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Bootstrap Approach for Estimating Seemingly Unrelated Regressions with Varying Degrees of Autocorrelated Disturbances

O. B. Ebukuyo[a], A. A. Adepoju[a] and E. I. Olamide[a],*

[a]Department of Statistics, University of Ibadan, Ibadan, Nigeria.

* Corresponding author.
Address: Department of Statistics, University of Ibadan, Ibadan, Nigeria; E-Mail: eiolamide@yahoo.com

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Abstract: The Seemingly Unrelated Regressions (SUR) model proposed in 1962 by Arnold Zellner has gained a wide acceptability and its practical use is enormous. In this research, two methods of estimation techniques were examined in the presence of varying degrees of first order Autoregressive [AR(1)] coefficients in the error terms of the model. Data was simulated using bootstrapping approach for sample sizes of 20, 50, 100, 500 and 1000. Performances of Ordinary Least Squares (OLS) and Generalized Least Squares (GLS) estimators were examined under a definite form of the variance-covariance matrix used for estimation in all the sample sizes considered. The results revealed that the GLS estimator was efficient both in small and large sample sizes. Comparative performances of the estimators were studied with 0.3 and 0.5 as assumed coefficients of AR(1) in the first and second regressions and these coefficients were further interchanged for each regression equation, it was deduced that standard errors of the parameters decreased with increase in the coefficients of AR(1) for both estimators with the SUR estimator performing better as sample size increased. Examining the performances of the SUR estimator with varying degrees of AR(1) using Mean Square Error (MSE), the SUR estimator performed better with autocorrelation coefficient of 0.3 than that of 0.5 in both regression equations with best MSE obtained to be 0.8185 using $\rho = 0.3$ in the second regression equation for sample size of 50.

Key words: Autocorrelation; Bootstrapping; Generalized least squares; Ordinary least squares; Seemingly unrelated regressions

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1. INTRODUCTION

Seemingly Unrelated Regression (SUR) is a system of regression equations which consists of a set of M regression equations, each of which contains different explanatory variables and satisfies the assumptions of the Classical Linear Regression Model (CLRM). The SUR estimation technique which allows for an efficient joint estimation of all the regression parameters was first reported by Zellner [21] which involves the application of Aitken's Generalised Least Squares (AGLS) [2] to the whole system of equations. Several scholars have also developed other estimators for diverse SUR models to address different situations being examined. Dwivedi and Srivastava [6], Zellner [21] cited in William [18] have shown that the estimation procedure of SUR model was based on Generalized Least Squares (GLS) approach. In answering how much efficiency is gained by using GLS instead of OLS, Zellner [21] has shown in his two-stage approach the gain in efficiency of SUR model over separate equation by equation, that efficiency would be attained when contemporaneous correlation between the disturbances is high and explanatory variables in different equations are uncorrelated. Youssef [19,20] studied the properties of seemingly unrelated regression equation estimators.

In an additional paper, he considered a general distribution function for the coefficients of seemingly unrelated regression equations (SURE) model when we unrestricted regression (SUUR) equations. Viraswami [17] presented a working paper on some efficiency results on SURE model. In his work, he considered a two equation seemingly unrelated regressions model in which the equations have some common independent variables and obtained the asymptotic efficiency of the OLS estimator of a parameter of interest relative to its FGLS estimator. He also provided the small-sample relative efficiency of the ordinary least squares estimator and the seemingly unrelated residuals estimator. Alaba *et al.* [3] recently examined the efficiency gain of the GLS estimator over the Ordinary Least Squares (OLS) estimator. This paper thus examines the performances of OLS and GLS estimators when the disturbances are both autoregressively and contemporaneously correlated.

The remainder of the paper is organized as follows. In section 2, the parametric SUR framework is presented while the simulation studies carried out in the work is discussed in Section 3. Results and detailed discussions are presented in Section 4 while Section 5 gives some concluding remarks.

2. PARAMETRIC SUR FRAMEWORK

Suppose there are M equations

$$Y_{it} = X_{it}\beta_i + \varepsilon_{it} \quad i = 1, 2, ..., M \tag{1}$$

Here, i represents the number of equation and $t = 1, ..., T$ is the observation index. The number of observations is assumed to be large, so that in the analysis we take $T \to \infty$ whereas the number of equations M remains fixed. Each equation i has a single response variable Y_{it} and a $K-$dimensional vector of regression X_{it}. The Seemingly Unrelated Regressions (SUR) model above is:

$$Y_i = X_i\beta_i + \varepsilon_i \tag{2}$$

where Y_i is a $T \times 1$ vector of observation on the response variables;
X_i is a $T \times K_i$ matrix of explanatory variables;
β_i is a $K_i \times 1$ vector of regression parameters; and
ε_i is a $T \times 1$ vector of disturbances.

