

Research on the Distribution of Mersenne Primes Based on Zhou's Conjecture

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Abstract: This paper presents an approximate expression of the number of Mersenne primes on the basis of Zhou's conjecture, predicts the position of each Mersenne prime, and compares the number of Mersenne primes derived from this expression to the number of Mersenne primes derived from actual values and the four existing approximate expressions.

Key words: Mersenne primes; Zhou's conjecture; Distribution; Approximate expression

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1. INTRODUCTION

In the modern number theory, research on Mersenne primes is an important topic, although it has been investigated for over two thousand years. The grid technology has greatly improved the research on Mersenne primes, and has resulted in the discovery of the forty-eighth Mersenne prime in 2013.

One of the important aspects to research the Mersenne prime is the study of its distribution. Many mathematicians have given their proposed conjectures on it. Regretfully, those conjectures are well have obvious deviation from the actual distribution. This paper presents an approximate expression of Mersenne primes based on the precise expression known as Zhou's conjecture, named after Chinese mathematician Haizhong Zhou , who first proposed this conjecture in 1992.

2. MERSENNE PRIMES AND ZHOU'S CONJECTURE

A prime number is a natural number greater than 1, which does not have any positive divisors other than 1 and itself^[1]. There are infinite prime numbers. A Mersenne prime is a prime number of the form $M_p=2^p-1$, where p is an assumed prime. They are called after the French mathematician Marin Mersenne who studied them in the early 17th century. Research on Mersenne primes is an essential topic of modern number theory. The study of Mersenne prime has been showed hot but difficult.

The Lucas–Lehmer test is a primality test for Mersenne numbers. The test was originally developed by French mathematician François Lucas in 1856^[2], subsequently improved by Lucas in 1878 and American mathematician Derrick Lehmer in the 1930s. Lucas-Lehmer test is that , for p an odd prime, the Mersenne number 2^p-1 is prime if and only if 2^p-1 divides $S(p-1)$ where $S(p-1)$ defined by induction as $S(n+1) = (S(n))^2 - 2$, and $S(1) = 4$. This approach is still playing an important role in computation method.

In 1952, American mathematician Raphael Robinson compiled computer program based on Lucas–Lehmer test. As of February 2013, 48 Mersenne primes are known. M25,964,951 has been found to be the 42th Mersenne prime. In modern times, the largest known prime has almost always been a Mersenne prime.

Mersenne primes distribution is extremely irregular, which leads to the irregular time intervals to discover Mersenne primes^[3]. Exploring the distribution of Mersenne primes seems to be more difficult than finding new Mersenne primes. Many fundamental questions about Mersenne primes remain unresolved. It is even unknown whether the number of Mersenne primes is finite or infinite, on which for a long time Mathematicians have made some conjectures.

Alexander Hurwitz reported that he had applied Lucas' test to investigate the primality of the Mersenne Numbers between $3,300 < p < 5,000$ and discovered that M4253 and M4423 are prime numbers^[4]. American mathematician Daniel Shanks thought that the approximate number of Mersenne primes is $\frac{1}{\log 2} \sum_{p=p_n}^{p_m} \frac{1}{p}$ in the range $p_n \leq p \leq p_m$. So he thought there are about five Mersenne primes in the interval $500 \leq p \leq 50,000$ ^[3]. Actually, seven Mersenne primes was noted in that interval.

Canadian mathematician Donald Gillies guessed that there are about two Mersenne primes when p is between x and $2x$. x is a positive integer greater than one^[5]. But the distribution of Mersenne primes derived from Donald Gillies' conjecture is quite different from the actual distribution. Sometimes, none of Mersenne prime can be found between x and $3x$, or even between x and $4x$. For example, none of Mersenne prime exists between x and $3x$ when $x = 22000$. There is not any Mersenne prime between x and $4x$ when $x = 128$.

American mathematician John Brillhart^[3] also made a conjecture which stated that p of the n -th Mersenne primes M_p is approximately equal to $(1.5)^n$. But the distribution of Mersenne primes derived from John Brillhart's conjecture is still quite different from the actual distribution.

Dutch mathematician Hendrik Lenstra and American mathematician Carl Pomerance gave a conjecture on the number of Mersenne primes in 1980 which can be repressed as the following three statements. The restated form is indicated by American mathematician Samuel S. Wagstaff in 1983^[6].

The number of Mersenne primes less than or equal to x is about $\frac{e^\gamma \log \log x}{\log 2}$, where γ is Euler's constant.

