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Strongly Lacunary Summable Generalized Difference Double Sequence Spaces in $n-$Normed Spaces Defined by Ideal Convergence and an Orlicz Function

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Abstract: The notion of n-normed space was studied at the initial stage by Gahler (Gahler, 1965), Gunawan (Gunawan, 2001) and many others. In this paper, we introduce some certain new generalized difference double sequence spaces via ideal convergence, double lacunary sequence and an Orlicz function in n-normed spaces and examine some properties of the resulting these spaces.

Key words: P-convergent; Double lacunary sequence; n-normed space; Orlicz function

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1. INTRODUCTION

Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\varnothing \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called *non-trivial ideal* if $I \neq \varnothing$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{X/A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called *admissible* if and only if $\{\{x\} . x \in X\} \subset I$. Further details on ideals of 2^X can be found in Kostyrko, *et al.* [1]. The notion was further investigated by Salat, *et al.* [2] and others.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > n$ [3]. We shall write more briefly as "*$P-$convergent*".

The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Let l^2_∞ the space of all bounded double such that

$$\|x_{k,l}\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called *double lacunary sequence* [4] if there exist two increasing of integers such that

$$k_o = 0,\ h_r = k_r - k_{r-1} \to \infty \quad \text{as } r \to \infty$$

$$\text{and } l_o = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \quad \text{as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}},\ \bar{q}_s = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \bar{q}_s.$$

Recall in [5] that an *Orlicz function* M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the *modulus function* and characterized by Ruckle [6]. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1. *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < Kt\delta^{-1}M(2)$ for some constant $K > 0$.*

A double sequence space X is said to be *solid* or *normal* if $(\alpha_{k,l}x_{k,l}) \in X$, and for all double sequences $\alpha = (\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|., ..., .\|$ on X satisfying the following four conditions:

(i) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,

(ii) $\|x_1, x_2, ..., x_n\|$ is invariant under permutation,

(iii) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|, \alpha \in \mathbb{R}$,

(iv) $\|x_1 + x_1^\imath, x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|x_1^\imath, x_2, ..., x_n\|$ is called an $n-$*norm* on X, and the pair $(X, \|., ..., .\|)$ is called an $n-$*normed space* [7,8]. Normed space was studied by Mursaleen and Mohiuddine [9,10], Mohiuddine and Lohani [11], Mohiuddine and Alghamdi [12] and many others from different aspects.

A trivial example of $n-$normed space is $X=\mathbb{R}$ equipped with the following Euclidean $n-$norm:

$$\|x_1,x_2,...,x_n\|_E = abs\left(\begin{vmatrix} x_{11}...x_{1n} \\ ... \\ x_{n1}...x_{nn} \end{vmatrix}\right)$$

where $x_i=(x_{i1},...,x_{in})\in\mathbb{R}^n$ for each $i=1,2,\ldots,n$.

2. MAIN RESULTS

Let I_2 be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$, $\theta_{r,s}$ be a double lacunary sequence, M be an Orlicz function, $p=(p_{k,l})$ be a bounded double sequence of strictly positive real numbers and $(X,\|.,...,.\|)$ be an $n-$normed space. Further $w(n-X)$ denotes $X-$valued sequence space. Now, we define the following double generalized difference sequence spaces:

$$w_{\theta_{r,s}}^{I_2}[M,\Delta^m,p,\|.,...,.\|]_o=\Big\{x=(x_{k,l})\in w(n-X):\forall\varepsilon>0,$$
$$\left\{(r,s)\in I_{r,s}:\ \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_2$$
$$\text{for some }\rho>0\text{ and for every }z_1,z_2,...,z_{n-1}\in X\Big\},$$

$$w_{\theta_{r,s}}^{I_2}[M,\Delta^m,p,\|.,...,.\|]=\Big\{x=(x_{k,l})\in w(n-X):\forall\varepsilon>0,$$
$$\left\{(r,s)\in I_{r,s}:\ \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_2$$
$$\text{for some }\rho>0,L\in X\text{ and for every }z_1,z_2,...,z_{n-1}\in X\Big\},$$

$$w_{\theta_{r,s}}^{I_2}[M,\Delta^m,p,\|.,...,.\|]_\infty=\Big\{x=(x_{k,l})\in w(n-X):\exists K>0,$$
$$\left\{(r,s)\in I_{r,s}:\ \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq K\right\}\in I_2$$
$$\text{for some }\rho>0\text{ and for every }z_1,z_2,...,z_{n-1}\in X\Big\},$$

