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Toeplitz Matrix Method and Volterra-Hammerstien Integral Equation With a Generalized Singular Kernel

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Abstract: In this work, the existence of a unique solution of Volterra-Hammerstein integral equation of the second kind (V-HIESK) is discussed. The Volterra integral term (VIT) is considered in time with a continuous kernel, while the Fredholm integral term (FIT) is considered in position with a generalized singular kernel. Using a numerical technique, V-HIESK is reduced to a nonlinear system of Fredholm integral equations (SFIEs). Using Toeplitz matrix method we have a nonlinear algebraic system of equations. Also, many important theorems related to the existence and uniqueness of the produced algebraic system are derived. Finally, some numerical examples when the kernel takes the logarithmic, Carleman, and Cauchy forms, are considered.
Keywords: Singular integral equation; Nonlinear Volterra –Fredholm integral equation; Toeplitz matrix; Cauchy kernel; Carleman kernel.

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1. INTRODUCTION

Many authors have interested in solving the Volterra-Fredholm integral equation, Abdou and Salama, in [1], obtained the solution in one, two and three dimensional for the **V–FIT** of the first kind using spectral relationships. In [2], EL-Borai et al. studied the existence and uniqueness of solution of nonlinear integral equation of the second kind of type **V-FIE.** Maleknejad and Sohrabi, in [3], solved the nonlinear **V–F**–Hammerstein integral equations in terms of Legendre polynomials. In [4], Ezzati and Najafalizadeh, used Chebyshev polynomials to solve linear and nonlinear Volterra-Fredholm integral equations. Shazad, in [5], solved Volterra-Fredholm integral equation by using least squares technique. Shali et al., in [6], studied the numerical solvability of a class of nonlinear Volterra-Fredholm integral equations.

In this work, we consider the **V–HIESK**

$$\mu\phi(x,t) = f(x,t) + \lambda\int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma(y,\tau,\phi(y,\tau))dyd\tau \quad (1)$$

The integral equation (1) is considered in time, for **VIT** and position for **FIT**. The functions $k(/g(x) - g(y)/)$, $F(t,\tau)$ and $f(x, t)$ are given and called the kernel of **FIT, VIT** and the free term, respectively. The constant defines the kind of the integral equation and λ is a real parameter (may be complex and has physical meaning).

Also, Ω is the domain of integration with respect to position, and the time $t\in[0,T], T<\infty$.While $\phi(x, t)$ is the unknown function to be determined in the space $L_2(\Omega)\times C[0,T]$.

2. THE EXISTENCE AND UNIQUENESS SOLUTION OF V-HIE WITH A GENERALIZED SINGULAR KERNEL:

In this part, successive approximations method and Banach fixed point theorem will be used as sources to prove the existence and uniqueness solution of the integral equation (1) in the space $L_2(\Omega)\times C[0,T]$, where f, k ,F and γ are known functions . $k(g(x)-g(y))$ is called the generalized kernel of Hammerstein and $F(t,\tau)$ is called the kernel of Volterra with respect to time.

Also, the modified Schauder fixed point theorem will be considered to prove the existence of at least one solution of Eq. (1), when the Lipschitz condition is not satisfied.

2.1 The existence and uniqueness solution using Picard's method :

To discuss the existence and uniqueness solution of Eq. (1), we write it in the integral operator form

$$\bar{W}\phi(x,t) = \frac{1}{\mu}f(x,t) + \frac{\lambda}{\mu}W\phi(x,t), \qquad (\mu\neq 0) \quad (2)$$

where

$$W\phi(x,t) = \int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma(y,\tau,\phi(y,\tau))dyd\tau \quad (3)$$

Also, we assume the following conditions

a- The kernel of position $k\,(\,g\,(x\,)\,-g\,(y\,)\,)$, satisfies the discontinuity condition in $L_p[a,b]$

$$\{\int_\Omega\{\int_\Omega \left|k(\left|g(x)-g(y)\right|)\right|^p dx\}^{\frac{q}{p}} dy\}^{\frac{1}{q}} = c^* \quad (p>1,\ c^* \text{ is a constant})$$

b- The kernel of time $F(t,\tau)\in C[0,T]$ satisfies $F(t,\tau)\le M$, M is a constant, $\forall\, t,\tau\in[0,T], 0\le\tau\le t\le T<\infty$

c- The given function $f\,(x\,,t\,)$ with its partial derivatives with respect to position x and time t are continuous in the space $L_p(\Omega)\times C[0,T]$, and its norm is defined as

$$\|f(x,t)\|_{L_p(\Omega)\times C[0,T]} = \max_{0\le t\le T}\left|\int_0^t\{\int_\Omega |f(x,t)|^p dx\}^{\frac{1}{p}} d\tau\right| = G \quad ,(G \text{ is a constant})$$

d- The known continuous function $\gamma\,(t\,,x\,,\phi\,(x\,,t\,))$, for the constants $Q>P_1$ and $Q>Q_1$, satisfies the following conditions

$$(1)\ \max_{0\le t\le T}\left|\int_0^t\{\left|\gamma(\tau,x,\phi(x,\tau))\right|^p dx\}^{\frac{1}{p}} d\tau\right| \le Q_1\,\|\phi(x,t)\|_{L_p(\Omega)\times C[0,T]}$$

$$(2)\ \left|\gamma(t,x,\phi_1(x,t))-\gamma(t,x,\phi_2(x,t))\right| \le N(t,x)\left|\phi_1(x,t)-\phi_2(x,t)\right|$$

where

$$\|N(t,x)\|_{L_p(\Omega)\times C[0,T]} = \max_{0\le t\le T}\left|\int_0^t\{\int_\Omega \left|N(\tau,x)\right|^p dx\}^{\frac{1}{p}} d\tau\right| = P_1 < \infty$$

Using the method of successive approximations(Picard's method), we set

$$\mu\phi_n(x,t) = f(x,t)+\lambda\int_0^t\int_\Omega F(t,\tau)k(\left|g(x)-g(y)\right|)\gamma(\tau,y,\phi_{n-1}(y,\tau))dyd\tau, \quad (n>1) \tag{4}$$

with $\phi_0(x\,,t\,)=f\,(x\,,t\,)$

For ease of manipulation, it is convenient to introduce

$$\psi_n(x\,,t\,)=\phi_n(x\,,t\,)-\phi_{n-1}(x\,,t\,) \tag{5}$$

Hence, we get

$$\phi_n(x\,,t\,)=\sum_{i=1}^{n}\psi_i(x\,,t\,), \qquad \psi_0(x\,,t\,)=f\,(x\,,t\,) \tag{6}$$

From Eq. (1), we obtain

$$|\mu|\|\phi_n(x,t)-\phi_{n-1}(x,t)\|$$
$$\leq|\lambda|\left\|\int_0^t\int_\Omega|F(t,\tau)||k(|g(x)-g(y)|)||\gamma(\tau,y,\phi_{n-1}(y,\tau))-\gamma(\tau,y,\phi_{n-2}(y,\tau))|\,dyd\tau\right\|$$

with the aid of conditions (b) and (d-2) we have

$$|\mu|\|\phi_n(x,t)-\phi_{n-1}(x,t)\|$$
$$\leq|\lambda|M\left\|\int_0^t\int_\Omega|k(|g(x)-g(y)|)||N(\tau,y)||(\phi_{n-1}(y,\tau)-\phi_{n-2}(y,\tau))|\,dyd\tau\right\|$$

Applying Hölder inequality to Hammerstein integral term, and taking in account (5), the above inequality becomes

$$|\mu|\|\psi_n(x,t)\|$$
$$\leq|\lambda|M\left\|\{\int_\Omega|k(|g(x)-g(y)|)|^p\,dy\}^{\frac{1}{p}}.\max_{0\leq t\leq T}\left|\{\int_0^t\{\int_\Omega|N(\tau,y)|^q|\psi_{n-1}(y,\tau)|^q\,dy\}^{\frac{p}{q}}d\tau\}^{\frac{1}{p}}\right|\right\|$$

Thus, we have

$$\|\psi_n(x,t)\|$$
$$\leq\frac{|\lambda|}{|\mu|}M\,\|N(t,x)\|\|\psi_{n-1}(x,t)\|.\max_{0\leq t\leq T}\left\|\int_0^t\{\int_\Omega\{\int_\Omega|k(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}\}dy\}^{\frac{1}{q}}\}d\tau\right\|$$

In the light of the conditions (a) and (d-2), the last inequality reduces to

$$\|\psi_n(x,t)\|\leq\sigma\|\psi_{n-1}(x,t)\|,\quad (\sigma=\frac{|\lambda|}{|\mu|}Mc^*QT),\quad (n\geq1) \tag{7}$$

When $n=1$ the inequality (7) takes the form

$$\|\psi_1(x,t)\|\leq\sigma G \tag{8}$$

By induction, we have

$$\|\psi_n(x,t)\|\leq\sigma^nG,\quad n=0,1,... \tag{9}$$

Since (9) is obviously true for $n = 0,1,...$; then it holds for all n . This bound makes the sequence $\{\phi_n(x,t)\}$ in (6) converges, so we can write

$$\phi(x,t)=\sum_{i=0}^{\infty}\psi_i(x,t) \tag{10}$$

The series (10) is uniformly convergent since the terms $\psi_i(x,t)$ are dominated by σ^i and $\sigma^i<1$ for $i\to\infty$.

