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On Invariant Statistically Convergence and Lacunary Invariant Statistical Convergence of Sequences of Sets

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Abstract: In this paper, we define invariant convergence, lacunary invariant convergence, invariant statistical convergence, lacunary invariant statistical convergence for sequences of sets. We investigate some relations between lacunary invariant statistical convergence and invariant statistical convergence for sequences of sets.

Key words: Lacunary invariant statistical convergence; Invariant statistical convergence; Sequences of sets

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1. INTRODUCTION AND PRELIMINARIES

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on m, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ is such that $x_n \geq 0$ for all n,
2. $\phi(e) = 1$,where $e = (1, 1, 1....)$,
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in m$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is translation mappings $\sigma(n) = n+1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown that

$$V_\sigma = \{x = (x_n) : \lim_m t_{mn}(x) = Le \text{ uniformly in } n, L = \sigma - \lim x\}$$

where,

$$t_{mn}(x) = \frac{x_n + Tx_n + ... + T^m x_n}{m+1}.$$

The concept of statistical convergence for sequences of real numbers was introduced Fast [1], Salat [2] and others. Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ real numbers is said to be statistically convergent to L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ will be denoted by such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_r - k_{r-1}]$. Freedman, Sember and Raphael [3] defined the space N_θ in the following way. For any lacunary sequence $\theta = (k_r)$,

$$N_\theta = \left\{ x = (x_k) : \text{ for some } L, \ \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \right\}$$

In [4], lacunary statistically convergent sequence is defined as follows:

Let θ be a lacunary sequence; the number sequence (x_k) is lacunary statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

A set E of positive integers said to have uniform invariant density of zero if and only if the number of elements of E which lie in the set $\{\sigma(m), \sigma^2(m), ..., \sigma^n(m)\}$ is $o(n)$ as $n \to \infty$, uniformly in m.

By using uniform invariant density, following definitions were given in [5] and [6].

A complex number sequence $x = (x_k)$ is said to be σ statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{0 \leq k \leq n : |x_{\sigma^k(m)} - L| \geq \varepsilon\}| = 0 \ uniformly \ in \ m = 1, 2 \ldots$$

In this case we write $S_\sigma - \lim x = L$ or $x_k \to L(S_\sigma)$.

Let be $\theta = (k_r)$ be a lacunary sequence; the number sequence $x = (x_k)$ is $S_{\sigma\theta}$ convergent to L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_{\sigma^k(m)} - L| \geq \varepsilon\}| = 0 \text{ uniformly in } m = 1, 2 \ldots$$

In this case we write $S_{\sigma\theta} - \lim x = L$ or $x_k \to L(S_{\sigma\theta})$.

Let (X, ρ) be a metric space. For any point $x \in X$ and non-empty subset A of X we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A).$$

Let (X, ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A).$$

for each $x \in X$. In this case we write $W - \lim A_k = A$ [7].

The concepts of Wijsman statistical convergence and Wijsman strong Cesaro summability were introduced by Nuray and Rhoades [8] as follows:

Let (X, ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, the sequence $\{A_k\}$ is said to be Wijsman strongly Cesaro summable to A if for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0.$$

Let (X, ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, the sequence $\{A_k\}$ is said to be Wijsman statistically convergent to A if for $\varepsilon > 0$ and each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

2. MAIN RESULT

In this section, we will generalize some convergence definitions known for number sequences to the sequences of sets.

Definition 1. *Let (X, ρ) be metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman invariant convergent to A, for each $x \in X$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_{\sigma^k(m)}) = d(x, A)$$

uniformly in m.

In this case we write $A_k \to A(WV_\sigma)$ and the set of all Wijsman invariant convergent sequences of sets will be denoted WV_σ.

Definition 2. *Let (X,ρ) be metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly invariant convergent to A, for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{\sigma^k(m)}) - d(x,A)| = 0$$

uniformly in m.

In this case we write $A_k \to A([WV_\sigma])$ and the set of all Wijsman strongly invariant convergent sequences of sets will be denoted $[WV_\sigma]$.

