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Computational Modeling of Thermoelastic Problems of a Thin Annular Disc

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Abstract

The paper is concerned with the inverse unsteady-state problem of thermolelastic deformation of a thin annular disc in the plane state of stress. Homogeneous boundary conditions of the third kind are maintained on curved surfaces of the disc while on the lower plane surface the heat flux is maintained at $u(r, t) = 0$which a known function of r and t is. The flux is prescribed also on the plane z = ξ which serves as the interior condition.

A mathematically this problem of determining the temperature, displacement and stress functions of a thin annular disc is studied. The finite Marchi-Zgrablich and Laplace transform techniques are used to find the solutions of the inverse transient themoelastic problems of a thin annular disc.

Key words

Boundary value problems; Thermoelasticity; Heat conduction

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1. INTRODUCTION

The inverse thermoelastic problem consists in the determination of temperature of the heating medium, the heat flux on the boundary surfaces of the solid when the conditions of the displacement and stresses are known at some points of the solid under consideration. The inverse problem is very important in view of its relevance to various industrial machines subjected to heating such as main shaft of the lathe and turbine and roll of a rolling mil.

In the present paper an attempt is made to determine the temperature, displacement and stress functions on upper plane surface of a thin annular disc occupying the space D: a ≤ r ≤ b, 0 ≤ z ≤ h by applying finite Marchi-Zgrablich (FMZ) transform and Laplace transform techniques. A brief note containing relevant results of the FMZ integral transform, although elementary, not easily found in text-books is provided in the Appendix. The inverse unsteady-state thermoelastic problem of a thin annular disc studied earlier in,[1] is reconsidered here in order to highlight some new features. A related problem of determining the temperature, displacement and stress functions due to partially distributed heat supply at z = $\xi(0 < \xi <$ h) in a thin annular disc[2]is also studied.

Sierakowski and Sun[3] studied the direct problem of an exact solution to the elastic deformation of a finite length hollow cylinder. In[4] the inverse transient thermoelastic problem of determining temperature, displacement and stress functions on the upper plane surface of a finite length hollow cylinder studied. The FMZ and Laplace transform techniques that are used to deal with the annular disc problem. The corresponding correct expressions are derived in the present paper. Also, the numerical results are obtained and presented graphically.

2. THERMOLEASTIC PROBLEM OF A THIN ANNULAR DISC IN THE PLANE STATE OF STRESS

Consider a thin annular isotropic disc of thickness h occupying the space D : $a \leq r \leq b$, $0 \leq z \leq h$. The differential equation governing the displacement function $U(r,z,t)$ is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)a_t T \tag{2.1}$$

where ν and a_t are Poisson's ratio and the linear coefficient of thermal expansion of the material of the disc respectively, and $T(r,z,t)$ is the temperature of the disc satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t}, \tag{2.2}$$

subject to the initial condition

$$T(r,z,0) = 0 \tag{2.3}$$

the boundary conditions

$$\left[T\,(r,z,t) + k_1 \frac{\partial}{\partial r} T(r,z,t)\right]_{r=a} = 0 \tag{2.4}$$

$$\left[T\,(r,z,t) + k_2 \frac{\partial}{\partial r} T(r,z,t)\right]_{r=b} = 0 \tag{2.5}$$

$$\left[\frac{\partial}{\partial z} T(r,z,t)\right]_{z=0} = u(r,t) \tag{2.6}$$

and the interior condition

$$\left[\frac{\partial}{\partial z} T(r,z,t)\right]_{z=\xi} = f(r,t) \tag{2.7}$$

where k is the diffusivity and k_1, k_2 are the radiation constants on the two curved surfaces of the disc.

