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New Soliton Solutions for Systems of Nonlinear Evolution Equations by the Rational Sine-Cosine Method

Marwan Alquran[1,*]; Kamel Al-Khaled[2]; Hasan Ananbeh[3]

[1]Department of Mathematics and Statistics, Jordan University of Science and Technology, IRBID 22110, Jordan
[2]Department of Mathematics and Statistics, Jordan University of Science and Technology, IRBID 22110, Jordan
Email: kamel@just.edu.jo
[3]Department of Mathematics and Statistics, Jordan University of Science and Technology, IRBID 22110, Jordan
Email:hasan_ennab@yahoo.com
[*]Corresponding author. Address: Department of Mathematics and Statistics, Jordan University of Science and Technology, IRBID 22110, Jordan. Email: marwan04@just.edu.jo
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Abstract: In this paper, we construct new solitary solutions to nonlinear PDEs by the rational Sine and Cosine method. Moreover, the periodic solutions and bell-shaped solitons solutions to the Benjamin-Bona-Mahony and the Gardner equations are obtained. New solutions to Broer-Kaup (BK) system are also obtained. Finally, the solution of a two-component evolutionary system of a homogeneous KdV equations of order 2 has been investigated by the proposed method.

Keywords: Wave variables; Rational Sine-Cosine Method; Nonlinear PDEs; Evolutionary equations

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INTRODUCTION

It is well known that many models in mathematics and physics are described by nonlinear differential equations. Nowadays, research in physics devotes much attention to nonlinear partial differential evolution model equations, appearing in various fields of science, especially fluid mechanics, solid-state physics, plasma physics, and nonlinear optics[5,9]. Among these nonlinear evolution equations, is the simplest mathematical known as Benjamin-Bona-Mahony equation, that produce a special kind of soliton solutions[2], and described by the following normalized system

$$u_t = u_{xxt} - u_x - uu_x. \tag{0.1}$$

The mathematical theory of nonlinear evolution equations starting form KdV equation and the modified KdV (mKdV) equation, contains some important equations, such as Gardner's equation, that is also known as the mixed KdV-mKdV equation is very widely studied in various area of physics. The Gardner equation shows up, particularly, in the context of internal gravity waves in a density-stratified ocean. The following version of this equation is going to be studied in this paper

$$u_t = u_{xxx} + 6uu_x. \tag{0.2}$$

Alongside the above two equations, there is also the two-component evolutionary system of a homogeneous KdV equations that arise quite frequently in mathematical physics, and has the following forms

$$u_t = -uu_x - v_x \tag{0.3}$$

$$v_t = -u_x - (uv)_x - u_{xxx}$$

and

$$u_t = -3v_{xx} \tag{0.4}$$

$$v_t = u_{xx} + 4u^2.$$

Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for exact solutions of nonlinear partial differential equations. Exact solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science, since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. In the past decade, many significant methods have been proposed for obtaining solutions of nonlinear partial differential equations such as the Sinc-Galerkin method[14,5], the finite difference method[4], the Adomian decomposition method[13], the Differential transform method[15], the extended tanh-function method[9,11,12,10], the sine-cosine method[3], the improved G'/G expansion method[16], the Exp-function method[1], the direct algebraic method, Hirota's method, inverse scattering method, Backlund transformation, the Wadati trace method, Hirota bilinear forms, pseudo spectral method, the tanh-sech method, the Riccati equation expansion method and so on.

The main aim of this paper is to apply the rational sine-cosine function method with the help of symbolic computation to obtain new soliton solutions of (0.1), (0.2) and the nonlinear systems (0.3), (0.4). By using rational sine-cosine function method, many kinds of nonlinear partial differential equations arising in mathematical physics have been solved successfully in[6,7,8].

1. THE RATIONAL SINE AND COSINE FUNCTIONS METHODS

Since we restrict our attention to traveling waves, we use the transformation $u(x,t) = u(\zeta)$, where the wave variable $z = x - ct$, converts the the nonlinear PDE to an equivalent ODE. The rational sine-cosine algorithm admits the use of the ansatze[6,7,8]

$$u(x,t) = \frac{a_0}{1 + a_1 \cos(\mu z)} \tag{1.1}$$

and the ansatze

$$u(x,t) = \frac{a_0}{1 + a_1 \sin(\mu z)} \tag{1.2}$$

where a_0, a_1, μ and c are parameters that will be determined. Substituting (1.1) or (1.2) into the reduced ODE gives a polynomial equation of cosine or sine terms. We then collect the coefficients of the resulting triangle functions and setting them to zeros, to get a system of algebraic equations among the unknowns a_0, a_1, μ and c. The problem is now completely reduced to an algebraic one. Having determined a_0, a_1, μ and c by algebraic calculations or by using Mathematica, the solutions proposed in (1.1) and in (1.2) follow immediately.