and

$$w_{\theta_{r,s}}[M,\Delta^m,p,\|.,...,.\|]_\infty=\Big\{x=(x_{k,l})\in w(n-X):\exists K>0,$$
$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq K$$
$$\text{for some }\rho>0\text{ and for every }z_1,z_2,...,z_{n-1}\in X\Big\},$$

where $\Delta^m x = (\Delta^m x_{k,l}) = \left(\Delta^{m-1}x_{k,l} - \Delta^{m-1}x_{k,l+1} - \Delta^{m-1}x_{k+1,l} + \Delta^{m-1}x_{k+1,l+1}\right)$, $\left(\Delta^1 x_{k,l}\right) = (\Delta x_{k,l}) = (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1})$, $\Delta^0 x = (x_{k,l})$ and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{k,l} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{k+i,l+j}.$$

If $m = 0$ and $\theta_{r,s} = \{(2^r, 2^s)\}$, we obtain

$$\begin{aligned} w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|., ..., .\|\right]_o &= w^{I_2}\left[M, p, \|., ..., .\|\right]_o, \\ w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|., ..., .\|\right] &= w^{I_2}\left[M, p, \|., ..., .\|\right], \\ w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|., ..., .\|\right]_\infty &= w^{I_2}\left[M, p, \|., ..., .\|\right]_\infty \\ \text{and } w_{\theta_{r,s}}\left[M, \Delta^m, p, \|., ..., .\|\right]_\infty &= w\left[M, p, \|., ..., .\|\right]_\infty \end{aligned}$$

which were defined and studied by Esi [13].

The following well-known inequality will be used in this study:

If $0 \le \inf_{k,l} p_{k,l} = H_o \le p_{k,l} \le \sup_{k,l} = H < \infty$, $D = \max\left(1, 2^{H-1}\right)$, then

$$|x_{k,l} + y_{k,l}|^{p_{k,l}} \le D\left\{|x_{k,l}|^{p_{k,l}} + |y_{k,l}|^{p_{k,l}}\right\}$$

for all $k, l \in \mathbb{N}$ and $x_{k,l}, y_{k,l} \in \mathbb{C}$. Also $|x_{k,l}|^{p_{k,l}} \le \max\left(1, |x_{k,l}|^H\right)$ for all $x_{k,l} \in \mathbb{C}$.

Theorem 1. *The sets* $w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|, ., \|\right]_o$, $w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|, ., \|\right]$ *and* $w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|, ., \|\right]_\infty$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. We will prove only for $w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|, ., \|\right]_o$ and the others can be proved similarly. Let $x, y \in w_{\theta_{r,s}}^{I_2}\left[M, \Delta^m, p, \|, ., \|\right]_o$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\left\{(r,s) \in I_{r,s}: \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \ge \frac{\varepsilon}{2}\right\} \in I_2,$$
for some $\rho_1 > 0$

and

$$\left\{(r,s) \in I_{r,s}: \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m y_{k,l}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \ge \frac{\varepsilon}{2}\right\} \in I_2,$$
for some $\rho_2 > 0$.

Since $\|., ..., .\|$ is a $n-$norm and M is an Orlicz function, the following inequality holds:

$$\begin{aligned} &\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m(\alpha x_{k,l} + \beta y_{k,l})}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \\ \le &\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2}M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \end{aligned}$$

$$+\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}M\left(\left\|\frac{\Delta^m y_{k,l}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$+\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m y_{k,l}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$

From the above inequality, we get

$$\left\{(r,s)\in I_{r,s}:\ \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m\left(\alpha x_{k,l}+\beta y_{k,l}\right)}{|\alpha|\rho_1+|\beta|\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}$$
$$\subset\left\{(r,s)\in I_{r,s}:\ \frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}$$
$$\cup\left\{(r,s)\in I_{r,s}:\ \frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m y_{k,l}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}.$$

Two sets on the right hand side belong to I_2 and this completes the proof.

It is also easy verify that the space $w_{\theta_{r,s}}[M,\Delta^m,p,\|.,...,.\|]_\infty$ is also a linear space. □

Theorem 2. *For fixed $(n,m)\in\mathbb{N}\times\mathbb{N}$, $w_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty$ paranormed space with respect to the paranorm defined by*

$$h_{(n,m)}(x)=\sum_{k,l=1,1}^{m,m}\|x_{k,l},z_1,z_2,...,z_{n-1}\|$$
$$+\inf\rho^{\frac{p_{n,m}}{H}}>0:\left(\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}}\leq 1,$$

for some $\rho>0$ and for every $z_1,z_2,...,z_{n-1}\in X$.