In order to solve the differential equation (2.2) subject to the conditions (2.3)–(2.7) we define the finite Marchi-Zgrablich (FMZ) transform of T as

$$\bar{T}(\mu_n,z,t) = \int_a^b rT(r,z,t)S_0(k_1,k_2,\mu_n r)dr \tag{2.8}$$

with the inversion formula

$$T(r,z,t) = \sum_{n=1}^{\infty} \frac{\overline{T}(\mu_n,z,t)S_0(k_1,k_2,\mu_n r)}{C_n} \tag{2.9}$$

where S_0 and C_n are given by Eqns. (19)–(21) and (25) in the Appendix and μ_n are the positive roots of the transcendental equation (22). Taking the FMZ transform of Eqns. (2.2)–(2.3), (2.7)–(2.7) and using the operational property (18) with p = 0 in the Appendix and conditions (2.4) –(2.5) we get

$$\frac{\partial^2 \overline{T}}{\partial z^2} - \mu_n^2 \overline{T} = \frac{1}{k}\frac{\partial \overline{T}}{\partial t} \tag{2.10}$$

$$\overline{T}(\mu_n, z, 0) = 0 \tag{2.11}$$

$$\left[\frac{\partial \overline{T}}{\partial z}\right]_{z=0} = \overline{u}(\mu_n, t); \quad \left[\frac{\partial \overline{T}}{\partial z}\right]_{z=\xi} = \overline{f}(\mu_n, t) \tag{2.12}$$

Now we introduce the Laplace transform with respect to the variable t and define

$$\overline{T}^*(\mu_n, z, s) = \int_0^{\infty} e^{-st}\overline{T}(\mu_n, z, t)dt. \tag{2.13}$$

Taking Laplace transform of Eqns. (2.10),(2.12) and using the condition (2.11) gives

$$\frac{\partial^2}{\partial z^2}\overline{T}^* - q^2\overline{T}^* = 0; \quad q^2 = \mu_n^2 + \frac{s}{k} \tag{2.14}$$

$$\left[\frac{\partial}{\partial z}\overline{T}^*\right]_{z=0} = \overline{u}*(\mu_n, s); \quad \left[\frac{\partial}{\partial z}\overline{T}^*\right]_{z=\xi} = \overline{f}^*(\mu_n, s) \tag{2.15}$$

Solving the differential equation (2.14) and using the conditions (2.15) yields

$$\overline{T}^*(\mu_n, z, s) = \overline{f}^*(\mu_n, s)\frac{\cosh(qz)}{q\ \sinh(q\xi)} - \overline{u}^*(\mu_n, s)\frac{\cosh[q(z-\xi)]}{q\ \sinh(q\xi)}. \tag{2.16}$$

In order to find the inverse Laplace transform of $\overline{T}^*$ in (2.16) we need to find the inverse Laplace transform for G(s) given by

$$G(s) = \frac{\cosh(qz)}{q\ \sinh(q\xi)}; \quad q^2 = \mu_n^2 + \frac{s}{k} \tag{2.17}$$

According to the method described in [5] we have
Inverse Laplace transform of G(s)= sum of residues of e^{st} G(s) at the poles (2.18).

The poles and residues of e^{st}G(s) are given by

(i)

$$q\xi = im\pi \quad or \quad s = -k(\mu_n^2 + \lambda_m^2) \tag{2.18}$$

$$\beta_{mn}^2 = \mu_n^2 + \lambda_m^2; \quad \lambda_m = \frac{m\pi}{\xi}, \quad m = 1, 2, \ldots \tag{2.19}$$

$$\underset{s=-k\beta_{mn}^2}{Res}\left[s^{st}G(s)\right] = \lim_{s\to -k\beta_{mn}^2}\frac{e^{st}\cosh(qz)}{\xi q\ \cosh(q\xi)(dq/ds)} = \frac{2k}{\xi}(-1)^m\cos(\lambda_m z)e^{-k\beta_{mn}^2 t} \tag{2.20}$$