2. APPLICATION I

In this section we apply the proposed method for two physical models that admit solitary solutions.

2.1 Benjamin-Bona-Mahony (BBM) Equation

Consider the BBM equation

$$u_t = u_{xxt} - u_x - uu_x. \tag{2.1}$$

Using the wave variable $\zeta = x - ct$ carries (2.1) into the ODE

$$(1-c)u + \frac{1}{2}u^2 + cu'', \tag{2.2}$$

obtained after integrating the ODE and setting the constant of integration to zero.

Substituting (1.1) into (2.2) gives

$$\begin{gathered} 2 + a_0 - 2c + a_1(4 + a_0 + 2c(-2+\mu^2))\cos(\mu z) + \\ 2a_1^2(1 + c(-1+\mu^2))\cos^2(\mu z) + 4a_1^2 c\mu^2 \sin^2(\mu z) = 0 \end{gathered} \tag{2.3}$$

The above equation is satisfied only if the following system of algebraic equations hold

$$0 = 2 + a_0 + c(-2 + 4a_1^2\mu^2) \tag{2.4}$$

$$0 = 4 + a_0 + 2c(-2+\mu^2)$$

$$0 = -1 + c + c\mu^2$$

which leads to

$$a_0 = -\frac{6\mu^2}{1+\mu^2}, \qquad a_1 = \mp 1, \qquad c = \frac{1}{1+\mu^2}, \tag{2.5}$$

where μ is any arbitrary constant. Therefore, the solution of (2.1) is

$$u_1(x,t) = -\frac{6\mu^2}{(1+\mu^2)(1 \mp \cos\left(\mu(x - \frac{1}{1+\mu^2}t)\right))}. \tag{2.6}$$

Now, if we use the ansatze (1.2) instead of (1.1), then we get the same system (2.4) and therefore, one more solution follows and given by

$$u_2(x,t) = -\frac{6\mu^2}{(1+\mu^2)(1 \mp \sin\left(\mu(x - \frac{1}{1+\mu^2}t)\right))}. \tag{2.7}$$

2.2 Gardner Equation

Consider the Gardner equation

$$u_t = u_{xxx} + 6uu_x. \tag{2.8}$$

Using the wave variable $\zeta = x - ct$ carries (2.8) into the ODE

$$cu + 3u^2 + u'' = 0, \tag{2.9}$$

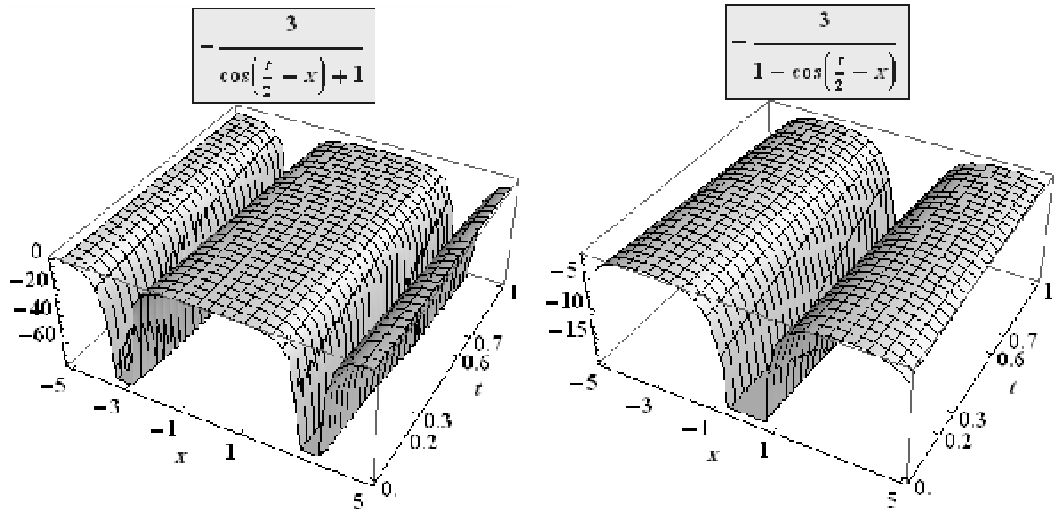

Figure 1
Plots of the first Obtained Solution for Equation (2.1) when $\mu = 1$

obtained after integrating the ODE and setting the constant of integration to zero.
Substituting (1.1) into (2.9) gives

$$3a_0 + c + a_1(3a_0 + 2c + \mu^2)\cos(\mu z) + a_1^2(c + \mu^2)\cos^2(\mu z) + 2a_1^2\mu^2\sin^2(\mu) = 0. \tag{2.10}$$

