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First Excursion Probabilities of Non-Linear Dynamical Systems by Importance Sampling

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Abstract: This paper suggests a procedure to estimate first excursion probabilities for non-linear dynamical systems subjected to Gaussian excitation. The approach is based on the mean up-crossing rate and importance sampling method. Firstly, by using of Poisson assumption and Rice formula, the equivalent linear system is carried out. The linearization principle is that non-linear and linear systems have the same up-crossing rate for a specified threshold. Secondly, an importance sampling technique is used in order to estimate excursion probabilities for the equivalent linear system. The variance of the failure probability estimates, the number of samples and the computational time are reduced significantly compared with direct Monte Carlo simulations.

Key words: First excursion probability; Importance sampling; Mean up-crossing rate

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1. INTRODUCTION

The first excursion probability is one of the most basic reliability measures in structural reliability assessment of dynamical systems. So far, no analytical solution has been obtained even in the case of a linear system. Pioneered by Rice (1945) [1,2], a class of numerical solution methods based on Fokker-plank equation has been developed, such as path integration (1976) [3], cell-mapping (1993) [4] and stochastic averaging (2003) [5,6], however these methods increase in complexity at least exponentially with the state space dimension. Over the past decade, Monte Carlo simulations (MCS) with variance reduction technique offered a robust methodology well suited for solving such reliability problems.

For the linear dynamical systems, S. K. Au (2001) developed an extremely powerful methodology by investigating analytically the failure region of linear systems and constructing an efficient importance sampling density [7]. Ka-VengYuen (2005) also presented a simulation method using simple additive rules of probability [8]. Lambros (2006) proposed the domain decomposition method [9]. K. M. Zuev (2011) proposed the Horseracing Simulation algorithm [10]. For the non-linear dynamical systems, H. J. Pradlwater (2004) suggested a procedure based on the so called "averaged excursion probability flow" [11]. A. I. Olsen (2007) put forward a two step iterative method [12].

This paper proposes a robust reliability methodology. The approach is focused on out crossing theory and importance sampling technique. By used of Poisson assumption and Rice formula, the equivalent linear system is obtained. Importance sampling density (ISD) function is a weighted sum of the probability density function.

2. THE PROCEDURE

In this section, numerical results for first passage probability of Duffing oscillators will be presented. The main idea is to estimate the excursion probability using a linear version of the non-linear Duffing oscillators. Recently, it has been shown that it is the most efficient linearization method for estimating failure probability, the mean up-crossing characteristics are far more important than the characteristics of the mean square response [12]. So it is reasonable to assume that both the non-linear system and the equivalent linear system have the same up-crossing rate for a specified threshold.

Consider a non-linear dynamic system subjected to Gaussian white noise excitation, Duffing oscillators. Where $w(t)$ is a Gaussian white noise process with spectral intensity G_0.

$$\begin{cases} \ddot{X}(t) + 2\beta\varpi_0\dot{X}(t) + \varpi_0^2[X(t) + \varepsilon X^3(t)] = \sqrt{\gamma}w(t), & 0 \le t \le T \\ X(0) = 0, \dot{X}(0) = 0 \end{cases} \tag{1}$$

$D = \{(x, \dot{x}) : x < b(t), \dot{x} \in R\}$ is the safe domain. Here β is the damping ratio and ϖ_0 is the frequency of free oscillations of the corresponding linear system ($\varepsilon = 0$). The non-linearity parameter is ε ($\varepsilon \ge 0$).

The first excursion probability of Duffing oscillators is given as:

$$P_F = P\{\forall t \in (0, T),\ X(t) \notin D\} = P\{\forall t \in (0, T),\ |X(t)| > b(t)\} \tag{2}$$

2.1. Equivalent Linearization

The stationary joint probability density of $X(t)$ and $\dot{X}(t)$ is obtained [13].