(ii) $q = 0$ or $s = -k\mu_n^2$

$$\underset{s \to -k\mu_n^2}{Res} \left[s^{st} G(s) \right] = \lim_{s \to -k\mu_n^2} \frac{e^{st} \cosh(qz)}{2 \sinh(q\xi)(dq/ds)} = \frac{k}{\xi} e^{-k\mu_n^2 t} \tag{2.21}$$

Now we can find the inverse Laplace transform of G(s) by using (2.18) – (2.21). Then, using (2.16) – (2.17) the inverse Laplace transform of $\bar{T}^*(\mu_n, z, s)$ is given by

$$\bar{T}(\mu_n, z, t) = \bar{T}_1(\mu_n, z, t) + \bar{T}_2(\mu_n, z, t), \tag{2.22}$$

$$\bar{T}_1(\mu_n, z, t) = \frac{k}{\xi} \int_0^t \bar{f}(\mu_n, t') e^{-k\mu_n^2 (t-t')} dt' + \frac{2k}{\xi} \sum_{m=1}^{\infty} (-1)^m \cos(\lambda_m z) \int_0^t \bar{f}(\mu_n, t') e^{-k\beta_{mn}^2 (t-t')} dt' \tag{2.23}$$

$$\bar{T}_2(\mu_n, z, t) = -\frac{k}{\xi} \int_0^t \bar{u}(\mu_n, t') e^{-k\mu_n^2 (t-t')} dt' - \frac{2k}{\xi} \sum_{m=1}^{\infty} (-1)^m \cos[\lambda_m (z-\xi)] \int_0^t \bar{u}(\mu_n, t') e^{-k\beta_{mn}^2 (t-t')} dt' \tag{2.24}$$

where we have made use of the convolution theorem of Laplace transform. Thus the temperature $T(r, z, t)$ satisfying the conditions (2.2) – (2.7) is given by (2.9), (2.22) – (2.24) and (2.19).

The problem of determining the temperature $T(r, z, t)$ satisfying Eqns. (2.2) – (2.7) was studied earlier by Khobragade and Durge [1]. However, due to erroneous inverse Laplace transform the first terms in (2.23) and (2.24) that do not involve infinite series are missing from the corresponding expressions of [1]. It is therefore important to show that $\bar{T}(\mu_n, z, t)$ given by (2.22) – (2.24) satisfies the conditions (2.10) – (2.12). To simplify the matters we do it for the case where$u(r, t) = 0$.

From the Fourier series expansion

$$\sum_{m=1}^{\infty} \frac{(-1)^m \cos(mx)}{m^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh(\alpha x)}{\sinh(\pi\alpha)} - \frac{1}{2\alpha^2}, \quad -\pi \le x \le \pi \tag{2.25}$$

we can deduce the formula

$$1 + 2\mu_n^2 \sum_{m=1}^{\infty} \frac{(-1)^m \cos(\lambda_m z)}{\beta_{mn}^2} = \frac{\xi\mu_n \cosh(\mu_n z)}{\sinh(\mu_n \xi)}, \quad -\xi \le z \le \xi \tag{2.26}$$

If we carry out an integration by parts in the expression for $\bar{T}_1(\mu_n, z, t)$ given by (2.23) and make use of the formula (2.26) we can write

$$\begin{aligned} \bar{T}_1(\mu_n, z, t) =& \frac{\cosh(\mu_n z)}{\mu_n \sinh(\mu_n \xi)} \bar{f}(\mu_n, t) - \frac{1}{\xi\mu_n^2} \left[\bar{f}(\mu_n, 0) e^{-k\mu_n^2 t} + \int_0^t \frac{\partial \bar{f}}{\partial t'} e^{-k\mu_n^2 (t-t')} dt' \right] \\ & - \frac{2}{\xi} \sum_{m=1}^{\infty} \frac{(-1)^m \cos(\lambda_m z)}{\beta_{mn}^2} \left[\bar{f}(\mu_n, 0) e^{-k\beta_{mn}^2 t} + \int_0^t \frac{\partial \bar{f}}{\partial t'} e^{-k\beta_{mn}^2 (t-t')} dt' \right] \end{aligned} \tag{2.27}$$