The equation is satisfied only if the following system of algebraic equations hold

$$0 = 3a_0 + c + 2a_1^2\mu^2 \tag{2.11}$$
$$0 = 3a_0 + 2c + \mu^2$$
$$0 = c - \mu^2$$

which leads to

$$a_0 = -\mu^2, \qquad a_1 = \mp 1, \qquad c = \mu^2, \tag{2.12}$$

where μ is any arbitrary constant. Therefore, the solution of (2.8) is

$$u_1(x,t) = -\frac{\mu^2}{1 \mp \cos\left(\mu(x - \mu^2 t)\right)}. \tag{2.13}$$

Using (1.2), one more solution follows and given by

$$u_2(x,t) = -\frac{\mu^2}{1 \mp \sin\left(\mu(x - \mu^2 t)\right)}. \tag{2.14}$$

3. APPLICATION II

In this section we apply the rational sine and cosine method for two systems of evolutionary equations.

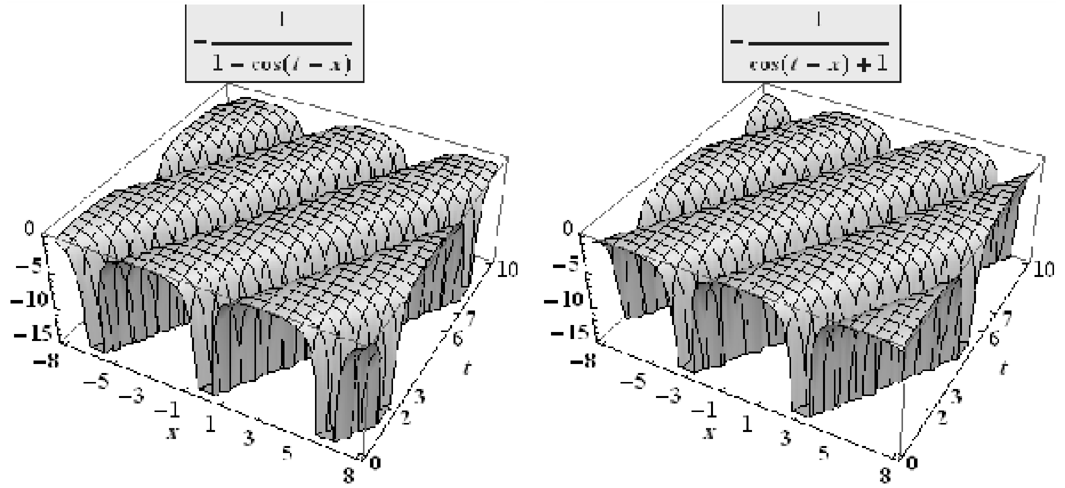

Figure 2
Plots of the First Obtained Solution for Equation (2.8) when $\mu = 1$

3.1 Broer-kaup System

Consider the Broer-Kaup system

$$u_t = -uu_x - v_x \tag{3.1}$$
$$v_t = -u_x - (uv)_x - u_{xxx}$$

Using the wave variable $\zeta = x - ct$ carries (3.1) into the ODEs

$$v = cu - \frac{1}{2}u^2 \tag{3.2}$$
$$v = \frac{u + u''}{c - u}$$

obtained after integrating the ODEs and setting the constant of integration to zero. From (3.2) we have

$$(c - u)(u - \frac{1}{2}u^2) - u - u'' = 0. \tag{3.3}$$

Substituting (1.1) into (3.3) gives

$$\begin{gathered} -2 + a_0^2 - 3a_0c + 2c^2 - a_1(4 + 3a_0c - 4c^2 + 2\mu^2)\cos(\mu z) + \\ 2a_1^2(-1 + c^2 - \mu^2)\cos^2(\mu z) - 4a_1^2\mu^2\sin^2(\mu z) = 0 \end{gathered} \tag{3.4}$$