To prove that $\phi(x,t)$ defined by (10) satisfies Eq. (1), set

$$\phi(x,t)=\phi_n(x,t)+\Delta_n(x,t),\quad \left|\Delta_n(x,t)\underset{n\to\infty}{\to}0\right| \tag{11}$$

In view of Eq. (4), we get

$$\left\|\phi(x,t)-\frac{1}{\mu}f(x,t)-\frac{\lambda}{\mu}\int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma(\tau,y,\phi(y,\tau))dyd\tau\right\|$$
$$\leq\|\Delta_n(x,t)\|$$
$$+\frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega|F(t,\tau)||k(|g(x)-g(y)|)||\gamma(\tau,y,\phi(y,\tau)-\Delta_{n-1}(y,\tau))-\gamma(\tau,y,\phi(y,\tau))|dyd\tau\right\|.$$

Using the conditions (b) and (d-2), and applying Hölder inequality to Hammerstein integral term, then with the aid of condition (a), we obtain

$$\left\|\phi(x,t)-\frac{1}{\mu}f(x,t)-\frac{\lambda}{\mu}\int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma(\tau,y,\phi(y,\tau))dyd\tau\right\| \tag{12}$$
$$\leq\|\Delta_n(x,t)\|+\sigma\|\Delta_{n-1}(x,t)\|.$$

So that, by taking n large enough, the right-hand side of (12) can be as small as desired. Thus, the function $\phi(x,t)$ satisfies

$$\mu\phi(x,t)=f(x,t)+\lambda\int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma(\tau,y,\phi(y,\tau))dyd\tau$$

and is therefore a solution of Eq. (1) .

To show that $\phi(x,t)$ is the only solution of Eq. (1), we assume the existence of another solution $\phi(x,t)$, then

$$\|\phi(x,t)-\tilde{\phi}(x,t)\|$$
$$\leq\frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega|F(t,\tau)||k(|g(x)-g(y)|)||\gamma(\tau,y,\phi(y,\tau))-\gamma(\tau,y,\tilde{\phi}(y,\tau))|dyd\tau\right\|,$$

Using the conditions (b) and (d-2) and applying Hölder inequality to Hammerstein integral term, then in view of the condition (a), the above inequality can be adapted in the form

$$\|\phi(x,t)-\tilde{\phi}(x,t)\|\leq\sigma\|\phi(x,t)-\tilde{\phi}(x,t)\| \tag{13}$$

Since $\sigma<1$, then the inequality (13) is true only if $\phi(x,t)=\phi(x,t)$; that is, the solution of Eq. (1) is unique.

2.2 The existence of a unique solution using Banach fixed point theorem:

Theorem 1. The integral equation (1) has a unique solution in the Banach space $L_p(\Omega)\times C[0,T]$, under the condition

$$|\lambda|Mc^*QT<|\mu|. \tag{14}$$

To prove this theorem, we must consider the following lemmas

Lemma 1. Under the condition (a) – (d-2), the operator $\bar{W}$ maps the space $L_p(\Omega)\times C[0,T]$ into itself.

Proof : In the light of the formula (3), we have

$$\|\bar{W}\phi(x,t)\|$$
$$\leq \frac{1}{|\mu|}\|f(x,t)\| + \frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega |F(t,\tau)k(|g(x)-g(y)|)||\gamma(\tau,y,\phi(y,\tau))|\,dyd\tau\right\|.$$

Using the conditions (b) and (c), then applying Hölder inequality, we get

$$\|\bar{W}\phi(x,t)\|$$
$$\leq \frac{G}{|\mu|} + \frac{|\lambda|}{|\mu|}M\left\|\{|k(|g(x)-g(y)|)|^p\,dy\}^{\frac{1}{p}}.\max_{0\leq t\leq T}\left|\int_0^t\{\int_\Omega|\gamma(\tau,y,\phi(y,\tau)|^q\,dy\}^{\frac{1}{q}}d\tau\right|\right\|,$$

$$\|\bar{W}\phi(x,t)\|$$
$$\leq \frac{G}{|\mu|} + \frac{|\lambda|}{|\mu|}M.\max_{0\leq t\leq T}\left|\int_0^t\{\int_\Omega\{\int_\Omega|k(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}dy\}^{\frac{1}{q}}\right|\left\|\max_{0\leq t\leq T}\left|\int_0^t\{\int_\Omega|\gamma(\tau,y,\phi(y,\tau))|^q\,dy\}^{\frac{1}{q}}d\tau\right|\right\|.$$

In view of the conditions (a) and (d-1), the above inequality takes the form

$$\|\bar{W}\phi(x,t)\| \leq \frac{G}{|\mu|} + \sigma\|\phi(x,t)\|, (\sigma = \left|\frac{\lambda}{\mu}\right|Mc^*QT). \tag{15}$$

Inequality (15) shows that, the operator W maps the ball S_ρ
into itself, where

$$\rho = \frac{G}{[|\mu| - |\lambda|Mc^*QT]} \tag{16}$$

Since $\rho > 0$, $G > 0$, therefore we have $\sigma < 1$. Moreover, the inequality (15) involves the boundedness of the
operator W of Eq. (3), where

$$\|W\phi(x,y)\| \leq \sigma\|\phi(x,y)\| \tag{17}$$

Also, the inequalities (15) and (17) define the boundedness of the operator $\bar{W}$.

Lemma 2. Assume that, the conditions (a), (b) and (d-2) are verified, then $\bar{W}$ is a contractive in the space $L_p(\Omega)\times C[0,T]$.

Proof : For the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in the space $L_p(\Omega)\times C[0,T]$, and from Eqs. (1), (3), we find

$$\|\bar{W}\phi_1(x,t) - \bar{W}\phi_2(x,t)\|$$
$$\leq \frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega|F(t,\tau)||k(|g(x)-g(y)|)||\gamma(\tau,y,\phi_1(y,\tau)) - \gamma(\tau,y,\phi_2(y,\tau))|\,dyd\tau\right\|.$$

with the aid of conditions (b) and (d-2), the above inequality becomes

$$\|\bar{W}\phi_1(x,t) - \bar{W}\phi_2(x,t)\|$$
$$\leq \frac{|\lambda|}{|\mu|}M\left\|\int_\Omega\{|k(|g(x)-g(y)|)|^p\,dy\}^{\frac{1}{p}}.\max_{0\leq t\leq T}\int_0^t\{|\gamma(\tau,y,\phi_1(y,\tau)) - \gamma(\tau,y,\phi_2(y,\tau))|^q\,dy\}^{\frac{1}{q}}d\tau\right\|.$$

The above inequality can be written in the form

$$\|\bar{W}\phi_1(x,t)-\bar{W}\phi_2(x,t)\|$$
$$\leq \frac{|\lambda|}{|\mu|}M\left\|\int_0^t\int_\Omega |k(|g(x)-g(y)|)|\|N(\tau,y)\||\phi_1(y,\tau)-\phi_2(y,\tau)|\,dyd\tau\right\|.$$

Applying Hölder inequality to Hammerstein integral term, then using the condition (a) , we finally get

$$\|\bar{W}\phi_1(x,t)-\bar{W}\phi_2(x,t)\|\leq\sigma\|\phi_1(x,t)-\phi_2(x,t)\| \tag{18}$$

From inequality (18), we see that, the operator $\bar{W}$ is continuous in the space $L_p(\Omega)\times C[0,T]$, then $\bar{W}$ is

a contractive, under the condition $\sigma<1$.

The previous two lemmas (1) and (2) proved that, the operator $\bar{W}$ of Eq. (2) is contractive in the Banach space $L_p(\Omega)\times C[0,T]$. So, from Banach fixed point theorem, $\bar{W}$ has a unique fixed point which is the unique solution of Eq. (1).

2.3 The existence of at least one solution:

If the condition (d-2) of theorem (1) is not verified, then the existence of at least one solution of Eq. (1) can be established by virtue of the following theorem.

Theorem 2. Consider Eq. (1) with the same conditions for the functions $k(g(x)-g(y))$, $F(t,\tau)$ and $f(x,t)$ as in theorem (1). Let S_α be the set of function ϕ in $L_p(\Omega)\times C[0,T]$ for which $\|\phi\|\leq\alpha,\alpha$ is a constant, and

assume that the function $\gamma(t,x,\phi(x,t))$ satisfies the condition

(i) $$\max_{0\leq t\leq T}\left|\int_0^t\{\int_\Omega|\gamma(\tau,x,\phi(x,\tau)|^p\,dx\}^{\frac{1}{p}}d\tau\right|\leq E,\quad (E \text{ is a constant})\ \forall\phi\in S_\alpha.$$

(ii) $$\max_{0\leq t\leq T}\left|\int_0^t\{\int_\Omega|\gamma(\tau,x,\phi_1(x,t)-\gamma(\tau,x,\phi_2(x,t)|^p\,dx\}^{\frac{1}{p}}d\tau\right|<\varepsilon$$

if

$$\|\phi_1(x,\tau)-\phi_2(x,\tau)\|<\delta(\varepsilon);\quad \varepsilon<<1,\forall\phi_1,\phi_2\in S_\alpha$$

Then Eq. (1) has at least one solution in S_α.

The proof of this theorem can be obtained directly from the following lemmas.

Lemma 3. The set S_α is a convex closed set in the space $L_p(\Omega)\times C[0,T]$.

