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Comparison Theorem for Oscillation of Nonlinear Delay Partial Difference Equations

LIU Guanghui[a],* and GAO Youwu[a]

[a]College of Science, Hunan Institute of Engineering, China.

* Corresponding author.
Address: College of Science, Hunan Institute of Engineering, East Fuxing Road, Xiangtan 411104, Hunan, China; E-Mail: gh29202@163.com

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Abstract: In this paper,we consider certain nonlinear partial difference equations

$$(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k - (dA_{m,n})^k + \sum_{i=1}^{u} p_i(m,n)A^k_{m-\sigma_i,n-\tau_i} = 0$$

where $a, b, c, d \in (0, \infty)$, $d > c$, $k = q/p$, p, q are positive odd integers, u is a positive integer, $p_i(m,n), (i = 0, 1, 2, \cdots u)$ are positive real sequences. $\sigma_i, \tau_i \in N_0 = \{1, 2, \cdots\}, i = 1, 2, \cdots, u$. A new comparison theorem for oscillation of the above equation is obtained.

Key words: Nonlinear partial difference equations; Comparison theorem; Eventually positive solutions

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1. INTRODUCTION

In this paper we consider nonlinear partial difference equation

$$(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k - (dA_{m,n})^k + \sum_{i=1}^{u} p_i(m,n)A^k_{m-\sigma_i,n-\tau_i} = 0, \quad (1.1)$$

where $a, b, c, d \in (0, \infty)$, $d > c$, $k = q/p$, p, q are positive odd integers, u is a positive integer, $p_i(m, n), (i = 0, 1, 2, \cdots u)$ are positive real sequences. $\sigma_i, \tau_i \in N_0 = \{1, 2, \cdots\}, i = 1, 2, \cdots, u$. The purpose of this paper is to obtain a new comparison theorem for oscillation of all solutions of (1.1).

2. MAIN RESULTS

To prove our main result we need several preparatory results.

Lemma 2.1 Assume that $\{A_{m,n}\}$ is a positive solution of (1.1). Then

$$\text{i}: \quad A_{m+1,n} \le \theta_1 A_{m,n}, A_{m,n+1} \le \theta_2 A_{m,n}, \tag{2.1}$$

and

$$\text{ii}: \quad A_{m-\sigma_i,n-\tau_i} \ge \theta_1^{\sigma_0-\sigma_i}\theta_2^{\tau_0-\tau_i} A_{m-\sigma_0,n-\tau_0}, \tag{2.2}$$

where $\theta_1 = \dfrac{d-c}{a}, \theta_2 = \dfrac{d-c}{b}, \sigma_0 = \min\limits_{1\le i\le u}\{\sigma_i\}, \tau_0 = \min\limits_{1\le i\le u}\{\tau_i\}$.

Proof. Assume that $\{A_{m,n}\}$ is eventually positive solutions of (1.1). From (1.1), we have

$$(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k - (dA_{m,n})^k = -\sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i}^k \le 0,$$

and so

$$(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k \le (dA_{m,n})^k.$$

Since $k = \dfrac{p}{q}, p, q$ are positive odd integers, then

$$aA_{m+1,n} + bA_{m,n+1} \le (d-c)A_{m,n}.$$

Hence $A_{m+1,n} \le \theta_1 A_{m,n}$ and $A_{m,n+1} \le \theta_2 A_{m,n}$. From the above inequality, we can find $A_{m,n} \le \theta_1^{\sigma_0} A_{m-\sigma_0,n} \le \theta_1^{\sigma_i} A_{m-\sigma_i,n}$, $A_{m-\sigma_0,n} \le \theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0}$, and

$$A_{m-\sigma_i,n} \le \theta_2^{\tau_0} A_{m-\sigma_i,n-\tau_0} \le \theta_2^{\tau_i} A_{m-\sigma_i,n-\tau_i}.$$

Hence

$$A_{m,n} \le \theta_1^{\sigma_0}\theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0} \le \theta_1^{\sigma_i}\theta_2^{\tau_i} A_{m-\sigma_i,n-\tau i}.$$

The proof of Lemma 2.1 is completed. □

Lemma 2.2 [1] If $x, y \in R^+$ and $x \ne y$, then

$$rx^{r-1}(x-y) > x^r - y^r > ry^{r-1}(x-y), \quad \text{for} \quad r > 1.$$

Theorem 2.1 If the difference inequality

$$aA_{m+1,n}+bA_{m,n+1}-(d-c)A_{m,n}+\sum_{i=1}^{u}\frac{\theta_1^{\sigma_0-k\sigma_i}\theta_2^{\tau_0-k\tau_i}}{kd^{k-1}}p_i(m,n)A_{m-\sigma_0,n-\tau_0} \le 0 \tag{2.3}$$

has no eventually positive solutions, then every solution of Equation (1.1) oscillates.