The expected number of Mersenne primes $2^p - 1$ with p between x and $2x$ is about e^γ .

The probability that $2^p - 1$ being a prime is about $\frac{e^\gamma \log ap}{p \log 2}$ where $a=2$ if $p \equiv 3 \pmod{4}$, and $a=6$ if $p \equiv 1 \pmod{4}$.

In 1992, Haizhong Zhou gave a precise expression about Mersenne primes' distribution^[7]. This important achievement has been named as "Zhou's conjecture". In fact, Zhou's conjecture is important and helpful to research the distribution of Mersenne primes.

Zhou's conjecture states that if $2^{2^n} < p < 2^{2^{n+1}}$, then there must be $2^{n+1} - 1$ Mersenne primes where p is the exponent in the expression of the Mersenne prime, and $n=0, 1, 2, 3, \dots$. Haizhong Zhou deduces that $\pi_M(2^{2^{n+1}}) = 2^{n+2} - n - 2$, where $\pi_M(2^{2^{n+1}})$ is the number of Mersenne primes when $p < 2^{2^{n+1}}$.

3. THE MAIN RESEARCH RESULTS

Based on Zhou's conjecture, we can deduce that the number of Mersenne primes is $2^{n+2} - n - 2 = 2 \log_2 2^{2^{n+1}} - \log_2 \log_2 2^{2^{n+1}} - 1 = 2 \log_2 N - \log_2 \log_2 N - 1$, where $N = 2^{2^{n+1}}$. Until now, there are 5 data which is verified to be the actual numbers: $N = 2^{2^0}, 2^{2^1}, 2^{2^2}, 2^{2^3}, 2^{2^4}$. Our conjecture numbers are coincident with the actual numbers in these five data.

When N is an arbitrary natural number except 1, Table 1 gives the comparison of definite numbers with the numbers of our conjecture and Lenstra and Pomerance's conjecture.

Table 1
Mersenne Prime Number Forecast

Actual numbers	The exponent p	Our conjecture numbers	Our conjecture's deviation	$L\&P$'s conjecture numbers	$L\&P$'s conjecture deviation
1	2	1	0	0.8	0.2
2	3	1.5	0.5	1.9	0.1
3	5	2.4	0.6	3.2	-0.2
4	7	3.1	0.9	4.1	-0.1
5	13	4.5	0.5	5.6	-0.6
6	17	5.2	0.8	6.3	-0.3
7	19	5.4	1.6	6.6	0.4
8	31	6.6	1.4	7.9	0.1
9	61	8.3	0.7	9.6	-0.6
10	89	9.3	0.7	10.6	-0.6
11	107	9.8	1.2	11.1	-0.1
12	127	10.2	1.8	11.5	0.5
13	521	13.9	-0.9	15.1	-2.1
14	607	14.3	-0.3	15.5	-1.5
15	1279	16.3	-1.3	17.4	-2.4
16	2203	17.7	-1.7	18.8	-2.8
17	2281	17.8	-0.8	18.9	-1.9
18	3217	18.8	-0.8	19.8	-1.8
19	4253	19.5	-0.5	20.5	-1.5
20	4423	19.6	0.4	20.6	-0.6
21	9689	21.8	-0.8	22.6	-1.6
22	9941	21.8	0.2	22.7	-0.7
23	11213	22.2	0.8	23.0	0.0
24	19937	23.7	0.3	24.5	-0.5
25	21701	24.0	1.0	24.7	0.3
26	23209	24.2	1.8	24.9	1.1
27	44497	25.9	1.1	26.6	0.4
28	86243	27.8	0.2	28.3	-0.3
29	110503	28.4	0.6	28.9	0.1
30	132049	28.9	1.1	29.4	0.6
31	216091	30.3	0.7	30.6	0.4
32	756839	33.8	-1.8	33.8	-1.8
33	859433	34.1	-1.1	34.2	-1.2
34	1257787	35.2	-1.2	35.2	-1.2
35	1398269	35.5	-0.5	35.4	-0.4
36	2976221	37.6	-1.6	37.4	-1.4
37	3021377	37.6	-0.6	37.4	-0.4
38	6972593	40.0	-2.0	39.6	-1.6
39	13466917	41.8	-2.8	41.2	-2.2
40	20996011	43.1	-3.1	42.4	-2.4
41	24036583	43.4	-2.4	42.7	-1.7
42	25964951	43.6	-1.6	42.9	-0.9
43*	30402457	44.1	-1.1	43.3	-0.3
44*	32582657	44.3	-0.3	43.5	0.5
45*	37156667	44.6	0.4	43.9	1.1
46*	42643801	45.0	1.0	44.2	1.8
47*	43112609	45.1	1.9	44.2	2.8
48*	57885161	45.9	2.1	45.0	3.0