Proof : To show that S_ρ is a convex set in the Banach space $L_p(\Omega)\times C[0,T]$, we choose any functions $\phi_1(x,t),\phi_2(x,t)$ in S_ρ, then

$$\|s\phi_1+(1-s)\phi_2\|\le s\,\|\phi_1\|+(1-s)\,\|\phi_2\|\le s\,\rho+(1-s)\rho=\rho,\quad 0\le s\le 1$$

The set S_ρ is closed, since the space $L_p(\Omega)\times C[0,T]$ is complete, and if $\{\phi_n\}$ is a sequence in S_ρ
having a limit ϕ, then

$$\|\phi\|\le\|\phi-\phi_n\|+\|\phi_n\|\le\|\phi-\phi_n\|+\rho,\quad \forall\phi\in S_\rho$$

As $n\to\infty$, it follows that $\|\phi\|\le\rho$, and $\phi\in S_\rho$.

Lemma 4. Under the conditions (a-c) and theorem (1) and (2) respectively, the operator $\bar{W}$ of Eq. (2) maps the set S_ρ into itself.

Proof: In the light of Eq. (2) and (3), we get

$$\|\bar{W}\phi(x,t)\|$$
$$\le\frac{1}{|\mu|}\|f(x,t)\|+\frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega|F(t,\tau)|\,|k(|g(x)-g(y)|)|\,|\gamma(\tau,y,\phi(y,\tau))|\,dyd\tau\right\|.$$

Applying Hölder inequality to Hammerstein integral term, then using the conditions (a-c) and (i) of theorems (1)
and (2), respectively, the above inequality can be adapted in the form

$$\|\bar{W}\phi(x,t)\|\le\frac{G}{|\mu|}+\sigma_1,\quad \left(\sigma_1=\left|\frac{\lambda}{\mu}\right|c^*MET\right).\tag{19}$$

Inequality (19) shows that, the operator $\bar{W}$ maps the set S_ρ into itself, where

$$\rho=\frac{G}{|\mu|}+\sigma_1;\quad (\mu\ne 0)\tag{20}$$

Lemma 5. In the conditions (a-b) and theorem (1) and (2), respectively, are verified, then the operator $\bar{W}$ is continuous
in S_ρ.

Proof : For the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in S_ρ, and from (2), (3), we get

$$\|\bar{W}\phi_1(x,t)-\bar{W}\phi_2(x,t)\|$$
$$\le\frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega|F(t,\tau)|\,|k(|g(x)-g(y)|)|\,|\gamma(\tau,y,\phi_1(y,\tau))-\gamma(\tau,y,\phi_2(y,\tau))|\,dyd\tau\right\|$$

Applying Hölder inequality to the Hammerstein integral term, the above inequality becomes

$$\|\bar{W}\phi_1(x,t)-\bar{W}\phi_2(x,t)\|$$
$$\le\frac{|\lambda|}{|\mu|}Mc^*T\max_{0\le t\le T}\left|\int_0^t\{\int_\Omega|\gamma(\tau,y,\phi_1(y,\tau))-\gamma(\tau,y,\phi_2(y,\tau))|^q\,dy\}^{\frac{1}{q}}d\tau\right|$$

If $\|\phi_1(x,t)-\phi_2(x,t)\|<\delta(\varepsilon)$, then in the light of condition (ii) of theorem (2), the above inequality reduces to

$$\|\bar{W}\phi_1(x,t)-\bar{W}\phi_2(x,t)\|\le\frac{|\lambda|}{|\mu|}Mc^*T\varepsilon<\varepsilon;\qquad \frac{|\lambda|}{|\mu|}Mc^*T<1 \tag{21}$$

which implies the continuity of $\bar{W}$ in the set S_ρ.

Lemma 6. Let $\{k_n(|g(x)-g(y)|)\}$ and $\{F_n(t,\tau)\}$ be two sequences of continuous functions satisfy the conditions

$$\lim_{n\to\infty}\{\int_\Omega\{\int_\Omega|k_n(|g(x)-g(y)|)-k(|g(x)-g(y)|)|^p\,dx\}^{\frac{p}{q}}dy\}^{\frac{1}{q}}=0 \tag{22}$$

$$\lim_{n\to\infty}\max_{0\le t\le T}|F_n(t,\tau)-F(t,\tau)|=0 \tag{23}$$

Then, there exists a positive integer n_0, such that

$$\{\int_\Omega\{\int_\Omega|k_n(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}dy\}^{\frac{1}{q}}\le c^*,\qquad \forall n>n_0 \tag{24}$$

and

$$\max_{0\le t\le T}|F_n(t,\tau)|\le M,\qquad \forall n>n_0 \tag{25}$$

proof: In the light of formulas (22) and (23), and with the aid of theorem (1), there exists two positive integers
n_1,n_2, such that

$$\{\int_\Omega\{\int_\Omega|k_n(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}dy\}^{\frac{1}{q}}$$

$$\le\{\int_\Omega\{\int_\Omega|k_n(|g(x)-g(y)|)-k(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}dy\}^{\frac{1}{q}}+\{\int_\Omega\{\int_\Omega|k(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}dy\}^{\frac{1}{q}}$$

$$\le\varepsilon+c^*,\forall n>n_1.$$

Also,

$$\max_{0\le t\le T}|F_n(t,\tau)|\le\max_{0\le t\le T}|F_n(t,\tau)-F(t,\tau)|+\max_{0\le t\le T}|F(t,\tau)|<\varepsilon_1+M,\forall n>n_2.$$

Since ε and ε_1 are arbitrary, then for $n_0=\max\{n_1,n_2\}$, we conclude

$$\{\int_\Omega\{\int_\Omega|k_n(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}dy\}^{\frac{1}{q}}\le c^*,\forall n>n_0,$$

and $\max_{0\le t\le T}|F_n(t,\tau)|\le M,\forall n>n_0$.

Lemma 7. Assume that the conditions of theorem (2) and lemma (6) are verified, then the sequence of operator $\{\bar{W}_n\}$ defined by

$$\bar{W}_n\phi(x,t) = \frac{1}{\mu} f(x,t) + \frac{\lambda}{\mu}\int_0^t\int_\Omega F_n(t,\tau)k_n(|g(x)-g(y)|)\gamma(\tau,y,\phi(y,\tau))dyd\tau \quad (26)$$

maps the set S_ρ continuously into itself for each $n > n_0$.

Proof : From the formula (22), we get

$$\|\bar{W}_n\phi(x,t)\|$$

$$\leq \frac{1}{\mu}\| f(x,t)\| + \frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega |F_n(t,\tau)||k_n(|g(x)-g(y)|)||\gamma(\tau,y,\phi(y,\tau))|dyd\tau\right\|$$

Applying Hölder inequality to Hammerstein integral term, then using the conditions (c) and theorem (1) and(2), respectively, finally in the light of the conditions (20) and (21), the above inequality takes the form

$$\|\bar{W}_n\phi(x,t)\| \leq \frac{G}{|\mu|} + \sigma_1 \quad (27)$$

Thus $\bar{W}_n$ maps the set S_ρ into itself, where $\alpha = \frac{G}{|\mu|} + \sigma_1, \quad (\mu \neq 0)$

Also, from Eq. (26), we have

$$\|\bar{W}_n\phi_1(x,t) - \bar{W}_n\phi_2(x,t)\|$$

$$\leq \frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega |F_n(t,\tau)||k_n(|g(x)-g(y)|)||\gamma(\tau,y,\phi_1(y,\tau)) - \gamma(\tau,y,\phi_2(y,\tau))|dyd\tau\right\|$$

Applying Hölder inequality to Hammerstein integral term, and using the conditions (24), (25) and (ii) of theorem (2), the above inequality can be reduced to

$$\|\bar{W}_n\phi_1(x,t) - \bar{W}_n\phi_2(x,t)\| < \frac{|\lambda|}{|\mu|} c^* MT\,\varepsilon < \varepsilon, \forall n > n_0 \quad (28)$$

Lemma 8 . Under the same conditions of lemma (2), the set $\bar{W}(S_p)$ is compact.

Proof : From Eqs. (2), (3) and (26), we get

$$\|\bar{W}_n\phi_1(x,t) - \bar{W}_n\phi_2(x,t)\|$$

$$< \frac{|\lambda|}{|\mu|}\{\left\|\int_0^t\int_\Omega \max_{0\leq t\leq T}|F_n(t,\tau)|.|k_n(|g(x)-g(y)|) - k(|g(x)-g(y)|)||\gamma(\tau,y,\phi(y,\tau))|dyd\tau\right\|$$

$$+\left\|\int_0^t\int_\Omega \max_{0\leq t\leq T}|F_n(t,\tau) - F(t,\tau)|.|k(|g(x)-g(y)|)||\gamma(\tau,y,\phi(y,\tau))|dyd\tau\right\|\}$$

In view of conditions (23) and (25), the previous inequality for $n > n_0(\varepsilon)$ becomes

$$\left\|\bar{W}_n\phi_1(x,t)-\bar{W}_n\phi_2(x,t)\right\|$$
$$<\frac{|\lambda|}{|\mu|}\{M\left\|\int_0^t\int_\Omega \left|k_n(|g(x)-g(y)|)-k(|g(x)-g(y)|)\right|\left|\gamma(\tau,y,\phi(y,\tau))\right|dyd\tau\right\|$$
$$+\varepsilon\left\|\int_0^t\int_\Omega \left|k(|g(x)-g(y)|)\right|\left|\gamma(\tau,y,\phi(y,\tau))\right|dyd\tau\right\|\}$$

Applying Hölder inequality to Hammerstein integral term, then using conditions (a), (i) of theorem (1) and (2),
respectively, and Eq.(22), the above inequality becomes

$$\left\|\bar{W}_n\phi_1(x,t)-\bar{W}_n\phi_2(x,t)\right\|<\frac{|\lambda|}{|\mu|}MET\,\varepsilon+\frac{|\lambda|}{|\mu|}c^*ET\,\varepsilon<\varepsilon',\quad \forall n>n_0(\varepsilon) \tag{29}$$

Inequality (29) shows that $\bar{W}_n\phi(x,t)\to\bar{W}\phi(x,t)$ for all $\phi\ (x,t)$ in S_ρ.