Proof. $h_{(n,m)}(\theta)=0$ and $h_{(n,m)}(-x)=h_{(n,m)}(x)$ are easy to prove, so we omit them. Let us take $x,y\in w_{\theta_{r,s}}[M,\Delta^m,p,\|.,...,.\|]_\infty$. Let

$$A(x)=\{\rho>0:\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq 1,$$
$$\forall z_1,z_2,...,z_{n-1}\in X\}$$

and

$$A(y)=\{\rho>0:\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\Delta^m y_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq 1,$$
$$\forall z_1,z_2,...,z_{n-1}\in X\}.$$

Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$. If $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M \left(\left\| \frac{\Delta^m (x_{k,l} + y_{k,l})}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right] \\
\leq &\frac{\rho_1}{\rho_1+\rho_2} \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M \left(\left\| \frac{\Delta^m x_{k,l}}{\rho_1}, z_1, z_2, ..., z_{n-1} \right\| \right) \right] \\
&+ \frac{\rho_2}{\rho_1+\rho_2} \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M \left(\left\| \frac{\Delta^m y_{k,l}}{\rho_1}, z_1, z_2, ..., z_{n-1} \right\| \right) \right].
\end{aligned}$$

Thus

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M \left(\left\| \frac{\Delta^m (x_{k,l} + y_{k,l})}{\rho_1+\rho_2}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq 1$$

and

$$\begin{aligned}
h_{(n,m)}(x+y) = &\sum_{k,l=1,1}^{m,m} \left\| x_{k,l} + y_{k,l}, z_1, z_2, ..., z_{n-1} \right\| \\
&+ \inf \left\{ (\rho_1+\rho_2)^{\frac{p_{n,m}}{H}} : \rho_1 \in A(x) \text{ and } \rho_2 \in A(y) \right\} \\
\leq &\sum_{k,l=1,1}^{m,m} \left\| x_{k,l}, z_1, z_2, ..., z_{n-1} \right\| + \inf \left\{ (\rho_1)^{\frac{p_{n,m}}{H}} : \rho_1 \in A(x) \right\} \\
&+ \sum_{k,l=1,1}^{m,m} \left\| y_{k,l}, z_1, z_2, ..., z_{n-1} \right\| + \inf \left\{ (\rho_2)^{\frac{p_{n,m}}{H}} : \rho_2 \in A(y) \right\} \\
= &\, h_{(n,m)}(x) + h_{(n,m)}(y).
\end{aligned}$$

Now, let $\lambda_{k,l}^u \to \lambda$, where $\lambda_{k,l}^u, \lambda \in \mathbb{C}$ and $h_{(n,m)}\left(x_{k,l}^u - x_{k,l}\right) \to 0$ as $u \to \infty$. We have to show that $h_{(n,m)}\left(\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}\right) \to 0$ as $u \to \infty$. Let $\lambda_{k,l} \to \alpha$, where $\lambda_{k,l}, \lambda \in \mathbb{C}$ and $h_{(n,m)}\left(x_{k,l}^u - x_{k,l}\right) \to 0$ as $u \to \infty$. Let

$$\begin{aligned}
A(x^u) = &\left\{ \rho_u > 0 : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M \left(\left\| \frac{\Delta^m x_{k,l}^u}{\rho_u}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq 1, \right. \\
&\left. \forall z_1, z_2, ..., z_{n-1} \in X \right\}.
\end{aligned}$$

and

$$\begin{aligned}
&A(x^u - x) \\
= &\left\{ \rho_u^\imath > 0 : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M \left(\left\| \frac{\Delta^m \left(x_{k,l}^u - x_{k,l}\right)}{\rho_u^\imath}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq 1, \right. \\
&\left. \forall z_1, z_2, ..., z_{n-1} \in X \right\}.
\end{aligned}$$