$$f(x,\dot{x}) = A\exp\{-\frac{1}{2\sigma_0^2}(x^2+\frac{\varepsilon}{2}x^4)-\frac{\dot{x}^2}{2\dot{\sigma}_0^2}\} \tag{3}$$

where $\sigma_0^2 = \dfrac{\pi G_0}{4\beta\varpi_0^3}$ and $\dot{\sigma}_0^2 = \varpi_0^2\sigma_0^2$, respectively, represent steady state variances of $X(t)$ and $\dot{X}(t)$ of linear system. A is the normalization constant. $K_{\frac{1}{4}}$ is a modified Bessel function, and

$$A^{-1} = \pi\sqrt{\frac{G_0}{4\beta\varepsilon}}\exp\left(\frac{1}{8\varepsilon\sigma_0^2}\right)K_{\frac{1}{4}}\left(\frac{1}{8\varepsilon\sigma_0^2}\right)$$

Based on Poisson assumption and Rice formula, the mean stationary up-crossing rate of the non-linear system is given:

$$\begin{aligned} V_X^+(x) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}P\left\{b(t)-\dot{X}(t)\Delta t < X(t) < b(t)\right\} \\ &= \dot{\sigma}_0^2 A\exp\left\{-\frac{1}{2\sigma_0^2}\left(x^2+\frac{\varepsilon}{2}x^4\right)\right\} \end{aligned} \tag{4}$$

The equivalent linear system is assumed as:

$$\ddot{Y}(t)+2\beta\varpi_0\dot{Y}(t)+\varpi_e^2 Y(t) = \sqrt{\gamma}w(t) \tag{5}$$

The mean stationary up-crossing rate of the linear system (5) is obtained.

$$V_Y^+(x) = \frac{\dot{\sigma}_Y}{2\pi\sigma_Y}\exp\left\{-\frac{1}{2\sigma_Y^2}x^2\right\} \tag{6}$$

where $\sigma_Y^2 = \dfrac{\pi G_0}{4\beta\varpi_0\varpi_e^2}$ and $\dot{\sigma}_Y^2 = \sigma_Y^2\varpi_e^2$, $G_0 = \dfrac{\gamma}{\pi}$.

The linearization coefficients ϖ_e can be found form the equation as follows:

$$V_Y^+(b(t)) = V_X^+(b(t)) \tag{7}$$

The first excursion probability of Duffing oscillators is given as:

$$P_F \approx P\left\{\forall t\in(0,T),\ |Y(t)| > b(t)\right\} \tag{8}$$

2.2. Procedure

Based on Duhamel's integral, the input output relationship of the equivalent linear system can be generally written as:

$$Y(t) = \int_0^t h(t,\tau)W(\tau)d\tau$$

where $h(t,\tau)$ is the unit impulse response function for the output at time t due to a unit impulse applied at the input at time τ. The duration of study is T, and $\Delta t = \dfrac{T}{n_t - 1}$, $t_k = (k-1)\Delta t$, $k = 1, ..., n_t$. $Z(1), ..., Z(n_t)$ are i.i.d. Gaussian random variables.

$g(k,s) \to h(t_k, t_s)$ as $\Delta t \to 0$. The input-output relationship can be written:

$$Y(k) = \sum_{s=1}^{k} g(k,s) Z_k(s) \sqrt{2\pi G_0 \Delta t} \tag{9}$$

In terms of Eq. (9), the failure event F is defined as the event of the absolute response of any one of the outputs beyond a given threshold level at any time step.

$$F = \bigcup_{k=1}^{n_t} F_k = \bigcup_{k=1}^{n_t} \{|Y(t_k)| > b(t)\} = \bigcup_{k=1}^{n_t} \{|Y(k)| > b(k)\}$$

F_k is the elementary failure event. The elementary failure event is completely described by a local design point, which can be obtained form unit impulse response functions [7–10].

Reliability index is given as

$$\min {\beta_k}^2 = \min\{{Z_k}^2(1) + \cdots + {Z_k}^2(k)\} \tag{10}$$

The local design point can be obtained as:

$$Z_k^*(s) = \frac{1}{\sigma_k^2} g(k,s) \sqrt{2\pi G_0 \Delta t} b(t), (k \geq s) \tag{11}$$

where $\sigma_k^2 = Var(Y(k)) = 2\pi G_0 \Delta t \sum\limits_{s=1}^{k} g^2(k,s)$, $s = 1, 2, ..., k$.

The probability of the elementary failure event is given as:

$$P(F_k) = 2\Phi(-||z_k^*||) = 2\Phi(-\beta_k) \tag{12}$$

where $Z_k^* = (Z_k^*(1), Z_k^*(2), \cdots, Z_k^*(k))$, $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution. $\phi(\cdot)$ is the joint probability density function.