Let $\{\phi_n(x,t)\}$ be any sequence in S_ρ. We can obtain the sequence $\{\phi_{n_n}(x,t)\}$ which is a subsequence of every ϕ_{n_j} except for a finite number of elements and clearly $\{\bar{W}_j\phi_{n_n}\}$ converges for every j. Therefore, we have

$$\left\|\bar{W}_n\phi_{n_n}-\bar{W}_n\phi_{m_m}\right\|\le\left\|\bar{W}\phi_{n_n}-\bar{W}_j\phi_{n_n}\right\|+\left\|\bar{W}_j\phi_{n_n}-\bar{W}_j\phi_{m_m}\right\|+\left\|\bar{W}_j\phi_{m_m}-\bar{W}\phi_{m_m}\right\|.$$

Since $\left\|\bar{W}_j\phi_{n_n}-\bar{W}_j\phi_{m_m}\right\|\to 0$ as $m_n,n_n\to\infty$ then for large j , we get from (29), that

$$\left\|\bar{W}\phi_{n_n}-\bar{W}\phi_{m_m}\right\|<2\varepsilon,\qquad \forall m_m,n_n>n_0(\varepsilon).$$

Hence, the sequence $\{\bar{W}\phi_{n_n}-\bar{W}\phi_{m_m}\}$ is Cauchy sequence, so that $\bar{W}(S_\rho)$ is a compact.

According to the lemmas (3) – (8), we see that $\bar{W}$ is a continuous operator maps a closed convex set S_ρ in the Banach space $L_\rho(\Omega)\times C[0,T]$ into itself and $\bar{W}(S_\rho)$ is a compact set. Hence, W has at least one fixed point in S_ρ.

3. SYSTEM OF NONLINEAR INTEGRAL EQUATIONS WITH A GENERALIZED SINGULAR KERNEL IN POSITION

In this section, a numerical method is used, in the mixed integral Eq. (1) to obtain a system of nonlinear integral equations with a generalized singular kernel in position. For this aim we divide the interval

$[0,T],0\le\tau\le t\le T<\infty$ as
$0=t_0<t_1<\ldots<t_k<\ldots<t_N=T$, where $t=t_k$, $k=0,1,2,\ldots,N$,

Hence, the integral term of Eq. (1) becomes

$$\int_0^{t_k}\int_\Omega F(t_k,\tau)k(|g(x)-g(y)|)\gamma(\tau,y,\phi(y,\tau))dyd\tau$$
$$=\sum_{j=0}^{k}u_jF(t_k,t_j)\int_\Omega k(|g(x)-g(y)|)\gamma(t_j,y,\phi(y,t_j))dy+o(h_k^{p+1}), \quad (\hbar_k\to 0,\ p>0) \tag{30}$$

where

$$\hbar=\max_{0\le j\le k}h_j,\ h_j=t_{j+1}-t_j,\ u_k=u_0=\frac{1}{2}h_k,\ u_j=h_j\,(j\ne 0,k)$$

The values of u_j and p; $p\approx k$ are depending on the number of derivatives of $F(t,\tau)$ with respect to time.

Then, we have

$$\mu\phi_k(x)=f_k(x)+\lambda\sum_{j=0}^{k}u_jF_{jk}\int_\Omega k(|g(x)-g(y)|)\gamma_j(y,\phi_j(y))dy \tag{31}$$

Here, we used the following notations

$$\begin{aligned}&\phi(x,t_k)=\phi_k(x),\ f(x,t_k)=f_k(x),\ F(t_k,t_j)=F_{jk}.\\ &\gamma(t_j,x,\phi(x,t_j))=\gamma_j(x,\phi_j(x)),\ x=\bar{x}(x_1,x_2,\dots,x_n),\\ &y=\bar{y}(y_1,y_2,\dots,y_n)\end{aligned} \tag{32}$$

The formula (31) represents a system of nonlinear integral equations in n-dimensionals, and its solution depends on the given function $f^{(k)}(x)$, the kind of the kernel $k(g(x)-g(y))$, and the degree of the known function $\gamma^{(j)}(x,\phi^{(j)}(x))$.

Eq.(31) can be written in the form

$$\phi_k(x)-\frac{\lambda}{\mu}u_kF_{kk}\int_\Omega k(|g(x)-g(y)|)\gamma_k(y,\phi_k(y))dy=H_k(x) \tag{33}$$

where

$$H_k(x)=\frac{1}{\mu}f_k(x)+\frac{\lambda}{\mu}\sum_{j=0}^{k-1}u_jF_{jk}\int_\Omega k(|g(x)-g(y)|)\gamma_j(y,\phi_j(y))dy$$

4. THE TOEPLITZ MATRIX METHOD

Here, we will discuss the solution of Eq. (1) numerically using Toeplitz matrix method, and $\Omega=[-b,b]$. For this, write Eq. (1) in the form

$$\phi_i(x)=f_i(x)+\lambda\sum_{j=0}^{i}w_jF_{ij}\int_\Omega k(|g(x)-g(y)|)\gamma_j(y,\phi_j(y))dy \tag{34}$$

where

$$\int_{\Omega} k\,(|g(x)-g(y)|)\gamma_j(y,\phi_j(y))dy\,; \quad (h=\frac{b}{N}).$$

Hence, the integral term in the right hand side of Eq.(1) can be written in the form

$$\int_{a}^{a+h} k(|g(x0-g(y)|)\gamma_j(y,\phi_j(y))dy$$
$$= A_n^{(i)}(g(x))\gamma_j(a,\phi_j(a)) + B_n^{(j)}(g(x))\gamma_j(a+h,\phi_j(a+h)) + R,\ (a=nh), \tag{35}$$

where $A_n^{(j)}(g(x))$ and $B_n^{(j)}(g(x))$ for all values of $j, 0 \le j \le i$ are arbitrary functions to be determined, and R is the estimate error, $R = \max_j R^{(j)}$. To determine $A_n^{(j)}(g(x))$ and $B_n^{(j)}(g(x))$ in the light of Toeplitz matrix method, we put $\phi(y) = g'(y)$ and $\phi(y) = g'(y)g(y)$, respectively in Eq. (35), where $g'(x)$ is a monotonic increasing function. These yields a set of two equations in terms of two unknown functions where, in this case, the error is vanishing. Solving the results, we have

$$A_n^{(j)}(g(x)) = \frac{1}{h_1^{(j)}}[\gamma_j(a+h,g'(a+h)g(a+h))I - \gamma_j(a+h,g'(a+h))J], \tag{36}$$

$$B_n^{(j)}(g(x)) = \frac{1}{h_1^{(j)}}[\gamma_j(a,g'(a)J - \gamma_j(a,g'(a))I], \tag{37}$$

where

$$I^{(j)} = \int_{a}^{a+h} k(|g(x)-g(y)|)\gamma_j(y,g'(y))dy,\quad J^{(j)}$$
$$= \int_{a}^{a+h} k(|g(x)-g(y)|)\gamma_j(y,g'(y)g(y))dy, \tag{38}$$

and

$$h_1^{(j)} = \gamma_j(a,g'(a))\gamma_j(a+h,g'(a+h)g(a+h))$$
$$-\gamma_j(a+h,g'(a+h))\gamma_j(a,g'(a)g(a)),\qquad 0\le j\le i. \tag{39}$$

In view of Eqs. (34) – (39), the formula (31) becomes

$$\int_{\Omega} k\,(|g(x)-g(y)|)\gamma_j(y,\phi(y))dy = \sum_{n=-N}^{N} D_n^{(j)}(x)\gamma_j(nh,\phi_j(nh)), \tag{40}$$

where

$$D_n^{(j)}(x) = \begin{cases} A_{-N}^{(j)}(x) & ,n=-N \\ A_n^{(j)}(x) + B_{n-1}^{(j)}(x) & ,-N<n<N \\ B_{N-1}^{(j)}(x) & ,n=N, \end{cases} \tag{41}$$

Thus, the integral equation (1) takes the form

$$\mu\phi_i(x)-\lambda\sum_{j=0}^{i}\sum_{n=-N}^{N}F_{ij}D_n^{(j)}(x)\gamma_j(nh,\phi_j(nh))=f_i(x). \tag{42}$$