The above equation is satisfied only if the following system of algebraic equations hold

$$0 = -2 + a_0^2 - 3a_0c + 2c^2 - 4a_1^2\mu^2 \tag{3.5}$$
$$0 = 4 + 3a_0c - 4c^2 + 2\mu^2$$
$$0 = -1 + c^2 + \mu^2$$

which leads to

$$a_0 = \frac{2(-1 + c^2)}{c}, \qquad a_1 = \pm\frac{1}{c}, \qquad \mu = \pm\sqrt{1 - c^2}, \tag{3.6}$$

where the constant $c$must be in the real open interval $(0, 1)$. Therefore, the solutions of (3.1) are

$$u_1(x,t) = \frac{2(-1+c^2)}{c \mp \cos\left(\sqrt{1-c^2}(x-ct)\right)} \tag{3.7}$$

$$v_1(x,t) = \frac{2(-1+c^2)^2}{\left(c \mp \cos\left(\sqrt{1-c^2}(x-ct)\right)\right)^2} + \frac{2c(-1+c^2)}{c \mp \cos\left(\sqrt{1-c^2}(x-ct)\right)}.$$

Using the ansatze (1.2) gives the same system obtained in (3.5) and then, the following solutions result as

$$u_2(x,t) = \frac{2(-1+c^2)}{c \mp \sin\left(\sqrt{1-c^2}(x-ct)\right)} \tag{3.8}$$

$$v_2(x,t) = \frac{2(-1+c^2)^2}{\left(c \mp \sin\left(\sqrt{1-c^2}(x-ct)\right)\right)^2} + \frac{2c(-1+c^2)}{c \mp \sin\left(\sqrt{1-c^2}(x-ct)\right)}.$$

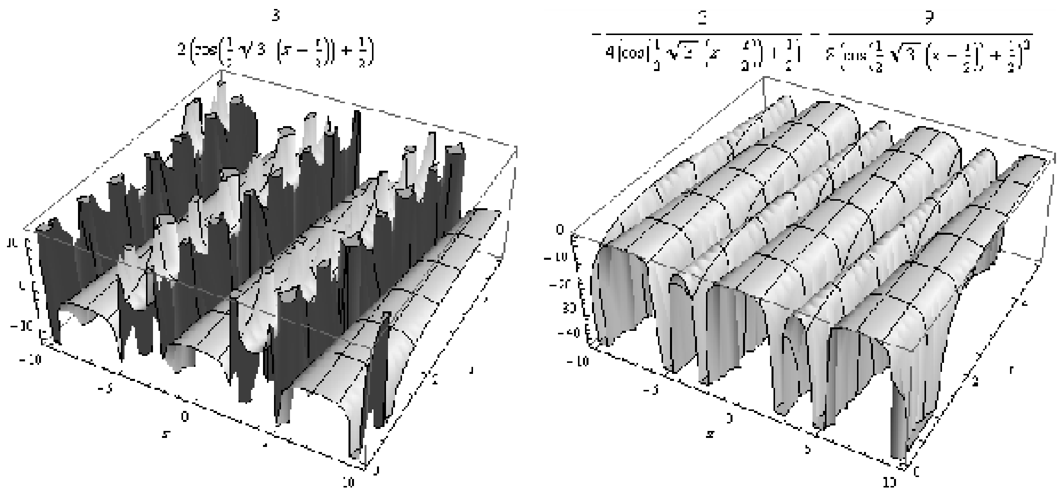

Figure 3
Plots of the First Obtained Solution for System (3.1) when $c = 0.5$

3.2 Two-component KdV Evolutionary System of Order 2

Consider the two-component evolutionary system of a homogeneous KdV equations of order 2

$$u_t = -3v_{xx} \tag{3.9}$$

$$v_t = u_{xx} + 4u^2$$

Using the wave variable $\zeta = x - ct$ carries (3.9) into the ODE

$$-cu' = -3v'' \tag{3.10}$$

$$-cv' = u'' + 4u^2.$$

From (3.10), we have

$$u = \frac{3}{c}v', \tag{3.11}$$

and therefore,

$$c^2u + 12u^2 + 3u'' = 0 \tag{3.12}$$

Substituting (1.1) into (3.12) gives

$$12a_0 + c^2 + a_1(12a_0 + 2c^2 + 3\mu^2)\cos(\mu z) + a_1^2(c^2 + 3\mu^2)\cos^2(\mu z) + 6a_1^2\mu^2\sin^2(\mu z) = 0. \tag{3.13}$$