The first excursion probability is given as:

$$\begin{aligned} P_F &= P\left\{\bigcup_{k=1}^{n_t} F_k\right\} = P\left\{\bigcup_{k=1}^{n_t} |Y(k)| > b(k)\right\} \\ &= P\left\{\bigcup_{k=1}^{n_t} \left|\sum_{s=1}^{k} g(k,s) Z_k(s) \sqrt{2\pi G_0 t}\right| > b(k)\right\} \\ &= \int_F \phi(z) dz = \int_\Omega \pi_F(z) \phi(z) dz = E[\pi_F(z)] \end{aligned}$$

where $\pi_F(z)$ is the indicator function, $\pi_F(z) = \begin{cases} 1, & z \in F \\ 0, & z \notin F \end{cases}$. F is the failure region.

Based on the importance sampling procedure, the first excursion probability is given as:

$$P_F = \int_{\Omega} \pi_F(z)\phi(z)dz = \int \frac{\phi(z)\pi_F(z)}{h(z)}h(z)dz$$

where $h(z)$ is the importance sampling density (ISD). The efficiency of the method relies on a proper choice of ISD.

The paper adopts ISD $h(z)$ which S. K. Au proposed in [7].

$$h(z) = \frac{\phi(z)}{\sum_{k=1}^{n_t} p(F_k)} \sum_{k=1}^{n_t} \pi(F_k) \tag{13}$$

and so

$$P_F = \sum_{k=1}^{n_t} P(F_k) \int \frac{\pi_F(z)}{\sum_{k=1}^{n_t} \pi_{F_k}(z)} h(z)dz = \sum_{k=1}^{n_t} P(F_k) \times E_h \left\{ \frac{1}{\sum_{k=1}^{n_t} \pi_{F_k}(z)} \right\} \tag{14}$$

The estimate of the first excursion probability is given as:

$$\hat{P}_F = (\sum_{k=1}^{n_t} P_k) \times \frac{1}{N} \sum_{r=1}^{n} \frac{1}{\sum_{k=1}^{n_t} \pi_{F_k}(z_r)} \tag{15}$$

To simulate a sample $Z_r (r = 1, ..., N)$ according to the ISD (13) is given as:

Step 1: Select a random number k of indices form the set $k = 1, ..., n_t$ according to $w(i) = \frac{P(F_i)}{\sum_{i=1}^{n_t} P(F_i)}$.

Step 2: Simulate z as a n_t dimensional standard Gaussian vector with independent components. Simulating random numbers U_2 and U_3 that is uniformly distributed in the interval $[0, 1]$ and computing $\alpha = \Phi^{-1}(U_2 + (1 - U_2)\Phi(\beta_k))$ and so

$$Z_r = \begin{cases} Z + (\alpha - < Z, u_k^* >)u_k^*, & U_3 \le 1/2 \\ -Z - (\alpha - < Z, u_k^* >)u_k^*, & U_3 > 1/2 \end{cases}$$

where $u_k^* = z_k^*/\beta_k$.

2.3. Results and Discussion

In this section, numerical examples are presented, which demonstrate that the efficiency of the proposed procedure by comparing with the crude MCS. The parameters of Duffing system are chosen: $\beta = 0.05$, $\gamma = 0.03$, $\varpi_0 = 1.0(\text{rad/s})$, $T = 15\text{s}$, $\Delta t = 0.05$, $b(t) = k\sigma_0$, $n_t = 301$. According to Equation (4), the mean up crossing rate is obtained as Figures 1 and 2, respectively, which has the different specified threshold and the different nonlinearity parameter ε. The impulse response function of the equivalent linear system is given as:

$$h(t) = \tfrac{1}{\omega_d} \exp(-\beta\omega_0 t) \sin(\omega_d t)$$

where $\omega_d = \sqrt{\omega_e^2 - (\beta\omega_0)^2}$, the linearization coefficients ϖ_e is calculated by Equation (7). The first failure probability is estimated respectively for the three threshold levels and the three non-linearity parameter. The procedure which the paper suggests is carried out 15 times simulation test. Every test simulated 100 samples.