Note. Our conjecture numbers: $2 \log_2 N - \log_2 \log_2 N - 1$

Our conjecture's deviation: $n - (2 \log_2 N - \log_2 \log_2 N - 1)$

$L\&P$'s conjecture numbers: $\frac{e^x \log \log x}{\log 2}$ $L\&P$'s conjecture deviation: $n - \frac{e^x \log \log x}{\log 2}$

* It is uncertain that whether any undiscovered Mersenne primes exist between the 42nd (M25,964,951) and the 48th (M57,885,161) in this Table. Therefore the order is provisional.

Theory number of our conjecture in this paper is very similar to the actual number. There are 25 numbers with deviation less than 1, but only one number of deviation greater than 3 where p is equal to 20,996,011. The first 42 numbers are determined but the last 6 numbers are undetermined. For the first 38 Mersenne primes, the maximum deviation of our conjecture is smaller than Lenstra and Pomerance's conjecture. If there is one or more other Mersenne primes when $25,964,951 < p < 57,885,161$, the maximum deviation of this article is smaller than Lenstra and Pomerance's conjecture for the last 10 Mersenne primes. If there is the only Mersenne prime, the maximum deviation of this article is larger. The authors guess that there is one other Mersenne prime at least when $25,964,951 < p < 57,885,161$.

In the interval of $[N_1, N_2]$, the number of Mersenne primes is $2 \log_2 N_2 - \log_2 \log_2 N_2 - 1 - (2 \log_2 N_1 - \log_2 \log_2 N_1 - 1) = 2 \log_2 \frac{N_2}{N_1} - \log_2 \frac{\ln N_2}{\ln N_1}$ where $N_1 = 2^{2^n} < p < 2^{2^{n+1}} = N_2$. N_1 and N_2 can be extended to the whole natural number, but the precision of this formula will quickly decrease with the reduction of $\frac{N_2}{N_1}$.

The number of Mersenne primes is

$$2 \log_2 \frac{N_2}{N_1} - \log_2 \frac{\ln N_2}{\ln N_1} = 2 \log_2 \frac{50000}{5000} - \log_2 \frac{\ln 50000}{\ln 5000} \approx 6.3, \text{ when } 5,000 \leq p \leq 50,000.$$

The actual number is seven. The number in this paper comes closer to the actual number than Shanks' conjecture.

Since the approximate number of Mersenne prime is

$$n = 2 \log_2 N - \log_2 \log_2 N - 1, \text{ then } 2 \log_2 N = n + 1 + \log_2 \log_2 N, \text{ hence } N = 2^{\frac{n+1+\log_2 \log_2 N}{2}}. \text{ According to } N \approx 2^{\frac{n+1}{2}}, p \text{ of the } n\text{th Mersenne prime is } N \approx 2^{\frac{n+1+\log_2 \log_2 2^{\frac{n+1}{2}}}{2}} = 2^{\frac{n+1+\log_2 \frac{n+1}{2}}{2}} = 2^{\frac{n+1}{2}} \sqrt{\frac{n+1}{2}}. \text{ Actual values of } p \text{ are compared with the values in our conjecture and Brillhart's conjecture showed in Table 2.}$$

Table 2
Values of p for n-th Mersenne Primes

Number	The exponent p	Our conjecture	Our conjecture's ratio	Brillhart's conjecture value	Brillhart's conjecture Ratio
1	2	2	1.0	1.5	1.3
2	3	3	1.0	2	1.5
3	5	6	0.8	3	1.7
4	7	9	0.8	5	1.4
5	13	14	0.9	8	1.6
6	17	21	0.8	11	1.5
7	19	32	0.6	17	1.1
8	31	48	0.6	26	1.2
9	61	72	0.8	38	1.6