Proof. Assume that $\{A_{m,n}\}$ a is positive solution of Equation (1.1). Then, by (1.1) and Lemma 2.2, we obtain

$$aA_{m+1,n} + bA_{m,n+1} - (d-c)A_{m,n} + \sum_{i=1}^{u} p_i(m,n)\frac{A^k_{m-\sigma_i,n-\tau_i}}{kd^{k-1}A^{k-1}_{m,n}} \leq 0 \tag{2.4}$$

Substituting (2.2) into (2.4), we have

$$aA_{m+1,n} + bA_{m,n+1} - (d-c)A_{m,n} + \sum_{i=1}^{u} \frac{\theta_1^{\sigma_0-k\sigma_i}\theta_2^{\tau_0-k\tau_i}}{kd^{k-1}} p_i(m,n)A_{m-\sigma_0,n-\tau_0} \leq 0.$$

This contradiction completes the proof. □

Define a set E by

$$E = \{\lambda > 0 | d - c - \lambda Q_{m,n} > 0, \quad \text{eventually}\}$$

where $Q_{m,n} = \sum_{i=1}^{u} \frac{\theta_1^{\sigma_0-k\sigma_i}\theta_2^{\tau_0-k\tau_i}}{kd^{k-1}} p_i(m,n)$.

Theorem 2.2 Assume that

(i) $\lim_{m,n\to\infty} \sup Q_{m,n} > 0$;

(ii) there exists $M \geq m_0$, $N \geq n_0$ such that if $\sigma_0 > \tau_0 > 0$,

$$\sup_{\lambda\in E, M\geq m, N\geq n} \lambda[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(d-c-\lambda Q_{m-i-j,n-i})]^{\frac{1}{\sigma_0-\tau_0}} < (\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\tau_0}\theta_1^{\tau_0-\sigma_0}, \tag{2.5}$$

and if $\tau_0 > \sigma_0 > 0$,

$$\sup_{\lambda\in E, M\geq m, N\geq n} \lambda[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(d-c-\lambda Q_{m-i,n-i-j})]^{\frac{1}{\tau_0-\sigma_0}} < (\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\sigma_0}\theta_2^{\sigma_0-\tau_0}. \tag{2.6}$$

Then every solution of (1.1) oscillates.

Proof. Suppose, to the contrary, $A_{m,n}$ is an eventually positive solution. We define a subset S of the positive numbers as follows:

$$S(\lambda) = \{\lambda > 0 | aA_{m+1,n} + bA_{m,n+1} - [(d-c-\lambda Q_{m,n}]A_{m,n} \leq 0, \quad \text{eventually}\}.$$

From (2.3) and Lemma 2.1, we have

$$aA_{m+1,n} + bA_{m,n+1} - (d-c-\theta_1^{-\sigma_0}\theta_2^{-\tau_0}Q_{m,n})A_{m,n} \leq 0,$$

which implies $\theta_1^{-\sigma_0}\theta_2^{-\tau_0} \in S(\lambda)$. Hence, $S(\lambda)$ is nonempty. For $\lambda \in S$, we have eventually that $d-c-\lambda Q_{m,n} > 0$, which implies that $S \subset E$, Due to condition (i), the set E is bounded, and hence, $S(\lambda)$ is bounded. Let $u \in S$. Then from Lemma 2.1, we have

$$\begin{aligned}(\frac{a}{\theta_2}+\frac{b}{\theta_1})A_{m+1,n+1} &\leq aA_{m+1,n} + bA_{m,n+1} \\ &\leq (d-c-uQ_{m,n})A_{m,n}.\end{aligned}$$

