEOREMS

Lemma 2.1.(see[6]). If ϕ is submultiplicative on $[0,\infty)$, then its inverse function Φ is supermultiplicative on $[0,\infty)$,i.e. $\Phi(xy) \geq \Phi(x)\Phi(y),\ x,y \geq 0$. Moreover, Φ satisfies $\Phi(\frac{y}{x}) \leq \frac{\Phi(y)}{\Phi(x)}$, for $x,y > 0$.

Lemma 2.2. If $y(t)$ is a nonoscillatory solution of (1), then $y(t)y^{\Delta}(t) > 0$, and there exist two positive constants c_1, c_2, such that

$$c_1 \leq\mid y(t) \mid\leq c_2 t$$

for t large enough.

Proof: Without loss of generality, we may assume that $y(t) > 0$, for $t \geq t_1 \geq t_0$, it follows from

$$(\phi(y^{\Delta}(t)))^{\Delta} = -f(t, y(g(t))) < 0$$

and ϕ is strictly increasing that $\phi(y^{\Delta}(t))$ is strictly decreasing on $[t_0,\infty)$. Thus, $y^{\Delta}(t)$ is strictly decreasing. $y^{\Delta}(t)$ is eventually positive, say, $y^{\Delta}(t) > 0$, for $t \geq t_1$, where t_1 is large enough. Otherwise, if there exists a t^* such that $y^{\Delta}(t^*) = 0$, then $t > t^*, y^{\Delta}(t) < 0$, Thus, it is easy to see that $y(t)$ must become negative, which contradicts our assumption.

Since $y^{\Delta}(t) \leq y^{\Delta}(t_1)$,for $t \geq t_1$, integrating it from t_1 to t,we obtain

$$y(t) - y(t_1) \leq y^{\Delta}(t_1)(t - t_1)$$

$$y(t) \leq y(t_1) + y^{\Delta}(t_1)(t - t_1) \leq c_2 t$$

for some $c_2 > 0$. On the other hand, since $y^{\Delta}(t) > 0$, for $t \geq t_1$,there exists a $c_1 > 0$ such that $y(t) > c_1$ for $t \geq t_1$. Hence, we complete the proof.

It follows from Lemma 2.2 and its proof that one and only one of the following three possibilities occurs for the asymptotic behavior of any nonoscillatory solution $y(t)$ of (1):

(I) : $\lim\limits_{t\to\infty} \frac{y(t)}{t} = const \neq 0$;

(II) : $\lim\limits_{t\to\infty} \frac{y(t)}{t} = 0, \lim\limits_{t\to\infty} \mid y(t) \mid= \infty$;

(III) : $\lim\limits_{t\to\infty} y(t) = const \neq 0$.

Theorem 2.1. The equation (1) has a positive solution of type I_1 if and only if

$$\int_{t_0}^{\infty} \mid f(t, cg(t)) \mid \Delta t < \infty, \tag{2}$$

for some constant $c \neq 0$.

Proof: (i) Necessity. Suppose that (1) has a nonoscillatory solution of type I_1. We may assume that $y(t)$ is eventually positive. Since $y(t) \to \infty$, as $t \to \infty$, it follows from Lemma 2.2 that $t \geq t_1, y(t) > 0$, and $y^{\Delta}(t) > 0$. Since

$$\lim_{t\to\infty} \frac{y(g(t))}{g(t)} = const \neq 0,$$

there exist three $c_1, c_2 > 0, t_2 > t_1$ such that

$$c_1 g(t) \leq y(g(t)) \leq c_2 g(t) \tag{3}$$

An integration of (1) yields

$$\int_{t_2}^{t} f(s, y(g(s)))\Delta s = \phi(y^{\Delta}(t_2)) - \phi(y^{\Delta}(t)) < \phi(y^{\Delta}(t_2)).$$

which implies

$$0 < \int_{t_2}^{\infty} f(s, y(g(s)))\Delta s < \infty \tag{4}$$

It follows that $\int_{t_2}^{\infty} f(s, cg(s))\Delta s < \infty$. A similar argument holds if $y(t) < 0$.