If $\rho_u \in A(x^u)$ and $\rho_u^\imath \in A(x^u - x)$ then we observe that

$$M\left(\left\|\frac{\Delta^m\left(\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}\right)}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\|\right)$$

$$\leq M\left(\left\|\frac{\Delta^m\left(\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}^u\right)}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\| + \left\|\frac{\Delta^m\left(\lambda x_{k,l}^u - \lambda x_{k,l}\right)}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\|\right)$$

$$\leq \frac{\rho_u\left|\lambda_{k,l}^u - \lambda\right|}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|} M\left(\left\|\frac{\Delta^m x_{k,l}^u}{\rho_u}, z_1, z_2, ..., z_{n-1}\right\|\right)$$

$$+ \frac{\rho_u^\imath|\lambda|}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|} M\left(\left\|\frac{\Delta^m\left(x_{k,l}^u - x_{k,l}\right)}{\rho_u^\imath}, z_1, z_2, ..., z_{n-1}\right\|\right).$$

From this inequality, it follows that

$$\left[M\left(\left\|\frac{\Delta^m\left(\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}\right)}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \leq 1$$

and consequently

$$h_{(n,m)}\left(\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}\right) = \sum_{k,l=1,1}^{m,m}\left\|\lambda_{k,l}^u x_{k,l} - \lambda x_{k,l}, z_1, z_2, ..., z_{n-1}\right\|$$

$$+ \inf\left\{\left(\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|\right)^{\frac{p_{n,m}}{H}} : \rho_u \in A(x^u) \text{ and } \rho_u^\imath \in A(x^u - x)\right\}$$

$$\leq \left|\lambda_{k,l}^u - \lambda\right| \sum_{k,l=1,1}^{m,m}\left\|x_{k,l}^u, z_1, z_2, ..., z_{n-1}\right\|$$

$$+ |\lambda| \sum_{k,l=1,1}^{m,m}\left\|x_{k,l}^u - x_{k,l}, z_1, z_2, ..., z_{n-1}\right\|$$

$$+ \left(\left|\lambda_{k,l}^u - \lambda\right|\right)^{\frac{p_{n,m}}{H}} \inf\left\{(\rho_u)^{\frac{p_{n,m}}{H}} : \rho_u \in A(x^u)\right\}$$

$$+ (|\lambda|)^{\frac{p_{n,m}}{H}} \inf\left\{(\rho_u^\imath)^{\frac{p_{n,m}}{H}} : \rho_u^\imath \in A(x^u - x)\right\}$$

$$\leq \max\left\{\left|\lambda_{k,l}^u - \lambda\right|, \left(\left|\lambda_{k,l}^u - \lambda\right|\right)^{\frac{p_{n,m}}{H}}\right\} h_{(n,m)}\left(x_{k,l}^u\right)$$

$$+ \max\left\{|\lambda|, (|\lambda|)^{\frac{p_{n,m}}{H}}\right\} h_{(n,m)}\left(x_{k,l}^u - x_{k,l}\right).$$

Hence by our assumption the right hand side tends to zero as $u \to \infty$. This completes the proof. □

Corollary 1. *It can be noted that* $h = \inf_{n,m\in\mathbb{N}} h_{(n,m)}$ *also gives a paranorm on the above sequence spaces. However if one consider the sequence space*

$$w_{\theta_{r,s}}[M, \Delta^m, p, \|.,...,.\|]_\infty$$

which is larger space than the space $w_{\theta_{r,s}}^{I_2}\left[M,\Delta^m,p,\|.,\ldots,.\|\right]_\infty$ *the construction of the paranorm is not clear and we leave it as an open problem. However it should be noted that for a fixed* $F \in I_2$*, the space*

$$\begin{aligned}
&w_{\theta_{r,s}}^{F}\left[M,\Delta^m,p,\|.,...,.\|\right]_\infty \\
&=\Big\{x=(x_{k,l}) \in w(n-X) : \exists K>0,\ \Big\{(n,m)\in\mathbb{N}\times\mathbb{N} : \\
&\sup_{(r,s)\in\mathbb{N}\times\mathbb{N}/F} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \geq K\Big\} \in I_2, \\
&\text{for some } \rho>0 \text{ and for every } z_1, z_2, ..., z_{n-1} \in X\Big\}
\end{aligned}$$

which is subspace of the space $w_{\theta_{r,s}}^{I_2}\left[M,\Delta^m,p,\|.,...,.\|\right]_\infty$ *is a paranormed space with the paranorms* $h_{(n,m)}$ *for* $(n,m) \notin F$ *and* $h_F = \inf_{(n,m)\in\mathbb{N}\times\mathbb{N}/F} h_{(n,m)}$.