If $\sigma_0 > \tau_0 > 0$, then

$$A_{m,n} \leq (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i,n-i}) A_{m-\tau_0,n-\tau_0},$$

and for $j = 1, 2, \cdots, \sigma_0 - \tau_0$, we have

$$\begin{aligned} A_{m-j,n} &\leq (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i}) A_{m-\tau_0-j,n-\tau_0} \\ &\leq \theta_1^{\sigma_0-\tau_0-j} (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i}) A_{m-\sigma_0,n-\tau_0}. \end{aligned} \tag{2.7}$$

Now, from Lemma 2.1 and (2.7), it follows that

$$A_{m,n}^{\sigma_0-\tau_0} \leq (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{-\tau_0(\sigma_0-\tau_0)} \theta_1^{(\sigma_0-\tau_0)^2} [\prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i})] A_{m-\sigma_0,n-\tau_0}^{\sigma_0-\tau_0},$$

i. e.,

$$A_{m,n} \leq \{(\frac{a}{\theta_2} + \frac{b}{\theta_1})^{-\tau_0(\sigma_0-\tau_0)} \theta_1^{(\sigma_0-\tau_0)^2} [\prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i})]\}^{\frac{1}{\sigma_0-\tau_0}} A_{m-\sigma_0,n-\tau_0}. \tag{2.8}$$

Similarly, if $\tau_0 > \sigma_0 > 0$, then

$$A_{m,n} \leq \{(\frac{a}{\theta_2} + \frac{b}{\theta_1})^{-\sigma_0(\tau_0-\sigma_0)} \theta_2^{(\tau_0-\sigma_0)^2} [\prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i,n-i-j})]\}^{\frac{1}{\tau_0-\sigma_0}} A_{m-\sigma_0,n-\tau_0}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.3), we get respectively, for $\sigma_0 > \tau_0$,

$$\begin{aligned} & aA_{m+1,n} + bA_{m,n+1} - (d - c)A_{m,n} \\ + \quad & Q_{m,n} (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{\tau_0} \theta_1^{\tau_0-\sigma_0} (\prod_{j=1}^{\sigma_0-\tau_0} [\prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i})]^{\frac{1}{\tau_0-\sigma_0}} A_{m,n} \leq 0, \end{aligned}$$

and for $\tau_0 > \sigma_0$,

$$\begin{aligned} & aA_{m+1,n} + bA_{m,n+1} - (d - c)A_{m,n} \\ + \quad & Q_{m,n} (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{\sigma_0} \theta_2^{\sigma_0-\tau_0} [\prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\sigma_0} (d - c - uQ_{m-i,n-i-j})]^{\frac{1}{\sigma_0-\tau_0}} A_{m,n} \leq 0. \end{aligned}$$

Hence, for $\sigma_0 > \tau_0$,

$$\begin{aligned} & aA_{m+1,n} + bA_{m,n+1} - \{d - c - Q_{m,n} (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{\tau_0} \theta_1^{\tau_0-\sigma_0} \\ & \times \sup_{m \geq M, n \geq N} [\prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i})]^{\frac{1}{\tau_0-\sigma_0}}\} A_{m,n} \leq 0, \end{aligned} \tag{2.10}$$

and for $\tau_0 > \sigma_0$,

$$aA_{m+1,n} + bA_{m,n+1} - \{d - c - Q_{m,n}(\frac{a}{\theta_2} + \frac{b}{\theta_1})^{\sigma_0}\theta_2^{\sigma_0-\tau_0} \times \sup_{m\geq M,n\geq N}[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(d - c - uQ_{m-i,n-i-j}])^{\frac{1}{\sigma_0-\tau_0}}\}A_{m,n} \leq 0. \tag{2.11}$$

From (2.10) and (2.11), we get

$$(\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\tau_0}\theta_1^{\tau_0-\sigma_0}\{\sup_{m\geq M,n\geq N}[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(d-c-uQ_{m-i-j,n-i})]^{\frac{1}{\tau_0-\sigma_0}}\} \in S \text{ for } \sigma_0 > \tau_0, \tag{2.12}$$

and

$$(\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\sigma_0}\theta_2^{\sigma_0-\tau_0}(\sup_{m\geq M,n\geq N}[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(d-c-uQ_{m-i,n-i-j})]^{\frac{1}{\sigma_0-\tau_0}}) \in S \quad \text{for } \tau_0 > \sigma_0. \tag{2.13}$$