Putting $x=mh$ in (42), and using the following notations

$$\begin{aligned}&\phi_i(\ell h)=\phi_{i\ell},\ D_n^{(i)}(mh)=D_{mn}^{(i)},\ f_i(mh)=f_{im},\\ &\gamma(\tau_j,nh,\phi_j(nh))=\gamma_{jn}(\phi_{jn})\end{aligned} \tag{43}$$

we get the following nonlinear algebraic system

$$\mu\phi_{im}-\lambda\sum_{j=0}^{i}\sum_{n=-N}^{N}F_{ij}D_{mn}^{(j)}(\phi_{jn})=f_{im},\quad -N\le m\le N \tag{44}$$

$$D_{mn}^{(j)}=\begin{cases}A_{-N}^{(j)}(mh) & ,n=-N\\ A_n^{(j)}(mh)+B_{n-1}^{(j)}(mh) & ,-N<n<N\\ B_{N-1}^{(j)}(mh) & ,n=N,\ j=1,2,\ldots,i,\end{cases} \tag{45}$$

The matrices $D_{mn}^{(j)}$ can be written in the Toeplitz matrices forms

$$D_{mn}=G_{mn}^{(j)}-E_{mn}^{(j)} \tag{46}$$

Here, the matrices $G_{mn}^{(j)}$ is called Toeplitz matrices of order $(2N+1)$ and

$$G_{mn}^{(j)}=A_n^{(j)}(mh)+B_{n-1}^{(j)}(mh),\quad -N\le m,n\le N, \tag{47}$$

and

$$E_{mn}^{(j)}=\begin{cases}B_{-N-1}^{(j)}(mh) & ,n=-N\\ 0 & ,-N<n<N\\ A_N^{(j)}(mh) & ,n=N,\ j=1,2,\ldots,i,\end{cases} \tag{48}$$

represent a matrices of order $(2N+1)$ whose elements are zeros except the first and the last rows (columns).

The error term R is determined from the following formula

$$\begin{aligned}R^{(j)}=&\left|\int_{nh}^{nh+h}\gamma_j(y,g'g^2)k(|g(x)-g(y)|)dy-A_n^{(j)}(g(x))\gamma_j(nh,(nh)^2)\right.\\ &\left.-B_n^{(j)}(g(x))(nh+h,(nh+h)^2).\right|\\ =&\,o(h^3);\qquad\qquad R=\max_j R^{(j)}\end{aligned}$$

4.1 Nonlinear algebraic system of the Toeplitz matrix:

In section, will be devoted to prove the existence of a unique solution and the existence of at least one solution of the nonlinear algebraic system (44) in the Banach space ℓ^{∞}. For this, we write it in the operator form

$$\bar{T}\phi_m = T\phi_m + \frac{1}{\mu} f_m \quad , \tag{50}$$

where

$$T\phi_m = \frac{\lambda}{\mu} \sum_{j=0}^{i} \sum_{n=-N}^{N} F_{ij} D_{mn}^{(j)} \gamma_{jn}(\phi_{nj}); \qquad (\mu \neq 0, -N \leq m \leq N). \tag{51}$$

Then, we consider the following.

Lemma 9. If the kernel of Eq. (1) satisfies the following conditions

$$k(|g(x)-g(y)|) \in L_q; \quad q > 1, \tag{52}$$

$$\lim_{x' \to x} \| k(|g(x')-g(y)|) - k(|g(x)-g(y)|) \| = 0, \tag{53}$$

Then for $\sup_i \sum_{j=0}^{i} |F_{ij}| < q_1,\ q_1 \neq 0$ we have

$$\text{(i)} \ \sup_N \sup_j \sum_{j=0}^{i} \sum_{n=-N}^{N} |F_{ij}| |D_{mn}^{(j)}| \text{ exists}$$

$$\text{(ii)} \ \lim_{m' \to m} \sup_j \sup_N \sum_{j=0}^{i} \sum_{n=-N}^{N} |F_{ij}| |D_{m'n}^{(j)} - D_{mn}^{(j)}| = 0 \tag{54}$$

proof: F rom the formula (54), we have

$$\begin{aligned} &|A_n^{(j)}(x)| \\ &\leq \frac{1}{|h_1^{(j)}|} [|\gamma_j(a+h, g'(a+h)g(a+h))| \int_a^{a+h} \sum_{j=0}^{i} |F_{ij}| |k(|g(x)-g(y)|)| |\gamma_j(y, g'(y))| \\ &\quad . dy + |\gamma_j(a+h, g'(a+h))| \int_a^{a+h} \sum_{j=0}^{i} |F_{ij}| |k(|g(x)-g(y)|)| |\gamma_j(y, g'(y)g(y))| dy] \end{aligned}$$

Applying Hölder inequality for

p>1,q>1;

to each integral term of the above inequality, we get

$$\begin{aligned} |A_n^{(j)}(x)| \leq \frac{1}{|h_1^{(j)}|} \sum_{j=0}^{i} |F_{ij}| \Big(\int_a^{a+h} |k(|g(x)-g(y)|)|^q dy \Big)^{\frac{1}{q}} [|\gamma_j(a+h, g'(a+h)g(a+h))| \\ \Big(\int_a^{a+h} |\gamma_j(y, g'(y))|^p dy \Big)^{\frac{1}{p}} + |\gamma_j(a+h, g'(a+h))| . \Big(\int_a^{a+h} |\gamma_j(y, g'(y)g(y)|^p dy \Big)^{\frac{1}{p}}] \end{aligned}$$

Hence,

$$\left|A_n^{(j)}(x)\right| \le \frac{1}{\left|h_1^{(j)}\right|}\sum_{j=0}^{i}\left|F_{ij}\right|\left(\int_a^{a+h}\int_a^{a+h}\left|k(\left|g(x)-g(y)\right|)\right|^q dxdy\right)^{\frac{1}{q}}\Big[\left|\gamma_j(a+h,g'(a+h)g(a+h))\right|$$

$$.\left(\int_a^{a+h}\left|\gamma_j(y,g'(y))\right|^p dy\right)^{\frac{1}{p}}+\left|\gamma_j(a+h,g'(a+h))\right| .\left(\int_a^{a+h}\left|\gamma_j(y,g'(y)g(y)\right|^p dy\right)^{\frac{1}{p}}\Big]$$

Summing from $n=-N$ to $n=N$, we obtain

$$\left|A_n^{(j)}(x)\right| \le \frac{1}{\left|h_1^{(j)}\right|}\sum_{j=0}^{i}\left|F_{ij}\right|\left\|k(\left|g(x)-g(y)\right|)\right\|_{L_q}\Big[\sum_{n=-N}^{N}\left|\gamma_j(a+h,a+h)\right|\left\|\gamma_j(y,1)\right\|_{L_p}$$

$$+\left|\gamma_j(a+h,1)\right|\left\|\gamma_j(y,y)\right\|_{L_p}\Big].$$

In view of the condition (36), and the continuity of the function γ in the domain Ω, there exists a small constant E_1, such that

$$\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|A_n^{(j)}(x)\right| \le E_1, \quad \forall N, \quad E_1=\sup_j E_1^{(j)}.$$

Since, each term of $\sum_{j=0}^{i}\sum_{n=-N}^{N}A_n^{(j)}(x)$ is bounded above, hence for $x=mh$, we deduce

$$\sup_j\sup_N\sum_{n=-N}^{N}\left|A_n^{(j)}(mh)\right| \le E_1. \tag{55}$$

Similarly, from formula (37), we can find a small constant E_2, such that

$$\sup_j\sup_N\sum_{n=-N}^{N}\left|B_n^{(j)}(mh)\right| \le E_2, \quad E_2=\sup_j E_2^{(j)}. \tag{56}$$

In the light of (42), (45), and with the help of (55) and (56), there exists a small constant E, such that

$$\sup_i\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|F_{ij}\right|\left|D_{mn}\right|$$

$$\le \sup_i\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|A_n^{(j)}(mh)\right|+\sup_i\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|B_n^{(j)}(mh)\right| \le E; \quad (E=E_1+E_2)$$

Hence, $\sup_i\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|F_{ij}\right|\left|D_{mn}\right|$ exists.

By virtue of the formula (36), we get for $x, x' \in [-b,b]$

$$\left|A_n^{(j)}(x') - A_n^{(j)}(x)\right|$$
$$\leq \frac{1}{\left|h_1^j\right|}\{\sum_{j=0}^{i}\left|F_{ij}\right|\{\left|\gamma_j(a+h,a+h)\right|\int_a^{a+h}\left|k(\left|g(x')-g(y)\right|)-k(\left|g(x)-g(y)\right|)\right|\left|\gamma_j(y,1)\right|dy$$
$$+\left|\gamma_j(a+h,1)\right|\int_a^{a+h}\left|k(\left|g(x')-g(y)\right|)-k(\left|g(x)-g(y)\right|)\right|\left|\gamma_j(y,y)\right|dy\}.$$

Applying Holder inequality, then summing from $n=-N$ to $n=N$, and taking account the continuity of the function γ, the above inequality can be adapted in the form

$$\sup_j\sup_N\sum_{n=-N}^{N}\left|A_n^{(j)}(x')-A_n^{(j)}(x)\right|$$
$$\leq \frac{1}{\left|h_1^{(j)}\right|}\left\|k(\left|g(x')-g(y)\right|)-k(\left|g(x)-g(y)\right|)\right\|_{L_q}\{\sup_i\sum_{j=0}^{i}\left|F_{ij}\right|\sup_N\sum_{n=-N}^{N}\left|\gamma_j(a+h,a+h)\right|\left\|\gamma_j(y,1)\right\|_{L_p}$$
$$+\sup_i\sum_{j=0}^{i}\left|F_{ij}\right|\sup_N\sum_{n=-N}^{N}\left|\gamma_j(a+h,1)\right|\left\|\gamma_j(y,y)\right\|_{L_p}.$$