The above equation is satisfied only if the following system of algebraic equations hold

$$0 = 12a_0 + c^2 + 6a_1^2\mu^2 \tag{3.14}$$

$$0 = 12a_0 + 2c^2 + 3\mu^2$$

$$0 = c^2 - 3\mu^2$$

which leads to

$$a_0 = \frac{-3\mu^2}{4}, \qquad a_1 = \pm 1, \qquad c = \pm\sqrt{3}\mu, \tag{3.15}$$

where μ is any arbitrary constant. Therefore, the solutions of (3.9) are

$$u_1(x,t) = -\frac{3\mu^2}{4(1-\cos\left(\mu(x+\sqrt{3}\mu t)\right)} \tag{3.16}$$

$$v_1(x,t) = -\frac{\sqrt{3}\mu^2}{4}\cot\left(\frac{\mu(x+\sqrt{3}\mu t)}{2}\right)$$

and

$$u_2(x,t) = -\frac{3\mu^2}{4(1+\cos\left(\mu(x+\sqrt{3}\mu t)\right)} \tag{3.17}$$

$$v_2(x,t) = \frac{\sqrt{3}\mu^2}{4}\tan\left(\frac{\mu(x+\sqrt{3}\mu t)}{2}\right).$$

Using the ansatze (1.2) gives the same system obtained in (3.14) and then, two more solutions follow

$$u_3(x,t) = -\frac{3\mu^2}{4(1-\sin\left(\mu(x+\sqrt{3}\mu t)\right)} \tag{3.18}$$

$$v_3(x,t) = -\frac{\sqrt{3}\mu^2}{2\left(-1+\cot\left(\frac{\mu(x+\sqrt{3}\mu t)}{2}\right)\right)}$$

and

$$u_4(x,t) = -\frac{3\mu^2}{4(1-\sin\left(\mu(x+\sqrt{3}\mu t)\right)} \tag{3.19}$$

$$v_4(x,t) = -\frac{\sqrt{3}\mu^2}{2\left(1+\cot\left(\frac{\mu(x+\sqrt{3}\mu t)}{2}\right)\right)}$$

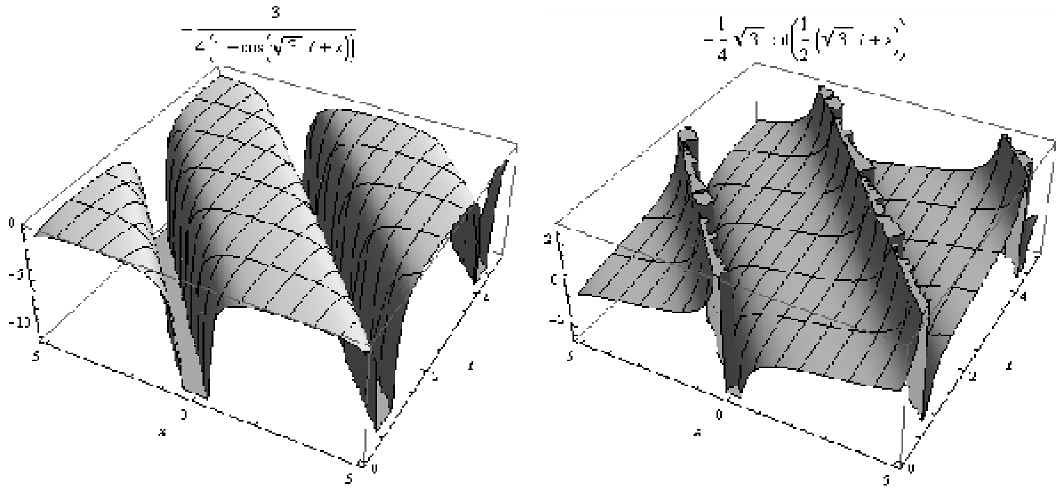

Figure 4
Plots of the First Obtained Solution for System (3.9) when $\mu = 1$

CONCLUSION

In this work we developed the rational sine-cosine method to handle some nonlinear evolution equations. The simplified form of the rational sine-cosine methods was applied to establish soliton solutions to nonlinear evolution equations. The method is applicable to several types of equations, easy to use, and may provide us a straightforward, effective and alternative mathematical tool for generating soliton solutions, and can be extended to other nonlinear problem in mathematical physics.

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