If we stack the above equations, we can further write (2) in a more compact and matrix form as

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon \tag{3}$$

where y is an $MT \times 1$ vector of response variables, X is an $MT \times k$ matrix of explanatory variables, β is a $k \times 1$ vector of parameters and ε is an $MT \times 1$ vector of disturbances. i.e,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix}, X = \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_M \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix}, \text{ and } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_M \end{bmatrix}.$$

and with $E(\varepsilon/X_1, X_2, ..., X_M) = 0$, $E(\varepsilon\varepsilon'/X_1, X_2, ..., X_M) = \Omega$.

We assume that a total of T observations are used in estimating the parameters of the M equations. Each equation involves Km regressors, for a total of $K = \sum_{i=1}^{M} K_i$. We will require $T > K_i$. The data are assumed to be well behaved. We also assume that disturbances are uncorrelated across observations. Therefore,

$$E[\varepsilon_{it}\varepsilon_{js}|X_1, X_2, ..., X_M] = \sigma_{ij}, \text{ if } t = s \text{ and 0 otherwise.}$$

The disturbance formulation is therefore,

$$E[\varepsilon_i\varepsilon_j'|X_1, X_2, ..., X_M] = \sigma_{ij}I_T$$

or

$$E[\varepsilon\varepsilon'|X_1, X_2, ..., X_M] = \Omega = \begin{bmatrix} \sigma_{11}l & \sigma_{12}l & \dots & \sigma_{1M}l \\ \sigma_{21}l & \sigma_{22}l & \dots & \sigma_{2M}l \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{M1}l & \sigma_{M2}l & \dots & \sigma_{MM}l \end{bmatrix} \tag{4}$$

The specification of the covariance structure is simplified by arranging the data by observation t, rather than by equation. The disturbance vector, $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}, ..., \varepsilon_{Mt})'$ is generated by a stationary, first-order autoregressive process.

$$\varepsilon = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{Mt} \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & \dots & 0 \\ 0 & \rho_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_M \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \vdots \\ \varepsilon_{Mt-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \\ \vdots \\ v_{Mt} \end{bmatrix} \tag{5}$$

or in matrix notation, $\varepsilon_t = R\varepsilon_{(t-1)} + v_{(t)}$, where the $v_{(t)}$ are Independent and Identically Distributed random variables (IID) with $E(v_{(t)}) = 0$ and covariance matrix

$$E(v_{(t)}v'_{(t)}) = \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1M} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{M1} & \sigma_{M2} & \dots & \sigma_{MM} \end{bmatrix} \tag{6}$$

The diagonal structure of the R matrix implies that each equation or cross-section unit exhibits its own serial correlation coefficient, and the innovations $v_{(t)}$ are contemporaneously correlated with covariance matrix Σ.

The most general model that is usually considered involves the diagonal R matrix, with M parameters, specifying the serial correlation together with a full, symmetric Σ matrix, with $M(M+1)/2$ parameters, specifying the contemporaneous covariance. This implies that in (4), $\Omega = \Sigma \otimes I$, and $\Omega^{-1} = \Sigma^{-1} \otimes l$.

If Ω is known and denoting the ijth element of Σ^{-1} by σ^{ij}, the generalized least squares estimator for the coefficients in this model is:

$$\hat{\beta} = \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}y \tag{7}$$

i.e.,

$$\hat{\beta} = \left[X'(\Sigma^{-1} \otimes l)X\right]^{-1} X'(\Sigma^{-1} \otimes l)y$$

Expanding the Kronecker products gives

$$\hat{\beta} = \begin{bmatrix} \sigma^{11}X'_1X_1 & \sigma^{12}X'_1X_2 & \dots & \sigma^{1M}X'_1X_M \\ \sigma^{21}X'_2X_1 & \sigma^{22}X'_2X_2 & \dots & \sigma^{2M}X'_2X_M \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{M1}X'_MX_1 & \sigma^{M2}X'_MX_2 & \dots & \sigma^{MM}X'_MX_M \end{bmatrix}^{-1} \begin{bmatrix} \sum_1^M j\sigma^{1j}X'_1y_j \\ \sum_1^M j\sigma^{2j}X'_2y_j \\ \vdots \\ \sum_1^M j\sigma^{Mj}X'_My_j \end{bmatrix} \tag{8}$$