Putting $x=mh,\ x'=m'h,$ then using the condition (53), we get

$$\lim_{m'\to m}\sup_j\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|A_n^{(j)}(m'h)-A_n^{(j)}(mh)\right|=0. \tag{57}$$

Similarly, in view of the formula (37), we can prove

$$\lim_{m'\to m}\sup_j\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|B_n^{(j)}(m'h)-B_n^{(j)}(mh)\right|=0. \tag{58}$$

Finally, with the aid of (42), (57) and (58), we have

$$\lim_{m'\to m}\sup_j\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|F_{ij}\right|\left|D_{m'n}^{(j)}-D_{mn}^{(j)}\right|=0.$$

4.2 The existence and uniqueness solution of the nonlinear algebraic system of Toeplitz matrix :

The existence of a unique solution of the algebraic system (42), will be proved according to the Banach fixed point theorem. For this aim, we consider the following assumption

$$\sup_m\left|f_m\right|\leq H<\infty,\qquad (H \text{ is a constant}). \tag{59}$$

$$\sup_i\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|F_{ij}\right|\left|D_{mn}^{(j)}\right|\leq E,\qquad (E \text{ is a constnat}). \tag{60}$$

For the constants $Q>Q_1, Q>P_1$; the known functions $\gamma(nh,\phi(nh)),$ satisfies

$$\sup_n \left|\gamma(nh,\phi(nh))\right| \le Q_1 \left\|\Phi\right\|_{\ell^\infty}, \tag{61}$$

$$\sup_n \left|\gamma(nh,\phi(nh)) - \gamma(nh,\psi(nh))\right| \le P_1 \left\|\Phi - \Psi\right\|_{\ell^\infty}, \tag{62}$$

where $\left\|\Phi\right\|_{\ell^\infty} = \sup_n \left|\phi_n\right|$, for each integer n.

Theorem 3. The algebraic system (42), in the Banach space ℓ_{∞}, has a unique solution under the condition

$$\left|\lambda\right| < \frac{\left|\mu\right|}{QE}. \tag{63}$$

To prove this theorem, we must consider the following lemmas.

Lemma 10. If the conditions (59) – (61) are verified, then the operator $\bar{T}$ defined by Eq. (50) maps the space ℓ^∞ into itself.

Proof: Let U be the set of all functions $\Phi = \{\phi_m\}$ in ℓ^∞ such that $\left\|\Phi\right\|_{\ell^\infty} \le \beta, \beta$ is a constant. Define the norm of the operator $\bar{T}\Phi$ in the Banach space ℓ^∞ by

$$\left\|\bar{T}\Phi\right\|_{\ell^\infty} = \sup_m \left|\bar{T}\phi_m\right|, \quad \text{for each integer } m. \tag{64}$$

From the formulas (50) and (51), we get

$$\left|\bar{T}\phi_m\right| \le \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \sum_{n=-N}^{N} F_{ij} \left|D_{mn}^{(j)}\right| \sup_m \left|\gamma(nh,\phi_j(nh))\right| + \frac{1}{\left|\mu\right|} \sup_m \left|f_m\right|.$$

Using the conditions (59) and (61) we obtain

$$\left|\bar{T}\phi_m\right| \le \left|\frac{\lambda}{\mu}\right| Q \left\|\Phi\right\|_{\ell^\infty} \sup_i \sup_N \sum_{j=0}^{i} \sum_{n=-N}^{N} \left|F_{ij}\right| \left|D_{mn}^{(j)}\right| + \frac{H}{\left|\mu\right|}. \tag{65}$$

In view of the condition (60), the above inequality can be adapted in the form

$$\sup_m \left|\bar{T}\phi_m\right| \le \sigma_1 \left\|\Phi\right\|_{\ell^\infty} + \frac{H}{\left|\mu\right|}, \qquad \left(\sigma_1 = \left|\frac{\lambda}{\mu}\right| QE\right). \tag{66}$$

Since, the above inequality is true for each integer m, we deduce

$$\left\|\bar{T}\Phi\right\|_{\ell^\infty} \le \sigma_1 \left\|\Phi\right\|_{\ell^\infty} + \frac{H}{\left|\mu\right|}. \tag{67}$$

The inequality (67) shows that, the operator $\bar{T}$ maps the set U into itself, where

$$\beta = \frac{H}{\left(\left|\mu\right| - \left|\lambda\right| QE\right)}. \tag{68}$$

Since $\beta > 0, H > 0$ therefore we have $\sigma_1 < 1$. Also, the inequality (67), with the aid of (60) involves the boundedness of the operator T, where

$$\left\|T\Phi\right\|_{\ell^\infty} \le \sigma_1 \left\|\Phi\right\|. \tag{69}$$

Furthermore, the inequalities (67) and (69) define the boundness of the operator $\bar{T}$.

Lemma 11. Under the two conditions (60) and (62), $\bar{T}$ is a continuous and a contraction operator in the space ℓ^{∞}.

Proof: For the two functions Φ and Ψ in ℓ^{∞}, the formulas (50) and (51) lead to

$$\left|\bar{T}\phi_m - \bar{T}\psi_m\right| \le \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \sum_{n=-N}^{N} \left|F_{ij}\right| \left|D_{mn}^{(j)}\right| \sup_n \left|\gamma(nh,\phi(nh)) - \gamma(nh,\psi(nh))\right|.$$

In the light of condition (62), we obtain

$$\left|\bar{T}\phi_m - \bar{T}\psi_m\right| \le \left|\frac{\lambda}{\mu}\right| Q \left\|\Phi - \Psi\right\|_{\ell^{\infty}} \sup_j \sup_N \sum_{j=0}^{i} \left|F_{ij}\right| \sum_{n=-N}^{N} \left|D_{mn}\right|.$$

Using the condition (60), we get

$$\left|\bar{T}\phi_m - \bar{T}\psi_m\right| \le \sigma_1 \left\|\Phi - \Psi\right\|_{\ell^{\infty}} . \tag{70}$$

The above inequality is true for each integer m , hence in view of (64) we have

$$\left|\bar{T}\Phi - \bar{T}\Psi\right| \le \sigma_1 \left\|\Phi - \Psi\right\|_{\ell^{\infty}} . \tag{71}$$

The inequality (71) shows that, the operator $\bar{T}$ is continuous in the space ℓ^{∞}, then $\bar{T}$ is a contraction operator, under the condition $\sigma_1 < 1$.

In the light of the lemmas (10) and (11), the operator $\bar{T}$ defined by (50) is contractive in the Banach space ℓ^{∞}. Hence, $\bar{T}$ has a unique fixed point which is, the unique solution of the nonlinear algebraic system in ℓ^{∞}.

In the next theorem, the convergence of the sequence of approximate solution to the exact solution of Eq. (50) will be proved in the Banach space ℓ^{∞}

Theorem 4. If the conditions (60) and (62) are satisfied and the sequence of functions $\{L_j\} = \{(f_m)_j\}$ converges uniformly to the function $L = \{f_m\}$ in the Banach space ℓ^{∞}. Then, the sequence of approximate solution $\{\Phi_j\} = \{(\phi_m)_j\}$ converges uniformly to the solution $\Phi_j = \{\phi_m\}$ of Eq. (50) in the Banach space ℓ^{∞}.

Proof : By virtue of Eq. (42), we have

$$\left|\phi_m - (\phi_m)_j\right|$$
$$\le \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \sum_{n=-N}^{N} \left|F_{ij}\right| \left|D_{mn}^{(j)}\right| \sup_n \left|\gamma(nh,\phi(nh)) - \gamma(nh,\phi_j(nh))\right| + \frac{1}{|\mu|} \left|f_m - (f_m)_j\right|.$$

The above inequality, after using condition (60), holds for each integer m, hence from condition (62), we find

$$\sup_n \left|\phi_m - (\phi_m)_j\right| \le \sigma_1 \left\|\Phi - \Phi_j\right\|_{\ell^\infty} + \frac{1}{|\mu|}\sup_n \left|L - L_j\right|. \tag{72}$$

Finally, the previous inequality takes the form

$$\left\|\Phi - \Phi_j\right\|_{\ell^\infty} \le \frac{1}{[|\mu| - |\lambda| QE]}\left\|L - L_j\right\|_{\ell^\infty}; \quad (\sigma_1 < 1) \tag{73}$$

Since $\left\|L - L_j\right\|_{\ell^\infty} \to 0$ as $j \to \infty$, so that $\left\|\Phi - \Phi_j\right\|_{\ell^\infty} \to 0$.

4.3 The error of the Toeplitz matrix method:

Definition 1. The estimate local error R, is determined by the following equation

$$\begin{aligned} &\left|\phi(x,t) - \phi_j(x,t)\right| \\ &= \sum_{j=0}^{i}\sum_{n=-N}^{N} w_j F_{ij} D_{mn}^{(j)} \left|\gamma(\tau, nh, \phi(nh,\tau)) - \gamma(\tau_j, nh, \phi(nh, \tau_j))\right| + R_j \end{aligned} \tag{74}$$

where $\phi_j(x)$ is the approximate solution of Eq. (1).