To be continued

Continued

Number	The exponent p	Our conjecture	Our conjecture's ratio	Brillhart's conjecture value	Brillhart's conjecture Ratio
10	89	106	0.8	58	1.5
11	107	157	0.7	86	1.2
12	127	231	0.5	130	1.0
13	521	339	1.5	195	2.7
14	607	496	1.2	292	2.1
15	1279	724	1.8	438	2.9
16	2203	1056	2.1	657	3.4
17	2281	1536	1.5	985	2.3
18	3217	2232	1.4	1478	2.2
19	4253	3238	1.3	2217	1.9
20	4423	4693	0.9	3325	1.3
21	9689	6792	1.4	4988	1.9
22	9941	9822	1.0	7482	1.3
23	11213	14189	0.8	11223	1.0
24	19937	20480	1.0	16834	1.2
25	21701	29537	0.7	25251	0.9
26	23209	42566	0.5	37877	0.6
27	44497	61303	0.7	56815	0.8
28	86243	88231	1.0	85223	1.0
29	110503	126910	0.9	127834	0.9
30	132049	182445	0.7	191751	0.7
31	216091	262144	0.8	287627	0.8
32	756839	376476	2.0	431440	1.8
33	859433	540424	1.6	647160	1.3
34	1257787	775432	1.6	970740	1.3
35	1398269	1112183	1.3	1456110	1.0
36	2976221	1594560	1.9	2184164	1.4
37	3021377	2285318	1.3	3276247	0.9
38	6972593	3274178	2.1	4914370	1.4
39	13466917	4689375	2.9	7371555	1.8
40	20996011	6714162	3.1	11057332	1.9
41	24036583	9610358	2.5	16585998	1.4
42	25964951	13751945	1.9	24878998	1.0
43*	30402457	19673030	1.5	37318497	0.8
44*	32582657	28136247	1.2	55977745	0.6
45*	37156667	40230350	0.9	83966617	0.4
46*	42643801	57509400	0.7	125949930	0.3
47*	43112609	82191237	0.5	188924890	0.2
48*	57885161	117440510	0.5	283387330	0.2

Note. Our conjecture: $2^{\frac{n+1}{2}} \sqrt{\frac{n+1}{2}}$ Our conjecture ratio: $\frac{p}{2^{\frac{n+1}{2}} \sqrt{\frac{n+1}{2}}}$

Brillhart's conjecture value : $(1.5)^n$ Brillhart's conjecture ratio : $\frac{p}{(1.5)^n}$

Values in this paper and Brillhart's conjecture have a certain gap with the actual values. Conjecture values are not more than the twice of actual values most of the time. Actual values are most 2.1 times of the conjecture values in this paper when $p \leq 6,972,593$. Actual values are right on most 3.4 times

of the Brillhart's conjecture values. The first 42 ratios have been determined and the last 6 ratio are undetermined. If there is no other Mersenne prime when $p < 57,885,161$, Brillhart's conjecture values are at most 5 times than actual values for the last 10 Mersenne primes. Nonetheless, actual values are at most 3.1 times than conjecture values in this paper. If there is one or more other Mersenne primes when $p < 57,885,161$, Brillhart's conjecture values are at most 8 times than actual values for the last 10 Mersenne primes. But actual values are still at most 3.1 times than conjecture values in this paper.

From the above deducing, we can guess that there are 2 Mersenne primes in average when $x \leq p \leq 2x$ where x is large. Actually, sometimes, there is no Mersenne prime between x and $2x$. But sometimes there are more than one Mersenne prime. For example, seven Mersenne primes exist at least when x is 24,036,583. When x is smaller, the average of Mersenne primes number is less than 2 between x and $2x$.

The probability that M_N is a Mersenne prime is

$$(2 \log_2 N - \log_2 \log_2 N - 1)' = 2 \frac{1}{N \log 2} - \frac{1}{N \log 2 \log N}.$$

So the probability that M_p is a Mersenne prime is

$$\left(2 \frac{1}{N \log 2} - \frac{1}{N \log 2 \log N} \right) \log N = 2 \frac{\log N}{N \log 2} - \frac{1}{N \log 2}.$$

4. SUMMARY

It is difficult to find the pattern of Mersenne primes because the distribution of Mersenne primes is very irregular. Based on Zhou's conjecture, this paper uses formula-transformation method, and derives the approximated expression of Mersenne primes. The numbers of Mersenne primes estimated by this expression are much nearer to the actual numbers than the numbers estimated by Lenstra and Pomerance. The position of each Mersenne prime is also of the view by this expression and the results are better than the estimation in Brillharts' conjecture. The results in this paper are closer to the actual value than Shanks and Gillies' conjecture.

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