(ii) Sufficiency. Without loss of generality, we may assume that the constant $c > 0$ in (2) is positive. Let $k > 0$ be a constant such that $\Phi(2k) \leq c$. it follows from (2) that there exists $t_1 > t_0$ large enough such that

$$0 < \int_{t_1}^{\infty} f(t, cg(t))\Delta t \leq k \tag{5}$$

Define $t_* = \min\{t_1, \inf_{t \geq t_1} g(t)\} \geq t_0$,

$$0 < \int_{t_*}^{\infty} f(t, cg(t))\Delta t \leq k$$

Define a set

$$Y := \{y \in C_{rd}[t_*, \infty) : \Phi(k)(t - t_1)_+ \leq y(t) \leq \Phi(2k)(t - t_1)_+, t \geq t_*\}$$

where

$$(t - t_1)_+ = \begin{cases} t - t_1, & t \geq t_1 \\ 0, & t_* \leq t < t_1. \end{cases}$$

and Y is a set with partial order $\leq$: $y_1 \leq y_2 \Leftrightarrow y_1(t) \leq y_2(t),\ t \geq t^*$. Clearly, for all $A \subset Y$, there exists $\inf A$ and $\sup A$, Let F be a mapping defined by

$$(Fy)(t) = \begin{cases} \int_{t_1}^{t} \Phi(k + \int_{s}^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s, & t \geq t_1 \\ 0, & t_* \leq t < t_1. \end{cases}$$

It follows from (5), Lemma 2.1 and

$$\int_{t_1}^{t} \Phi(k)\Delta s \leq (Fy)(t) \leq \int_{t_1}^{t} \Phi(2k)\Delta s \quad t \geq t_1$$

i.e.

$$\Phi(k)(t - t_1) \leq (Fy)(t) \leq \Phi(2k)(t - t_1) \quad t \geq t_1$$

that $FY \subseteq Y$, and F is rd-continuous and increasing. Therefore, by the *Knaster's* fixed-point theorem, there exists a fixed point $y \in Y$, such that $Fy = y$, i.e.,

$$y(t) = \int_{t_1}^{t} \Phi(k + \int_{s}^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s \quad t \geq t_1$$

It is clear that $y(t)$ is a positive solution of Eq.(1) and $\lim_{t\to\infty} \frac{y(t)}{t} = \Phi(k) = const$. This completes the proof.

Theorem2.2. The equation (1) has a positive solution of type (III) if and only if

$$\int_{t_0}^{\infty} \Phi(\int_{t}^{\infty} | f(s, c) | \Delta s)\Delta t < \infty, \quad c \neq 0 \tag{6}$$

Proof: (i) Necessity. Suppose that (1) has a positive solution $y(t)$ of type (III). it follows from Lemma 2.2 that $t \geq t_1, y(t) > 0$, and $y^{\Delta}(t) > 0$.where t_1 is large enough. Since $\lim_{t\to\infty} y(t) = const \neq 0$, there exist three $c_1, c_2 > 0, t_2 > t_1$ such that

$$c_1 \leq y(g(t)) \leq c_2 \quad t \geq t_2 \tag{7}$$

Since $y^{\Delta}(t) \to 0$ as $t \to \infty$, integrating (1) from t to ∞, we obtain:

$$\phi(y^{\Delta}(t)) = \int_t^{\infty} f(s, y(g(s)))\Delta s \qquad t \geq t_2$$

i.e.,

$$y^{\Delta}(t) = \Phi(\int_t^{\infty} f(s, y(g(s)))\Delta s \qquad t \geq t_2$$

Since $y^{\Delta}(t) > 0$ and $\lim_{t\to\infty} y(t) = const$.Therefore

$$0 < y(\infty) - y(t_2) = \int_{t_2}^{\infty} y^{\Delta}(s)\Delta s = \int_{t_2}^{\infty} \Phi(\int_s^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s < \infty \tag{8}$$