Differentiating (2.27) term by term twice with respect to z, we can show that

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \bar{T}_1 - \mu_n^2 \bar{T}_1 =& \frac{1}{\xi} \left[\bar{f}(\mu_n, 0) e^{-k\mu_n^2 t} + \int_0^t \frac{\partial \bar{f}}{\partial t'} \bar{e}^{k\mu_n^2 (t-t')} dt' \right] \\ & + \frac{2}{\xi} \sum_{m=1}^{\infty} (-1)^m \cos(\lambda_m z) \left[\bar{f}(\mu_n, 0) e^{-k\beta_{mn}^2 t} + \int_0^t \frac{\partial \bar{f}}{\partial t'} e^{-k\beta_{mn}^2 (t-t')} dt' \right] \end{aligned} \tag{2.28}$$

Also, differentiating (2.27) with respect to t term by term we get

$$\frac{1}{k}\frac{\partial}{\partial t}\overline{T}_1 = \frac{\cosh(\mu_n z)}{k\mu_n \sinh(\mu_n \xi)} \cdot \frac{\partial \overline{f}}{\partial t} + \frac{1}{\xi}\left[\overline{f}(\mu_n, 0)e^{-k\mu_n^2 t} - \frac{1}{k\mu_n^2}\frac{\partial \overline{f}}{\partial t} + \int_0^t \frac{\partial \overline{f}}{\partial t'} e^{-k\mu_n^2(t-t')}dt'\right]$$
$$+ \frac{2}{\xi}\sum_{m=1}^{\infty}(-1)^m \cos(\lambda_m z)\left[\overline{f}(\mu_n, 0)e^{-k\beta_{mn}^2 t} - \frac{1}{k\beta_{mn}^2}\frac{\partial \overline{f}}{\partial t} + \int_0^t \frac{\partial \overline{f}}{\partial t'} e^{-k\beta_{mn}^2(t-t')}dt'\right] \tag{2.29}$$

From (2.28) – (2.29) we obtain

$$\frac{\partial^2}{\partial z^2}\overline{T} - \mu_n^2\overline{T} - \frac{1}{k}\frac{\partial}{\partial t}\overline{T}_1 = \frac{1}{k\xi\mu_n^2}.\frac{\partial \overline{f}}{\partial t}\left[1 + 2\mu_n^2\sum_{m=1}^{\infty}\frac{(-1)^m\cos(\lambda_m^z)}{\beta_{mn}^2} - \frac{\mu_n\xi\cosh(\mu_n z)}{\sinh(\mu_n\xi)}\right] = 0, \tag{2.30}$$

where the extreme right equation follows from the formula (2.26). Eqn. (2.30) shows that $\overline{T}_1$ given by (2.27) satisfies the differential equation (2.10). Using (2.26) we can show also that $\overline{T}$ given by (2.27) satisfies the initial condition (2.11) and the boundary conditions (2.12). In this way we have shown that $\overline{T}$ given by (2.22) – (2.24) with $u(r, t) = 0$ satisfies the Eqns. (2.10) – (2.12). It is obvious that the restriction $u(r, t) \equiv 0$ can be dropped from the above analysis. This means that $\overline{T}(\mu_n, z, t)$ is the correct solution of Eqns. (2.10)–(2.12).