Theorem 3. *Let* M, M_1 *and* M_2 *be Orlicz functions. Then we have*

(i) $w_{\theta_{r,s}}^{I_2}\left[M_1,\Delta^m,p,\|.,...,.\|\right]_o \subset w_{\theta_{r,s}}^{I_2}\left[MoM_1,\Delta^m,p,\|.,...,.\|\right]_o$ *provided that* $p=(p_{k,l})$ *is such that* $H_o>0$.

(ii) $w_{\theta_{r,s}}^{I_2}\left[M_1,\Delta^m,p,\|.,...,.\|\right]_o \cap w_{\theta_{r,s}}^{I_2}\left[M_2,\Delta^m,p,\|.,...,.\|\right]_o$
$\subset w_{\theta_{r,s}}^{I_2}\left[M_1+M_2,\Delta^m,p,\|.,...,.\|\right]_o$.

Proof. **(i).** For given $\varepsilon>0$, we first choose $\varepsilon_o>0$ such that $\max\left\{\varepsilon_o^H, \varepsilon_o^{H_o}\right\} < \varepsilon$. Now using the continuity of M, choose $0<\delta<1$ such that $0<t<\delta$ implies $M(t)<\varepsilon_o$. Let $x \in w_{\theta_{r,s}}^{I_2}\left[M_1,\Delta^m,p,\|.,...,.\|\right]_o$. Now from the definition of the space $w^{I_2}\left[M_1,\Delta^m,p,\|.,...,.\|\right]_o$, for some $\rho>0$

$$A(\delta)=\left\{(r,s)\in I_{r,s} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M_1\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \geq \delta^H\right\} \in I_2.$$

Thus if $(n,m)\notin A(\delta)$ then

$$\begin{aligned}
&\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M_1\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} < \delta^H \\
&\Rightarrow \sum_{(k,l)\in I_{r,s}} \left[M_1\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} < h_{r,s}\delta^H, \\
&\Rightarrow \left[M_1\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} < \delta^H \text{ for all } (k,l)\in I_{r,s}, \\
&\Rightarrow M_1\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right) < \delta \text{ for all } (k,l)\in I_{r,s}.
\end{aligned}$$

Hence from above inequality and using continuity of M, we must have

$$M\left(M_1\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right) < \varepsilon_o \text{ for all } (k,l)\in I_{r,s}$$

which consequently implies that

$$\sum_{(k,l)\in I_{r,s}} \left[M\left(M_1\left(\left\|\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)\right]^{p_{k,l}} < h_{r,s}\max\left\{\varepsilon_o^H, \varepsilon_o^{H_o}\right\} < h_{r,s}\varepsilon,$$

$$\Rightarrow \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(M_1\left(\left\| \frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right) \right]^{p_{k,l}} < \varepsilon.$$

This shows that

$$\{(r,s) \in I_{r,s} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(M_1\left(\left\| \frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right) \right]^{p_{k,l}} \geq \varepsilon\} \subset A(\delta)$$

and so belongs to I_2. This completes the proof.

(ii) Let $x \in w^{I_2}_{\theta_{r,s}} [M_1, \Delta^m, p, \|., ..., .\|]_o \cap w^{I_2}_{\theta_{r,s}} [M_2, \Delta^m, p, \|., ..., .\|]_o$. Then the fact that

$$\begin{aligned} &\frac{1}{h_{r,s}} \left[(M_1 + M_2) \left(\left\| \frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ \leq &\frac{D}{h_{r,s}} \left[M_1 \left(\left\| \frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ &+ \frac{D}{h_{r,s}} \left[M_2 \left(\left\| \frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \end{aligned}$$

gives us the result. □

Theorem 4. ***(i)*** *If* $0 < H_o \leq p_{k,l} < 1$, *then*

$$w^{I_2}_{\theta_{r,s}} [M, \Delta^m, p, \|.,...,.\|]_o \subset w^{I_2}_{\theta_{r,s}} [M, \Delta^m, \|.,...,.\|]_o$$

.

(ii) *If* $1 \leq p_{k,l} \leq H < \infty$, *then*

$$w^{I_2}_{\theta_{r,s}} [M, \Delta^m, \|., ..., .\|]_o \subset w^{I_2}_{\theta_{r,s}} [M, \Delta^m, p, \|., ..., .\|]_o$$

.