This is the asymptotic covariance matrix for the GLS estimator. Assume that $X_i = X_j = X$, so that $X'_iX_j = X'X \ \forall i,j$, in (8), the inverse matrix becomes $\left(\Sigma^{-1} \otimes X'X\right)^{-1} = \left[\Sigma \otimes (X'X)^{-1}\right]$ and each term $X'y_j = X'Xb_j$. if we then move the common term $X'X$ out of the summations, we obtain

$$\hat{\beta} = \begin{bmatrix} \sigma_{11}(X'X)^{-1} & \sigma_{12}(X'X)^{-1} & \dots & \sigma_{1M}(X'X)^{-1} \\ \sigma_{21(X'X)^{-1}} & \sigma_{22}(X'X)^{-1} & \dots & \sigma_{2M}(X'X)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{M1}(X'X)^{-1} & \sigma_{M2}(X'X)^{-1} & \dots & \sigma_{MM}(X'X)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} (X'X)\sum_1^M j\sigma^{1l}b_l \\ (X'X)\sum_1^M j\sigma^{2l}b_l \\ \vdots \\ (X'X)\sum_1^M j\sigma^{Ml}b_l \end{bmatrix} \tag{9}$$

3. SIMULATION STUDY

This study considers two-equation SUR model with autoregressively and contemporaneously correlated error terms.

$$y_1 = 0.5 + 0.7x_{i1} + e_{i1}$$
$$y_2 = 0.6 + 0.8x_{i2} + e_{i2}$$

Using the true value of the variance covariance as

$$\Sigma = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix} \tag{10}$$

Thus, the new Cholesky decomposition is computed as:

$$K = \begin{bmatrix} 1 & 0 \\ 0.6 & 0.8 \end{bmatrix}$$

$\varepsilon^* = [\varepsilon_1^* \quad \varepsilon_2^*]$, the new correlated errors whose elements are determined by the product

$$\varepsilon^* = P^*\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 \\ 0.6\varepsilon_1 & 0.8\varepsilon_2 \end{bmatrix}.$$

The explanatory variables are generated from uniform distribution for the various sample sizes of 20, 50, 100, 500 and 1000. Then, $\varepsilon = (\varepsilon_1, \varepsilon_2)'$ are series of random normal deviates of required lengths of 20, 50, 100, 500 and 1000 that were generated, these series were then standardized and appropriately transformed using (10).

4. RESULTS AND DISCUSSIONS

4.1. Results

The summary of the results when the model is estimated by interchanging the coefficients of autocorrelated errors are presented below.

4.2. Discussion of Results

Table 1 gives the comparative performance of the estimators with the coefficient of AR(1) for the first regression equation to be 0.3 (i.e., $\rho_1 = 0.3$) and that of the second regression equation to be 0.5 (i.e., $\rho_2 = 0.5$).

Table 2 gives the comparative performance of the estimators with the coefficient of AR(1) for the first regression equation to be 0.5 (i.e., $\rho_1 = 0.5$) and that of the second regression equation to be 0.3 (i.e., $\rho_2 = 0.3$).

From Table 1 and 2, we found that the standard errors of the parameter estimates decreased as sample size increased with varying degrees of coefficients of AR(1) with the SUR estimator performing better than the OLS estimator in both cases (that is, the standard errors of the parameter estimates obtained using the SUR estimator were consistently lower than that of the OLS estimator in both cases as the sample size increased). Significantly, we found that higher coefficient of first order autoregressive scheme accounts for better efficiencies of the estimators.

From Table 3, we found on the basis of the Mean Square Error (MSE) for the SUR estimator that the MSE of the regression equations increased with increase in the coefficients of AR(1) with the second model for sample size of 50 performing best with coefficients of AR(1) $[0.3, 0.5]$ and MSE $[0.8185, 0.8852]$ respectively. Thus, the best MSE obtained for the SUR estimator is 0.8185 using $\rho = 0.3$ for the second regression for sample size of 50.

Table 1
Comparative Study of the Estimators Across Different Sample Sizes with $\rho_1 = 0.3$ and $\rho_2 = 0.5$