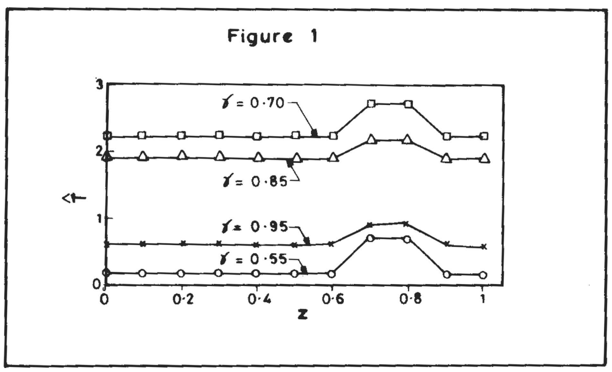

Figure 1
Variation of $\hat{T} = -T(r, z, t)$ with z where T is given by Eqn. (3.9), (3.8), (3.6), (3.5), (3.3) in Section 3 and Eqns. (25), (19), (20), (21) in Appendix. The values of various parameters are given by t = 1, a = 0.5, b = 1.0, k = 0.38, $k_1 = k_2 = 0.25$, $\xi = 0.75$, h = 1 and r = 0.55, 0.70, 0.85, 0.95

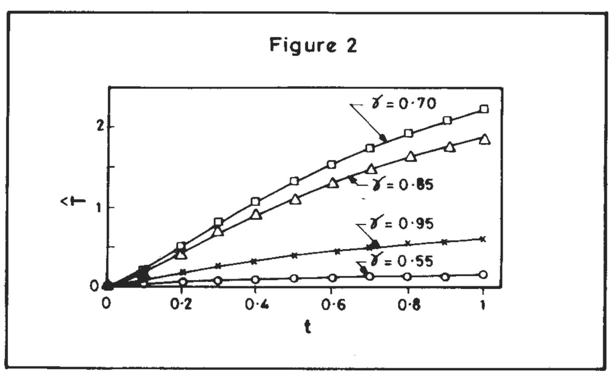

Figure 2
Variation of $\hat{T}$ with t when z = 1 and the values of other parameters are the same as in Fig.1

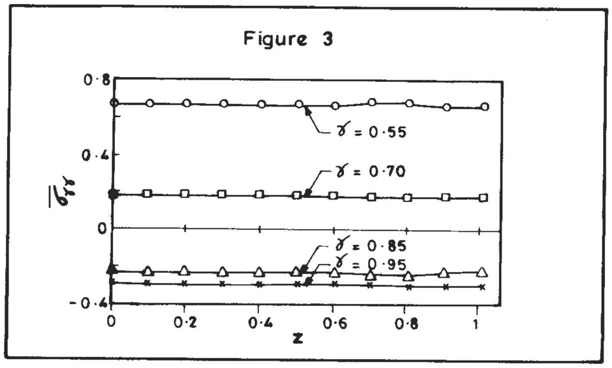

Figure 3
Variation of $\hat{\sigma}_{rr} = [2\mu(1+\nu)a_t]^{-1}\sigma_{rr}$ with z where σ_{rr} is given by Eqns. (3.14), (3.8), (3.6), (3.5). The values of other parameters are the same as in Fig. 1

3. SPECIAL CASE AND NUMERICAL RESULTS

In order to obtain some numerical results we choose special values of the prescribed functions and various parameters. As in [1] we set

$$f(r,t) = (1-e^{-t})(h-\xi)\delta(r-0.9); \quad u(r,t) = (1-e^{-t})h\,\delta(r-0.9), \tag{3.1}$$

then we have

$$\overline{f}(\mu_n,t) = \int_a^b r f(r,t) S_0(k_1,k_2,\mu_n r)dr = \alpha_n(1-e^{-t}), \tag{3.2}$$

$$\alpha_n = (h-\xi)(0.9)S_0(k_1,k_2,0.9\mu_n) \tag{3.3}$$

and we can show that

$$\int_0^t \overline{f}(\mu_n,t')e^{-k\beta_{mn}^2(t-t')}dt' = \alpha_n B_{mn}(t)/k, \tag{3.4}$$

$$B_{mn}(t) = \frac{k(1-e^{-t}) + (e^{-k\beta_{mn}^2 t}-1)/\beta_{mn}^2}{(k\beta_{mn}^2-1)}. \tag{3.5}$$