(iii) *If* $0 < p_{k,l} < q_{k,l} < \infty$ *and* $\dfrac{q_{k,l}}{p_{k,l}}$ *is bounded, then*

$$w^{I_2}_{\theta_{r,s}} [M, \Delta^m, p, \|., ..., .\|]_o \subset w^{I_2}_{\theta_{r,s}} [M, \Delta^m, q, \|., ..., .\|]_o$$

.

Proof. The proof is standard, so we omit it. □

Theorem 5. *The sequence spaces* $w^{I_2}_{\theta_{r,s}} [M, \Delta^m, p, \|., ..., .\|]_o$, $w^{I_2}_{\theta_{r,s}} [M, \Delta^m, p, \|., ..., .\|]$, $w^{I_2}_{\theta_{r,s}} [M, \Delta^m, p, \|., ..., .\|]_\infty$ *and* $w_{\theta_{r,s}} [M, \Delta^m, p, \|., ..., .\|]_\infty$ *are solid.*

Proof. We give the proof for only $w^{I_2}_{\theta_{r,s}} [M, \Delta^m, p, \|., ..., .\|]_o$. The others can be proved similarly. Let $x \in w^{I_2}_{\theta_{r,s}} [M_1, \Delta^m, p, \|., ..., .\|]_o$ and $\alpha = (\alpha_{k,l})$ be a double sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$. Then we have

$$\left\{ (r,s) \in I_{r,s} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\left\| \frac{\Delta^m (\alpha_{k,l} x_{k,l})}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq \varepsilon \right\}$$

$$\subset \left\{ (r,s) \in I_{r,s} : \frac{T}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\left\| \frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq \varepsilon \right\} \in I_2,$$

where $T = \max_{k,l} \left\{ 1, |\alpha_{k,l}|^H \right\}$.

Hence $\alpha x \in w^{I_2}_{\theta_{r,s}} [M_1, \Delta^m, p, \|.,...,.\|]_o$ for all double sequences $\alpha = (\alpha_{k,l})$ with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$ whenever $x \in w^{I_2}_{\theta_{r,s}} [M_1, \Delta^m, p, \|.,...,.\|]_o$. $\square$

REFERENCES

[1] Kostyrko, P., Salat, T., & Wilczynski, W. (2000/2001). I-convergence. *Real Analysis Exchange, 26*(2), 669-686.

[2] Salat, T., Tripathy, B. C., & Ziman, M. (2005). On I-convergence field. *Italian J. Pure and Appl. Math., 17*, 45-54.

[3] Pringsheim, A. (1900). Zur theori der zweifach unendlichen zahlenfolgen. *Math. Ann., 53*, 289-321.

[4] Savaş, E., & Patterson, R. F. (2011). Some double lacunary sequence spaces defined by orlicz functions. *Southeast Asian Bulletin of Mathematics, 35*(1), 103-110.

[5] Krasnoselski, M. A., & Rutickii, Y. B. (1961). *Convex function and orlicz spaces*. Nederland: Groningen.

[6] Ruckle, W. H. (1973). FK-spaces in which the sequence of coordinate vectors is bounded. *Canad. J. Math., 25*, 973-978.

[7] Gunawan, H. (2001). On n-inner product, n-norms and the Cauchy-Schwarz inequality. *Scientiae Mathematicae Japonicae Online, 5*, 47-54.

[8] Gahler, S. (1965). Lineare 2-normierte. *Rume, Math. Nachr., 28*(164), 1-43.

[9] Mursaleen, M., & Mohiuddine, S. A. (2009). On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. *J. Comput. Appl. Math., 233*, 142-149.

[10] Mursaleen, M., & Mohiuddine, S. A. (2012). On ideal convergence in probabilistic normed spaces. *Math. Slovaca, 62*(1), 49-62.

[11] Mohiuddine, S. A., & Danish Lohani, Q. M. (2009). On generalized statistical convergence in intuitionistic fuzzy normed space. *Chaos, Solitons Fractals, 42*, 1731-1737.

[12] Mohiuddine, S. A., & Alghamdi, M. A. (2012). Statistical summability through a lacunary sequence in locally solid Riesz spaces. *J. Inequal. Appl., 2012*, 225.

[13] Esi, A. (2012). Strongly summable double sequence spaces in n-normed space defined by ideal convergence and an orlicz function. *Kyungpook Math. J., 52*, 137-147.