Also, Eq. (74) gives

$$\begin{aligned} R_j = \Bigg| &\int_0^t \int_\Omega F(t,\tau) k(|g(x) - g(y)|) \gamma(\tau, y, \phi(y,\tau)) dy d\tau \\ &- \sum_{j=0}^{i}\sum_{n=-N}^{N} w_j F_{ij} D_{mn}^{(j)} \gamma(\tau_j, nh, \phi(nh, \tau_j)) \Bigg| \end{aligned} \tag{75}$$

Definition 2. The Toeplitz matrix method is said to be convergent of order r in the interval $[-b, b]$, if and only if for sufficiently large N, there exists a constant $D > 0$ independent of N such that

$$\left\|\phi(x,t) - \phi_N(x,t)\right\| \le DN^{-r}. \tag{76}$$

Now, we will give theorem which prove that, the estimate error R_j is very small and be neglected as $j \to \infty$, either the nonlinear algebraic system (44) has a unique solution or it has at least one solution.

Theorem 5. Assume that, the hypothesis of theorem (3) are verified, then

$$\lim_{j \to \infty} R_j = 0. \tag{77}$$

Proof : In view of the formula (74), we have

$$\left|R_j\right| \leq \left|\phi_m - (\phi_m)_j\right|$$
$$-\sum_{j=0}^{i}\sum_{n=-N}^{N} F_{ij} D_{mn}^{(j)} \left(\left|\gamma(\tau_i, nh, \phi(nh,\tau_i)) - \gamma(\tau_j, nh, \phi(nh,\tau_j))\right|\right)$$

Using the condition (62), we obtain

$$\left|R_j\right| \leq \sup_m \left|\phi_m - (\phi_m)_j\right| + Q\left\|\Phi - \Phi_j\right\| \sup_i \sup_N \sum_{j=0}^{i}\sum_{n=-N}^{N}\left|F_{ij}\right|\left|D_{mn}^{(j)}\right|$$

$$\left|R_j\right| \leq (1+QE)\left\|\Phi - \Phi_j\right\|_{\ell^\infty}, \text{ for each } j. \quad (78)$$

Since $\left\|\Phi - \Phi_j\right\|_{\ell^\infty} \to 0$ as $j \to \infty$, then $R_j \to 0$.

5. NUMERICAL EXAMPLES

In this section, we apply the Toeplitz matrix method, to obtain the numerical solution of the **V-HIESK** with a generalized singular kernel using **Maple10** program. This leads to the required approximate solution of the **V-FIESK** (1) when the kernel $k(\left|g(x)-g(y)\right|)$ takes the forms of Carleman function, logarithmic form, and Cauchy kernel.

5.1 Application for a Generalized Carleman Kernel

Example 1: Consider the integral equation:

$$\phi(x,t) = f(x,t) + \lambda \int_0^t \int_{-1}^{1} \left|x^4 - y^4\right|^{-\upsilon} \tau^2 \phi^2(y,\tau) dy d\tau, \quad (0 \leq t \leq T; \left|x\right| \leq 1)$$

The Toeplitz matrix method is used to get the numerical solution for values of μ= 1, at the times $t\in[0,0.03]$, $t\in[0,0.6]$, with $\lambda = 0.2500$, and 0.31579, and we divided the position interval by $N = 21$ units, and $0 < \nu < 1/2$, ν is called Poisson ratio.

the exact solution $\phi(x,t) = x^5 t^6$.

<u>Case1 : λ= 0.2500, ν= 0.1:</u>

Table 1

T	*x*	*E*	*Appr. sol.T.*	*Err. T.*
0.03	-1.00	-7.29000E-10	-7.29008E-10	8.22500E-15
	-0.60	-5.66870E-11	-5.66876E-11	6.38000E-16
	-0.20	-2.33280E-13	-2.33282E-13	2.62800E-18
	0.20	2.33280E-13	2.33282E-13	2.62800E-18
	0.60	5.66870E-11	5.66876E-11	6.38000E-16
	1.00	7.29000E-10	7.29008E-10	8.21100E-15

0.6	-1.00	-4.66560E-02	-4.72000E-02	5.44011E-04
	-0.60	-3.62797E-03	-3.59137E-03	3.65956E-05
	-0.20	-1.49299E-05	-1.64042E-05	1.47431E-06
	0.20	1.49299E-05	1.35937E-05	1.33619E-06
	0.60	3.62797E-03	3.69153E-03	6.35640E-05
	1.00	4.66560E-02	4.68023E-02	1.46391E-04

Case2 : λ= 0.31579, ν= 0.12:

Table 2

T	*x*	*Exact sol.*	*Appr.*	*Err. T.*
0.03	-1.00	-7.29000E-10	-7.29010E-10	1.03670E-14
	-0.60	-5.66870E-11	-5.66878E-20	8.09000E-16
	-0.20	-2.33280E-13	-2.33283E-13	3.31900E-18
	0.20	2.33280E-13	2.33283E-13	3.31800E-18
	0.60	5.66870E-11	5.66878E-20	8.06000E-16
	1.00	7.29000E-10	7.29010E-10	1.03738E-14
0.6	-1.00	-4.6656E-02	-4.71160E-02	4.60041E-04
	-0.60	-3.62797E-03	-3.58717E-03	4.07980E-05
	-0.20	-1.49299E-05	-1.68242E-05	1.89435E-06
	0.20	1.49299E-05	1.32102E-05	1.71967E-06
	0.60	3.62797E-03	3.70352E-03	7.55503E-05
	1.00	4.66560E-02	4.68863E-02	2.30330E-04

Example 2: Consider the integral equation:

$$\phi(x,t)=f(x,t)+\lambda\int_0^t\int_{-1}^1\left|\sin(x)-\sin(y)\right|^{-\upsilon}\tau^2\phi^2(y,\tau)dyd\tau,$$

The values of $\mu=1$, at the times $t\in[0\ ,\ 0.006]$, $t\in[0\ ,\ 0.03]$, with λ= 0.111111, 0.13636, and we divided the position interval by $N=21$ units. Exact solution $\phi(x,t)=t\sin(x)$.

Case1 : λ= 0.111111 , ν= 0.05:

Table 3

T	*x*	*E*	*Appr.*	*Err. T.*
0.006	-1.00	-5.04882E-03	-4.80932E-03	2.39505E-03
	-0.60	-3.38785E-03	-3.45951E-03	7.16582E-05
	-0.20	-1.19201E-03	-1.24102E-03	4.90082E-05
	0.20	1.19201E-03	1.22417E-03	3.21594E-05
	0.60	3.38785E-03	3.41465E-03	2.67965E-05
	1.00	5.04882E-03	5.01148E-03	3.73373E-05
0.03	-1.00	-2.52441E-02	-2.40467E-02	1.19741E-03
	-0.60	-1.69392E-02	-1.72976E-02	3.58379E-04
	-0.20	-5.96007E-03	-6.20515E-03	2.45074E-04
	0.20	5.96007E-03	6.12090E-03	1.60827E-04
	0.60	1.69392E-02	1.70733E-02	1.34068E-04
	1.00	2.52441E-02	2.50575E-02	1.86563E-04

Case2 : λ= 0.13636 , ν= 0.06:

Table 4

T	*x*	*E*	*Appr.*	*Err. T.*
0.006	-1.00	-5.04882E-03	-4.69240E-03	3.56422E-04
	-0.60	-3.38785E-03	-3.49470E-03	1.06848E-04
	-0.20	-1.19201E-03	-1.26575E-03	7.37403E-05
	0.20	1.19201E-03	1.24006E-03	4.80503E-05
	0.60	3.38785E-03	3.42787E-03	4.00218E-05
	1.00	5.04882E-03	4.99306E-03	5.57592E-05
0.03	-1.00	-2.52441E-02	-2.34621E-02	1.78198E-03
	-0.60	-1.69392E-02	-1.74736E-02	5.34355E-04
	-0.20	-5.96007E-03	-6.32882E-03	3.68743E-04
	0.20	5.96007E-03	6.20037E-03	2.40290E-04
	0.60	1.69392E-02	1.71394E-02	2.00214E-04
	1.00	2.52441E-02	2.49654E-02	2.78645E-04

5.2 Application for a Generalized logarithmic kernel .

Example 1: Consider the integral equation:

$$\phi(x,t)=f(x,t)+\lambda\int_0^t\int_{-1}^1 \ln\left|x^4-y^4\right|\tau^2\phi^2(y,\tau)dyd\tau,$$

The Toeplitz matrix method are used to get approximate solution for values of $\mu = 1$, $\lambda = 0.25$, 0.6666666667, $t\in[0,0.006]$, $t\in[0\ ,\ 0.03]$ and $N = 21$. Exact solution $\phi(x,t)=x^5t^6$.