Let

$$D_n(t) = \frac{k(1-e^{-t}) + (e^{-k\mu_{mn}^2 t}-1)/\mu_n^2}{(k\mu_n^2-1)}, \tag{3.6}$$

then using (2.22) – (2.24) and (3.2) – (3.6) we have

$$\overline{T}(\mu_n,z,t) = -(h-\xi)^{-1}\alpha_n E_n(z,t), \tag{3.7}$$

$$E_n(z,t) = D_n(t) - \frac{2}{\xi}\sum_{m=1}^{\infty}(-1)^m\left[(h-\xi)\cos(\lambda_m z) - h\ \cos\left[\lambda_m(z-\xi)\right]\right]B^{(}_{mn}t) \tag{3.8}$$

Inverting the FMZ transform in (3.7) we get

$$T(r,z,t) = -\frac{1}{(h-\xi)}\sum_{n=1}^{\infty}\frac{\alpha_n E_n(z,t) S_0(k_1,k_2,\mu_n r)}{C_n}, \tag{3.9}$$

where E_n(z,t) is given by (3.8), (3.6) and (3.5). The constants α_n and C_n are given by (3.3) and Eqns.(25), (19) – (21) in the Appendix. Also, μ_n in all these equations are the positive roots of transcendental equation (22). By taking a = 0.5, b=1, k = 0.38, $k_1 = k_2$ =0.25, ξ = 0.75 and h = 1, the variation of T(r, z, t) is shown in Fig. 1 with z when t = 1 and r = 0.55, 0.70, 0.85 and 0.95. Fig. 2 shows the variation of T(r, z, t) with t when z = 1 and the values of other parameters are the same as in Fig. 1.

The displacement function U(r, z, t) satisfies the relation (2.1). Since we have

$$\frac{\partial^2 S_0}{\partial r^2} + \frac{1}{r}\frac{\partial S_0}{\partial r} = -\mu_n^2 S_0 \tag{3.10}$$

we can write

$$U(r,z,t) = \frac{(1+\nu)a_t}{(h-\xi)}\sum_{n=1}^{\infty}\frac{\alpha_n E_n(z,t) S_0(k_1,k_2,\mu_n r)}{\mu_n^2 C_n}, \tag{3.11}$$

which is a particular solution of (2.1). The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu\frac{1}{r}\frac{\partial U}{\partial r}\ ; \quad \sigma_{\theta\theta} = -2\mu\frac{\partial^2 U}{\partial r^2} \tag{3.12}$$

where μ is Lame' constant, while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the disc in the plane state of stress. From (2.1) and (3.12) it follows that

$$\sigma_{rr} + \sigma_{\theta\theta} = -2\mu(1+\nu)a_t T \tag{3.13}$$

so that $\sigma_{\theta\theta}$ is a linear combination of σ_{rr} and T. Now, using (3.11) – (3.12) we can write

$$\sigma_{rr}(r,z,t) = -\frac{2\mu(1+\nu)a_t}{(h-\xi)}\sum_{n=1}^{\infty}\frac{\alpha_n E_n(z,t) S_0'(k_1,k_2,\mu_n r)}{(r\mu_n)C_n} \tag{3.14}$$

where E_n(z,t) is given by (3.8), (3.6) and (3.5). It may be noted that

$$(r\mu_n)^{-1}S_0'(k_1,k_2,\mu_n r) = A_n\left[-(r\mu_n)^{-1}J_1(\mu_n r)\right] - B_n\left[-(r\mu_n)^{-1}Y_1(\mu_n r)\right] \tag{3.15}$$

so that $\hat{\sigma}_{rr} = [2\mu(1+\nu)a_t]^{-1}\sigma_{rr}$ can be obtained from T simply by replacing $J_0(\mu_n r)$ and $Y_0(\mu_n r)$ by $-(\mu_n r)^{-1}J_1(\mu_n r)$ and $-(r\mu_n)^{-1}Y_1(\mu_n r)$ respectively. The variation of $\hat{\sigma}_{rr}$with z is shown in Fig.3 for r = 0.5,0.6,0.7,0.8,1.0 when the values of other parameters are the same as in Fig.1.