Definition 3. *Let (X,ρ) be metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A, for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} d(x, A_{\sigma^k(m)}) = d(x,A).$$

uniformly in m.

In this case we write $A_k \to A(WN_{\sigma\theta})$ and the set of all Wijsman lacunary invariant convergent sequences of sets will be denoted $WN_{\sigma\theta}$.

Definition 4. *Let (X,ρ) be metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly lacunary invariant convergent to A, for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_{\sigma^k(m)}) - d(x,A)| = 0.$$

uniformly in m.

In this case we write $A_k \to A([WN_{\sigma\theta}])$ and the set of all Wijsman strongly lacunary invariant convergent sequences of sets will be denoted $[WN_{\sigma\theta}]$.

Definition 5. *Let (X,ρ) be metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman invariant statistically convergent to A,for each $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{0 \le k \le n : |d(x, A_{\sigma^k(m)}) - d(x,A)| \ge \varepsilon\}| = 0$$

uniformly in m.

In this case we write $A_k \to A(WS_\sigma)$ and and the set of all Wijsman invariant statistically convergent sequences of sets will be denoted WS_σ.

Definition 6. *Let (X,ρ) be metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A, for each $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x,A)| \ge \varepsilon\}| = 0$$

uniformly in m.

In this case we write $A_k \to A(WS_{\sigma\theta})$ and and the set of all Wijsman lacunary invariant statistically convergent sequences of sets will be denoted $WS_{\sigma\theta}$.

Now we prove some relations between $[WN_{\sigma\theta}]$ convergence and $WS_{\sigma\theta}$ convergence and show that these are equivalent for bounded sequences of sets. We also study relation between WS_σ convergence and $WS_{\sigma\theta}$ convergence.

Theorem 1. *$\theta = (k_r)$ be lacunary sequence and let (X, ρ) be a metric space then for any non empty closed subsets A, $A_k \subseteq X$*

i . $A_k \to A([WN_{\sigma\theta}])$ *implies* $A_k \to A(WS_{\sigma\theta})$.

ii . $\{A_k\} \in L_\infty$ *and* $A_k \to A(WS_{\sigma\theta})$ *implies* $x_k \to A([WN_{\sigma\theta}])$.

iii . $WS_{\sigma\theta} \cap L_\infty = [WN_{\sigma\theta}]$.

where L_∞ denotes the set of bounded sequences of sets.

Proof. (i). If $\varepsilon > 0$ and $A_k \to A([WN_{\sigma\theta}])$, we can write,

$$\begin{aligned}\sum_{k\in I_r} |d(x, A_{\sigma^k(m)}) - d(x, A)| &\geq \sum_{\substack{k\in I_r \\ |d(x,A_{\sigma^k(m)})-d(x,A)|\geq\varepsilon}} |d(x, A_{\sigma^k(m)}) - d(x, A)| \\ &\geq \varepsilon. \left|\{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}\right|\end{aligned}$$

which yields result.

(ii). Suppose that $A_k \to A(WS_{\sigma\theta})$ and $\{A_k\} \in L_\infty$, say $|d(x, A_{\sigma^k(m)}) - d(x, A)| \leq M$ for all k and m. Given $\varepsilon > 0$, we get

$$\begin{aligned}&\frac{1}{h_r}\sum_{k\in I_r} |d(x, A_{\sigma^k(m)}) - d(x, A)| \\ &= \frac{1}{h_r} \sum_{\substack{k\in I_r \\ |d(x,A_{\sigma^k(m)})-d(x,A)|\geq\varepsilon}} |d(x, A_{\sigma^k(m)}) - d(x, A)| \\ &+ \frac{1}{h_r} \sum_{\substack{k\in I_r \\ |d(x,A_{\sigma^k(m)})-d(x,A)|<\varepsilon}} |d(x, A_{\sigma^k(m)}) - d(x, A)| \\ &\leq \frac{M}{h_r}\left|\{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}\right| + \varepsilon\end{aligned}$$

from which result follows.