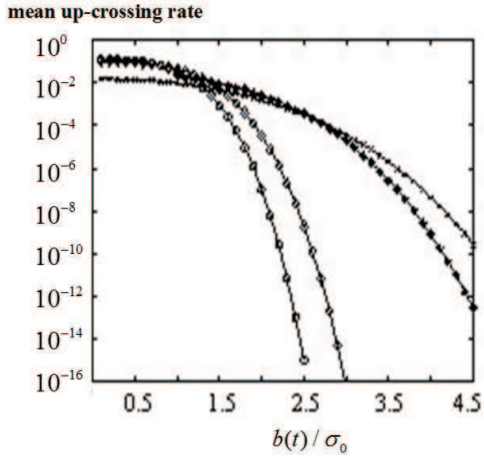

Figure 1
The Up-Crossing Rate vs Critical Threshold Depending on Different of the Non-Linearity Parameter:
$\circ : \varepsilon = 2, \Diamond : \varepsilon = 1, \star : \varepsilon = 0.1, \chi : \varepsilon = 0.05$

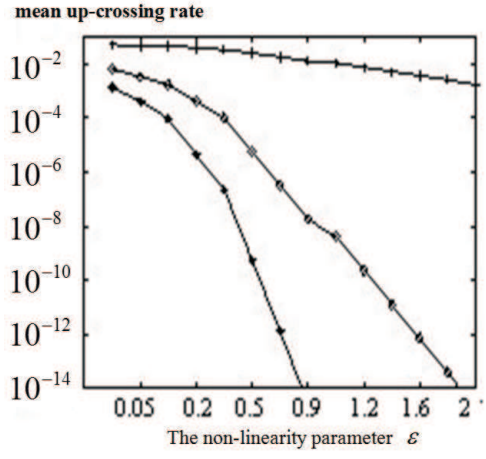

Figure 2
The Up-Crossing Rate vs the Non-Linearity Parameter Depending on Different Critical Threshold:
$\dagger : b(t) = 1.5\sigma_0, \Diamond : b(t) = 2.5\sigma_0, \star : b(t) = 3\sigma_0$

Test 1: The specified threshold is $b(t) = 3\sigma_0 = 3.6742$, $\varepsilon = 0.1$. The linearization coefficients is $\varpi_e = 1.310634$. Simulation result is shown below.

The value of failure probability $\hat{P}_F$:

$$\begin{array}{lll} 3.8726586E-05 & 3.5221183E-05 & 4.0286777E-05 \\ 3.7422466E-05 & 3.6562760E-05 & 3.5540073E-05 \\ 3.6227011E-05 & 3.6374826E-05 & 3.7412210E-05 \\ 3.6447880E-05 & 3.8154383E-05 & 3.6847210E-05 \\ 3.7608606E-05 & 3.6462850E-05 & 3.9022085E-05 \end{array}$$

Calculation time of every test took less than 1s, far less than the computational time of crude MCS.

Test 2: The specified threshold is $b(t) = 1.5\sigma_0 = 1.837117,\ \varepsilon = 2$.

Test 3: The specified threshold is $b(t) = 2.5\sigma_0 = 3.061862,\ \varepsilon = 0.5$.

For comparison, the results computed by crude MCS with 10^6 samples are shown in Table 1. Arithmetic average value of 15 tests results is considered the estimate value of failure probability.

Table 1
The Comparison Table of Failure Probability

	Crude MCS	**Procedure**	**Linearization coefficients ϖ_e**
Test 1	4.20944E-05	3.9221E-05	1.310634
Test 2	2.011945E-03	0.0019	2.164058
Test 3	8.4373016E-07	8.3933E-07	1.858361

3. CONCLUSIONS

Based on the mean up-crossing rate and importance sampling technique, a method has been proposed for solving the first excursion problem of non-linear system. Simulation results show the procedure to be correct and effective by comparing with the crude Monte Carlo method. The number of samples and the computational time are reduced significantly compared with crude MCS.

For estimating the first failure probability, due to simplicity and computational efficiency, equivalent linearization method has become a standard tool of stochastic structural dynamics. When the mean up crossing rate cannot be evaluated analytically, some numerical extrapolation can be implemented to assess it.

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