APPENDIX

FINITE MARCHI-ZGRABLICH TRANSFORM

Let the Bessel differential equation of order p be given by

$$x^2y'' + xy' + (x^2\mu^2 - p^2)y = 0 \tag{1}$$

and let the boundary conditions be given by

$$y(a) + k_1y'(a) = 0\ ; \quad y(b) + k_2y'(b) = 0 \tag{2}$$

The general solution of Eqn.(1) is

$$y(x) = A\ J_p(\mu x) + B\ Y_p(\mu x) \tag{3}$$

where $J_p(\mu x)$ and $Y_p(\mu x)$ are Bessel functions of the first and the second kind, respectively, of order p. Then the boundary conditions (2) yield

$$A\ J_p(k_1, \mu a) + B\ Y_p(k_1, \mu a) = 0 \tag{4}$$

$$A\ J_p(k_2, \mu b) + B\ Y_p(k_2, \mu b) = 0 \tag{5}$$

where, for i =1,2, we have

$$J_p(k_i, \mu x) = J_p(\mu x) + k_i \mu\ J'_p(\mu x), \tag{6}$$

$$Y_p(k_i, \mu x) = Y_p(\mu x) + k_i \mu\ Y'_p(\mu x). \tag{7}$$

The pair of Eqns.(4) – (5) has a nontrivial solution only if

$$J_p(k_1, \mu a) Y_p(k_2, \mu b) - J_p(k_2, \mu b) Y_p(k_1, \mu a) = 0 \tag{8}$$

Let μ_n be the n^{th} positive root of the transcendental equation (8). Using (4) – (5) we can write the solution (3) in the following two forms

$$y_n(x) = \frac{A}{Y_p(k_1, \mu_n a)} \left[J_p(\mu_n x) Y_p(k_1, \mu_n a) - Y_p(\mu_n x) J_p(k_1, \mu_n a) \right], \tag{9}$$

and

$$y_n(x) = \frac{A}{Y_p(k_2, \mu_n b)} \left[J_p(\mu_n x) Y_p(k_2, \mu_n b) - Y_p(\mu_n x) J_p(k_2, \mu_n b) \right]. \tag{10}$$

Define a function S_p as follows:

$$S_p(k_1, k_2, \mu_n x) = A_n\ J_p(\mu_n x) - B_n\ Y_p(\mu_n x), \tag{11}$$

$$A_n = Y_p(k_1, \mu_n a) + Y_p(k_2, \mu_n b); \quad B_n = J_p(k_1, \mu_n a) + J_p(k_2, \mu_n b). \tag{12}$$

In view of (9) – (10) the function S_p is a solution of Bessel's differential equation (1) of order p and satisfies the boundary conditions (2). Because of this fact, the sequence of functions $\left\{ S_p(k_1, k_2, \mu_n x) \right\}_{n=1}^{\infty}$ is orthogonal in the interval (a, b). Finite Marchi-Zgrablich (FMZ) integral transform is defined as (see [1,2,4,7])

$$\bar{f}(\mu_n) = \int_a^b x f(x) S_p(k_1, k_2, \mu_n x) dx \tag{13}$$

and its inversion formula is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{\bar{f}(\mu_n) S_p(k_1, k_2, \mu_n x)}{C_n} \tag{14}$$

where the norm C_n may be written

$$C_n = \int_a^b x\, S_p^2(k_1, k_2, \mu_n x) dx \tag{15}$$

The value of the integral C_n is given by (see Sneddon [8], Prob. 8 – 14)

$$C_n = \left\{ \frac{x^2}{2} \left[\left\{ S'_p(k_1, k_2, \mu_n x) \right\}^2 + \left(1 - \frac{p^2}{\mu_n^2 x^2} \right) S_p^2(k_1, k_2, \mu_n x) \right] \right\}_a^b \tag{16}$$