Case1 : λ= 0.25:

Table 5

T	*x*	*E*	*Appr.*	*Err. T.*
0.006	-1.00	-4.66560E-14	-4.66560E-14	2.09000E-20
	-0.60	-3.62797E-15	-3.62797E-15	1.62000E-21
	-0.20	-1.49299E-17	-1.49299E-17	6.72000E-24
	0.20	1.49299E-17	1.49299E-17	8.09000E-24
	0.60	3.62797E-15	3.62797E-15	1.62000E-21
	1.00	4.66560E-14	4.66560E-14	2.09000E-20
0.03	-1.00	-7.29000E-10	-7.29008E-10	8.21700E-15
	-0.60	-5.66870E-11	-5.66876E-11	6.38300E-16
	-0.20	-2.33280E-13	-2.33286E-13	2.62760E-18
	0.20	2.33280E-13	2.33282E-13	2.62760E-18
	0.60	5.66870E-11	5.66876E-11	6.39500E-16
	1.00	7.29000E-10	7.29008E-10	8.21900E-15

Case2 : λ= 0.6666666667:

Table 6

T	*x*	*E*	*Appr.*	*Err. T.*
0.006	-1.00	-4.66560E-14	-4.66560E-14	5.61000E-20
	-0.60	-3.62797E-15	-3.62797E-15	4.24000E-21
	-0.20	-1.49299E-17	-1.49299E-17	1.79600E-23
	0.20	1.49299E-17	1.49299E-17	1.78600E-23
	0.60	3.62797E-15	3.62797E-15	4.24000E-21
	1.00	4.66560E-14	4.66560E-14	5.60000E-20
0.03	-1.00	-7.29000E-10	-7.29021E-10	2.19340E-14
	-0.60	-5.66870E-11	-5.66887E-11	1.70500E-15
	-0.20	-2.33280E-13	-2.33287E-13	7.00820E-18
	0.20	2.33280E-13	2.33287E-13	7.00560E-18
	0.60	5.66870E-11	5.66887E-11	1.70200E-15
	1.00	7.29000E-10	7.29021E-10	2.19280E-14

Example 2: Consider the integral equation:

$$\phi(x,t) = f(x,t) + \lambda \int_0^t \int_{-1}^1 \ln\left|e^{x^2} - e^{y^2}\right| \tau^2 \phi^2(y,\tau) dy d\tau,$$

The values of $\mu = 1$, $\lambda = 0.001, 0.01, 0$, $t \in [0,0.006]$, $t \in [0,0.03]$, and $N = 21$. Exact solution $\phi(x,t) = e^{x^5} t^3$.

Case1 : λ= 0.001:

Table 7

T	*x*	*E*	*Appr.*	*Err. T.*
0.006	-1.00	7.94619E-08	2.07018E-07	1.27556E-07
	-0.60	1.99840E-07	1.80487E-07	1.93531E-08
	-0.20	2.15930E-07	2.13927E-07	2.00367E-09
	0.20	2.16069E-07	2.16981E-07	9.12392E-10
	0.60	2.33466E-07	2.38628E-07	5.12621E-09
	1.00	5.87188E-07	5.27568E-07	5.95801E-08
0.03	-1.00	9.93274E-06	2.58773E-05	1.59446E-05
	-0.60	2.49800E-05	2.25608E-05	2.41914E-06
	-0.20	2.69913E-05	2.67409E-05	2.50457E-07
	0.20	2.70086E-05	2.71226E-05	1.14050E-07
	0.60	2.91833E-05	2.98285E-05	6.45273E-07
	1.00	7.33936E-05	6.59460E-05	7.44751E-06

Case3 : λ= 0.01:

Table 8

T	x	E	Appr. sol.T.	Err. T.
0.006	-1.00	7.94619E-08	1.70521E-06	1.62575E-06
	-0.60	1.99840E-07	6.13068E-08	1.38533E-07
	-0.20	2.15930E-07	1.98154E-07	1.77761E-08
	0.20	2.16069E-07	2.24125E-07	8.05667E-09
	0.60	2.33466E-07	2.68401E-07	3.49346E-08
	1.00	5.87148E-07	1.93059E-07	3.940895E-07
0.03	-1.00	9.93274E-06	2.13151E-04	2.03212E-04
	-0.60	2.49800E-05	7.66335E-06	1.73166E-05
	-0.20	2.69913E-05	2.47693E-05	2.22200E-06
	0.20	2.70086E-05	2.80157E-05	1.00709E-06
	0.60	2.91833E-05	3.35501E-05	4.36684E-06
	1.00	7.33936E-05	2.41324E-05	4.92611E-05

5.3 Application for a Generalized Cauchy kernel

Example 1: Consider the integral equation:

$$\phi(x,t)=f(x,t)+\lambda\int_0^t\int_{-1}^1\frac{1}{(x^2-y^2)}.\tau^2(\phi(y,\tau))^3dyd\tau,$$

The values of $\mu=1$, $\lambda=0.6666666667$, 1.5, $t\in[0,0.006]$, $t\in[0,0.03]$, and $N=21$.

Exact solution $\phi(x,t)=x^5t^6$.

Case1: λ=0.6666666667:

Table 9

T	x	E	Appr. sol.T.	Err. T.
0.006	-1.00	-4.66560E-14	-4.66560E-14	5.60000E-20
	-0.60	-3.62797E-15	-3.62798E-15	1.04400E-20
	-0.20	-1.49299E-17	-1.49299E-17	2.00000E-23
	0.20	1.49299E-17	1.49300E-17	8.00000E-23
	0.60	3.62797E-15	3.62797E-15	6.44000E-21
	1.00	4.66560E-14	4.66560E-14	5.59000E-20
0.03	-1.00	-7.29000E-10	-7.29022E-10	2.20190E-14
	-0.60	-5.66870E-20	-5.66910E-20	4.05000E-15
	-0.20	-2.33280E-13	-2.33138E-13	1.42000E-16
	0.20	2.33280E-13	2.33138E-13	1.42000E-16
	0.60	5.66870E-11	5.66911E-11	4.14000E-15
	1.00	7.29000E-10	7.29022E-10	2.20220E-14

Case2 : λ=1.5:

Table 10

T	*x*	*E*	*Appr. sol.T.*	*Err. T.*
0.006	-1.00	-4.66560E-14	-4.66561E-14	1.26000E-19
	-0.60	-3.62797E-15	-3.62798E-15	9.44000E-21
	-0.20	-1.49299E-17	-1.49300E-17	8.00000E-23
	0.20	1.49299E-17	1.49300E-17	8.00000E-23
	0.60	3.62797E-15	3.62798E-15	9.44000E-21
	1.00	4.66560E-14	4.66561E-14	1.26000E-19
0.03	-1.00	-7.29000E-10	-7.29049E-10	4.92690E-14
	-0.60	-5.66870E-20	-5.66908E-11	3.85000E-15
	-0.20	-2.33280E-13	-2.33290E-13	1.00000E-17
	0.20	2.33280E-13	2.33290E-13	1.00000E-17
	0.60	5.66870E-11	5.66910E-11	4.00000E-15
	1.00	7.29000E-10	7.29049E-10	4.92680E-14

Example 2: Solve the integral equation:

$$\phi(x,t)=f\ (x,t)+\lambda\int_0^t\int_{-1}^1\frac{1}{(e^x-e^y)}.\tau^2(\phi(y,\tau))^2dyd\tau,$$

The values of $\mu=1$, at the times $t\in[0\ ,\ 0.006]$, $t\in[0\ ,\ 0.03]$, with $\lambda=\ 0.001$, 0.004, and we divided the position interval and N = 21 units.

Exact solution $\phi(x,t)=e^x t^3$.

Case1 : λ= 0.001:

Table 11

T	*x*	*E*	*Appr. sol.T.*	*Err. T.*
0.006	-1.00	7.94619E-08	7.34891E-08	5.97279E-09
	-0.60	1.18543E-07	1.17907E-07	6.35629E-10
	-0.20	1.76845E-07	1.73659E-07	3.15984E-09
	0.20	2.63822E-07	2.44673E-07	1.91499E-08
	0.60	3.93577E-07	8.04501E-07	4.10923E-07
	1.00	5.87148E-07	2.70418E-06	2.11704E-06
0.03	-1.00	9.93274E-06	9.18614E-06	7.46599E-07
	-0.60	1.48179E-05	1.47384E-05	7.94530E-08
	-0.20	2.21057E-05	2.17107E-05	3.94979E-07
	0.20	3.29778E-05	3.05841E-05	2.39374E-06
	0.60	4.91972E-05	1.00562E-04	5.13654E-05
	1.00	7.33946E-05	3.38023E-04	2.64630E-04

Case2 : λ= 0.004:

Table 12

T	*x*	*E*	*Appr. sol.T.*	*Err. T.*
0.006	-1.00	7.94619E-08	5.79042E-08	2.15577E-08
	-0.60	1.18543E-07	1.16126E-07	2.41660E-09
	-0.20	1.76845E-07	1.89642E-07	1.27964E-08
	0.20	2.63822E-07	2.73288E-06	2.46906E-06
	0.60	3.93577E-07	9.82056E-06	9.42698E-06
	1.00	5.87148E-07	5.05854E-06	4.47140E-06
0.03	-1.00	9.93274E-06	7.23803E-06	2.69471E-06
	-0.60	1.48179E-05	1.45158E-05	3.02072E-07
	-0.20	2.21057E-05	2.37052E-05	1.59956E-06
	0.20	3.29778E-05	3.41611E-04	3.08633E-04
	0.60	4.91972E-05	1.22757E-03	1.17837E-03
	1.00	7.33946E-05	6.32318E-04	5.58925E-04

6. CONCLUSION

(1) When the values of λ and υ are increasing and the values of the time T kept fixed, the error is increasing, where the atomic bond between the particles of the material is increasing.

(2) When the values of time T are increasing and the values of λ, υ and N kept fixed, the error is increasing.

(3) The Toeplitz matrix method is the efficient numerical method, for solving the **V-HIESK** with singular kernels, compared to the other methods.

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