(iii). This is an immediate consequence of (i) and (ii).

This completes proof of theorem. □

We now give a lemma which will be used in the proof of Theorem 2.

Lemma 1. *Let (X, ρ) be a metric space and A, A_k be closed subsets of X. Suppose for given $\varepsilon_1 > 0$ and every $\varepsilon > 0$, there exists n_0 and m_0 such that;*

$$\frac{1}{n}|\{0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\} < \varepsilon_1$$

for all $n \geq n_0$ and $m \geq m_0$, then $A_k \in WS_\sigma$.

Proof. Let $\varepsilon_1 > 0$ be given. For every $\varepsilon > 0$, choose n_0', m_0 such that,

$$\frac{1}{n}|\{0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\} < \frac{\varepsilon_1}{2} \tag{2.1}$$

for all $n \geq n_0$, $m \geq m_0$. It is enough to prove that there exist n_0'', such that for $n \geq n_0$, $0 \leq m \leq m_0$,

$$\frac{1}{n}|\{0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\} < \varepsilon_1 \tag{2.2}$$

Since taking$n_0 = max(n_0', n_0'')$, (2.2) will hold for $n \geq n_0$ and for all m, which gives result. Once m_0 has been chosen, $0 \leq m \leq m_0$, m_0 is fixed. So put

$$K = |\{0 \leq k \leq m_0 - 1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}|$$

Now taking $0 \leq m \leq m_0$ and $n \geq m_0$, by (2.1) we have

$$\begin{aligned}
&\frac{1}{n}|\{0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\} \\
&\leq \frac{1}{n}|\{0 \leq k \leq m_0 - 1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\} \\
&+ \frac{1}{n}|\{m_0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\} \\
&\leq \frac{K}{n} + \frac{1}{n}|\{m_0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m_0)}) - d(x, A)| \geq \varepsilon\} \\
&\leq \frac{K}{n} + \frac{\varepsilon_1}{2},
\end{aligned}$$

and taking n sufficiently large we can write

$$\leq \frac{K}{n} + \frac{\varepsilon_1}{2}$$

which gives (2.2) and hence the result follows. □

Theorem 2. *Let (X, ρ) be a metric space, then;*

$$WS_{\sigma\theta} = WS_\sigma$$

for every lacunary sequence θ.

Proof. Let $\{A_k\} \in WS_{\sigma\theta}$. Then from Definition 5, for given $\varepsilon_1 > 0$, there exists r_0 and set A such that

$$\frac{1}{h_r}|\{0 \leq k \leq h_r - 1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}| \leq \varepsilon_1$$

for $r \geq r_0$ and $m = k_{r-1} + 1 + u$, $u \geq 0$.

Let $n \geq h_r$, write $n = ih_r + t$ where $0 \leq t \leq h_r$, i is an integer. Since $n \geq h_r$, $i \geq 1$. Now

$$\begin{aligned}
&\frac{1}{n}|\{0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}| \\
&\leq \frac{1}{n}|\{0 \leq k \leq (i+1)h_r - 1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}|
\end{aligned}$$

$$= \frac{1}{n}\sum_{j=0}^{i} |\{jh_r \leq k \leq (j+1)h_r - 1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}|$$
$$\leq \frac{1}{n} < (i+1)h_r\varepsilon_1 \leq 2ih_r\frac{\varepsilon_1}{n}\ (i \geq 1)$$

for $\frac{h_r}{n} \leq 1$ and since $\frac{ih_r}{n} \leq 1$,

$$\frac{1}{n}|\{0 \leq k \leq n-1 : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}| \leq 2\varepsilon_1.$$

Then by Lemma 1, $WS_{\sigma\theta} \subset WS_\sigma$. It is easy to see that $WS_\sigma \subset WS_{\sigma\theta}$.
This completes the proof of the theorem. □

By using the same techniques as in Theorem 2, we can prove the following theorem.

Theorem 3. $[WN_{\sigma\theta}] \Leftrightarrow [WV_\sigma]$ *for every lacunary sequence* θ.

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