Since $S_p$satisfies the boundary conditions (2) we can write

$$C_n = \frac{b^2}{2}\left[\left(1+\frac{1}{k_2^2\mu_n^2}-\frac{p^2}{\mu_n^2 b^2}\right)S_p^2(k_1,k_2,\mu_n b)\right]-\frac{a^2}{2}\left[\left(1+\frac{1}{k_1^2\mu_n^2}-\frac{p^2}{\mu_n^2 a^2}\right)S_p^2(k_1,k_2,\mu_n a)\right] \tag{17}$$

Thus the FMZ integral transform and its inverse are given by (13) – (14) and (17). Also, if we carry out integration by parts twice we can prove the operational property

$$\int_a^b x\left[\frac{\partial^2 f}{\partial x^2}+\frac{1}{x}\frac{\partial f}{\partial x}-\frac{p^2}{x^2}f\right]S_p(k_1,k_2,\mu_n x)dx = \frac{b}{k_2}S_p(k_1,k_2,\mu_n b)\left[f+k_2\frac{\partial f}{\partial x}\right]_{x=b}$$
$$-\frac{a}{k_1}S_p(k_1,k_2,\mu_n a)\left[f+k_1\frac{\partial f}{\partial x}\right]_{x=a}-\mu_n^2\bar{f}(\mu_n). \tag{18}$$

SPECIAL CASE P = 0

In our analysis we need the results only for the special case p = 0. For this special case we have

$$S_0(k_1,k_2,\mu_n r) = A_n J_0(\mu_n r) - B_n Y_0(\mu_n r), \tag{19}$$

$$A_n = Y_0(\mu_n a) - k_1\mu_n Y_1(\mu_n a) + Y_0(\mu_n b) - k_2\mu_n Y_1(\mu_n b), \tag{20}$$

$$B_n = J_0(\mu_n a) - k_1\mu_n J_1(\mu_n a) + J_0(\mu_n b) - k_2\mu_n J_1(\mu_n b), \tag{21}$$

where μ_n is the nth positive root of the transcendental equation

$$[J_0(\mu_n a) - k_1\mu_n J_1(\mu_n a)]\,[Y_0(\mu_n b) - k_2\mu_n Y_1(\mu_n b)]$$
$$-[J_0(\mu_n b) - k_2\mu_n J_1(\mu_n b)]\,[Y_0(\mu_n a) - k_1\mu_n Y_1(\mu_n a)] = 0 \tag{22}$$

For a = 0.5, b = 1.0 and $k_1 = k_2 = 0.25$ the first 20 roots are listed in the Table. With p = 0 the FMZ transform may be rewritten

$$\bar{f}(\mu_n) = \int_a^b r f(r) S_0(k_1,k_2,\mu_n r)dr \tag{23}$$

together with the inversion formula

$$f(r) = \sum_{n=1}^{\infty}\frac{\bar{f}(\mu_n)S_0(k_1,k_2,\mu_n r)}{C_n} \tag{24}$$

where

$$C_n = \frac{b^2}{2}\left[1+\frac{1}{k_2^2\mu_n^2}\right]S_0^2(k_1,k_2,\mu_n b)-\frac{a^2}{2}\left[1+\frac{1}{k_1^2\mu_n^2}\right]S_0^2(k_1,k_2,\mu_n a) \tag{25}$$

Finally, the operational property is given by (18) with p = 0 where $\bar{f}(\mu_n)$ is now given by (23).

Table 1
First Positive 20 Roots of Equation (22) for a = 0.5, b = 1.0, $k_1 = k_2 = 0.25$

4.100035	21.114158	38.493181	55.917084
7.399590	24.582173	41.976075	59.403888
10.781796	28.055419	45.460120	62.891112
14.206770	31.532288	48.945081	66.378691
17.654071	35.011765	52.430780	69.866573

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