	$N = 20$				**$N = 50$**				**AR(1)**
Regressions	**OLS**		**SUR**		**OLS**		**SUR**		
	Estimate	**SE**	**Estimate**	**SE**	**Estimate**	**SE**	**Estimate**	**SE**	
y_1									0.3
$\beta_{10} = 0.5$	0.9334	0.3862	0.7496	0.3661	0.4436	0.2292	0.4823	0.2217	
$\beta_{11} = 0.7$	-0.6360	0.7912	-0.1761	0.7286	0.6221	0.4874	0.5193	0.4622	
y_2									0.5
$\beta_{20} = 0.6$	0.1710	0.4354	0.3473	0.4238	0.1339	0.1660	0.1289	0.1630	
$\beta_{21} = 0.8$	0.8003	1.0003	1.1115	0.9212	1.3440	0.3772	1.3630	0.3577	
	$N = 100$				**$N = 500$**				**AR(1)**
Regressions	**OLS**		**SUR**		**OLS**		**SUR**		
	Estimate	**SE**	**Estimate**	**SE**	**Estimate**	**SE**	**Estimate**	**SE**	
y_1									0.3
$\beta_{10} = 0.5$	0.1741	0.1643	0.0899	0.1509	0.3389	0.0769	0.3979	0.0671	
$\beta_{11} = 0.7$	1.1386	0.3596	1.3699	0.3124	0.6675	0.1689	0.4974	0.1299	
y_2									0.5
$\beta_{20} = 0.6$	0.2805	0.1435	0.2833	0.1370	0.3543	0.0651	0.3676	0.0602	
$\beta_{21} = 0.8$	0.8714	0.3479	0.8602	0.3022	0.7246	0.1569	0.6710	0.1207	
	$N = 1000$								**AR(1)**
Regressions	**OLS**		**SUR**						
	Estimate	**SE**	**Estimate**	**SE**					
y_1									0.3
$\beta_{10} = 0.5$	0.3191	0.0524	0.3088	0.0460					
$\beta_{11} = 0.7$	0.7389	0.1140	0.7683	0.0887					
y_2									0.5
$\beta_{20} = 0.6$	0.3021	0.0452	0.3013	0.0418					
$\beta_{21} = 0.8$	0.8360	0.1084	0.8393	0.0843					

5. CONCLUSION

The results obtained showed that the standard errors of the SUR estimator is consistently lower than that of the OLS estimator when the model is estimated with varied coefficients of AR(1). It is revealed that higher coefficient of AR(1) accounts for better efficiencies of the estimators.

Table 2
Comparative Study of the Estimators Across Different Sample Sizes with $\rho_1 = 0.5$ and $\rho_2 = 0.3$

	$N = 20$				$N = 50$				AR(1)
Regressions	OLS		SUR		OLS		SUR		
	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE	
y_1									0.5
$\beta_{10} = 0.5$	0.6426	0.3359	0.4562	0.3169	0.3268	0.1974	0.3450	0.1933	
$\beta_{11} = 0.7$	-0.5000	0.8027	0.1311	0.7086	0.5674	0.4850	0.5008	0.4620	
y_2									0.3
$\beta_{20} = 0.6$	0.3084	0.4836	0.5562	0.4532	0.1998	0.1915	0.1914	0.1865	
$\beta_{21} = 0.8$	1.5841	1.0033	0.8920	0.8857	1.3123	0.3931	1.3354	0.3745	
	$N = 100$				$N = 500$				AR(1)
Regressions	OLS		SUR		OLS		SUR		
	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE	
y_1									0.5
$\beta_{10} = 0.5$	0.1039	0.1424	0.0369	0.1343	0.2371	0.0680	0.2767	0.0624	
$\beta_{11} = 0.7$	1.1985	0.3612	1.4546	0.3127	0.6922	0.1697	0.5325	0.1301	
y_2									0.3
$\beta_{20} = 0.6$	0.3997	0.1611	0.4148	0.1495	0.4815	0.0726	0.5070	0.0638	
$\beta_{21} = 0.8$	0.8659	0.3481	0.8222	0.3012	0.7630	0.1558	0.6892	0.1195	
	$N = 1000$								AR(1)
Regressions	OLS		SUR						
	Estimate	SE	Estimate	SE					
y_1									0.5
$\beta_{10} = 0.5$	0.2257	0.0462	0.2112	0.0424					
$\beta_{11} = 0.7$	0.7471	0.1142	0.8052	0.0880					
y_2									0.3
$\beta_{20} = 0.6$	0.4224	0.0508	0.4116	0.0447					
$\beta_{21} = 0.8$	0.8374	0.1087	0.8683	0.0838					

Table 3
Comparative Performance of the SUR Estimator with Varying Degrees of Coefficients of AR(1) Using MSE

	$N = 20$		$N = 50$		$N = 100$	
Regressions	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.3$	$\rho = 0.5$
y_1	1.0023	1.1721	0.9425	1.0756	0.9883	1.1399
y_2	2.1512	2.5458	0.8185	0.8852	1.1522	1.3157
	$N = 500$		$N = 1000$			
Regressions	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.3$	$\rho = 0.5$		
y_1	1.2392	1.4294	1.1542	1.3181		
y_2	1.1796	1.3638	1.1428	1.3095		

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