It follows from (7),(8) that

$$\int_{t_2}^{\infty} \Phi(\int_t^{\infty} f(s, c)\Delta s)\Delta t < \infty.$$

(ii)Sufficiency. Suppose that (6)holds. We need only consider the case where the constant $c > 0$ in (6) is positive.Choose $t_1 > t_0$ so large that

$$t_* = \min\{t_1, \inf_{t\geq t_1} g(t)\} \geq t_0,$$

and

$$\int_{t_1}^{\infty} \Phi(\int_t^{\infty} f(s, c)\Delta s)\Delta t \leq \frac{c}{2} \tag{9}$$

Define a set U by

$$U := \{y \in C_{rd}[t_*, \infty) : \frac{c}{2} \leq y(t) \leq c, t \geq t_*\}$$

and a mapping F on U by

$$(Fy)(t) = \begin{cases} c - \int_t^{\infty} \Phi(\int_s^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s, & t \geq t_1 \\ c - \int_{t_1}^{\infty} \Phi(\int_s^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s, & t_* \leq t < t_1. \end{cases}$$

It can be shown that $FU \subseteq U$ and FU is relatively compact. Consequently, by the Schauder fixed-point theorem, there exists an element $y \in U$ such that $y = Fy$. i.e.,

$$y(t) = c - \int_t^{\infty} \Phi(\int_s^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s \quad t \geq t_1$$

Clearly y is a solution of type (III) of (1), and $\lim_{t\to\infty} y(t) = c$. This completes the proof.

Theorem 2.3. The equation (1) has a positive solution of type (II) if (2) holds and

$$\int_{t_0}^{\infty} \Phi(\int_{t}^{\infty} \mid f(s,\xi) \mid \Delta s)\Delta t = \infty, \tag{10}$$

Where $0 <\mid \xi \mid<\mid c \mid$ with $c\xi > 0$, where c is given as in (2).
Proof: We only consider the case where $c > 0$ in (2). Choose $k > 0$ small enough such that $0 < k < c$. choose t_1 large enough such that

$$t_* = \min\{t_1, \inf_{t\geq t_1} g(t)\} \geq t_0,$$

and

$$\int_{t_1}^{\infty} f(s,k(g(t)+1))\Delta t \leq \phi(k)$$

Let

$$W := \{y \in C_{rd}[t_*,\infty), k \leq y(t) \leq k(t+1), t \geq t_*\}.$$

Where the set W with the partial order $\leq$: $y_1 \leq y_2 \Leftrightarrow y_1(t) \leq y_2(t)$, for $t \geq t_*$. clearly, if $A \subset W$, there exsits $\inf A$ and $\sup A$, define a mapping F on W by

$$(Fy)(t) = \begin{cases} k + \int_{t_1}^{t} \Phi(\int_{s}^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s, & t \geq t_1 \\ k, & t_* \leq t < t_1. \end{cases}$$

Obviously, if $y \in W$, then, for $t_* \leq t$,

$$k \leq (Fy)(t) \leq k + \int_{t_1}^{t} \Phi(\int_{s}^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s$$

$$\leq k + \int_{t_1}^{t} \Phi(\phi(k))\Delta s \leq k(t+1)$$

i.e., $FW \subseteq W$, and F is rd-continuous. Therefore, by the *Knaster's* fixed-point theorem, there exist a fixed point $y \in W$, such that $Fy = y$, i.e.,

$$y(t) = k + \int_{t_1}^{t} \Phi(\int_{s}^{\infty} f(\sigma, y(g(\sigma)))\Delta\sigma)\Delta s, t \geq t_1 \tag{11}$$

which implies $y(t)$ is a positive solution of (1) on $[t_1,\infty)$. It follows from (10),(11) that

$$\lim_{t\to\infty} \frac{y(t)}{t} = \lim_{t\to\infty} y^{\Delta}(t) = \lim_{t\to\infty} \Phi(\int_{t}^{\infty} f(s, y(g(s)))))\Delta s = 0$$

$$\lim_{t\to\infty} y(t) \geq \lim_{t\to\infty} [k + \int_{t_1}^{t} \Phi(\int_{s}^{\infty} f(\sigma, k)\Delta\sigma)\Delta s] = \infty$$

This shows that $y(t)$ is a solution of (1). This completes the proof.

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