On the other hand, (2.5) implies that there exists $a_1 \in (0,1)$ (we can choose the same) such that for $\sigma_0 > \tau_0$

$$\sup_{\lambda\in E,m\geq M,n\geq N}\lambda[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(d-c-\lambda Q_{m-i-j,n-i})]^{\frac{1}{\sigma_0-\tau_0}} \leq a_1(\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\tau_0}\theta_1^{\tau_0-\sigma_0}, \tag{2.14}$$

and (2.6) implies that there exists $a_1 \in (0,1)$ (we can choose the same) such that for $\tau_0 > \sigma_0 > 0$,

$$\sup_{\lambda\in E,M\geq m,N\geq n}\lambda[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(d-c-\lambda Q_{m-i,n-i-j})]^{\frac{1}{\tau_0-\sigma_0}} \leq a_1(\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\sigma_0}\theta_2^{\sigma_0-\tau_0}. \tag{2.15}$$

In particular, (2.14) and (2.15) lead to (when $\lambda = u$), respectively,

$$(\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\tau_0}\theta_1^{\tau_0-\sigma_0}\sup_{\lambda\in E,m\geq M,n\geq N}[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(d-c-uQ_{m-i-j,n-i})]^{\frac{1}{\tau_0-\sigma_0}} \geq \frac{u}{a_1} \text{for } \sigma_0 > \tau_0, \tag{2.16}$$

and

$$(\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\sigma_0}\theta_2^{\sigma_0-\tau_0}\sup_{\lambda\in E,M\geq m,N\geq n}[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(d-c-uQ_{m-i,n-i-j})]^{\frac{1}{\sigma_0-\tau_0}} \geq \frac{u}{a_1} \text{for } \tau_0 > \sigma_0. \tag{2.17}$$

Since $u \in S$ and $u' \leq u$ implies that $u' \in S$, it follows from (2.12) and (2.16) for $\sigma_0 > \tau_0$, (2.13) and (2.17) for $\tau_0 > \sigma_0$ that $\frac{u}{a_1} \in S$. Repeating the above arguments with u replaced by $\dfrac{u}{a_1}$, we get $\frac{u}{a_1a_2} \in S$, where $a_2 \in (0,1)$. Continuing in this way, we obtain $\dfrac{u}{\prod_{i=1}^{\infty} a_i} \in S$, where $a_i \in (0,1)$. This contradicts the boundedness of S. The proof is complete. □

Corollary 2.1 In addition to (i) of Theorem 2.1, assume that for $\sigma_0 > \tau_0 > 0$,

$$\lim_{m,n\to\infty} \inf \frac{1}{(\sigma_0-\tau_0)\tau_0} \sum_{j=1}^{\sigma_0-\tau_0} \sum_{i=1}^{\tau_0} Q_{m-i-j,n-i} > \frac{(d-c)^{\tau_0+1}\tau_0^{\tau_0}}{(\tau_0+1)^{\tau_0+1}} (\frac{a}{\theta_2}+\frac{b}{\theta_1})^{-\tau_0}\theta_1^{\sigma_0-\tau_0},$$

and for $\tau_0 > \sigma_0 > 0$,

$$\lim_{m,n\to\infty} \inf \frac{1}{(\tau_0-\sigma_0)\sigma_0} \sum_{j=1}^{\tau_0-\sigma_0} \sum_{i=1}^{\sigma_0} Q_{m-i,n-i-j} > \frac{(d-c)^{\sigma_0+1}\sigma_0^{\sigma_0}}{(\sigma_0+1)^{\sigma_0+1}} (\frac{a}{\theta_2}+\frac{b}{\theta_1})^{-\sigma_0}\theta_2^{\tau_0-\sigma_0}.$$

Then every solution of (1.1) oscillates.

Proof. We note that

$$\max_{\frac{(d-c)}{e}>\lambda>0} \lambda(d-c-\lambda e)^{\tau_0} = \frac{(d-c)^{\tau_0+1}\tau_0^{\tau_0}}{e(\tau_0+1)^{\tau_0+1}}.$$

We shall use this for

$$e = \frac{1}{(\sigma_0-\tau_0)\tau_0} \sum_{j=1}^{\sigma_0-\tau_0} \sum_{i=1}^{\tau_0} Q_{m-i-j,n-i}.$$

Clearly,

$$\begin{aligned}
& \lambda[\prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (d-c-\lambda Q_{m-i-j,n-i})]^{\frac{1}{\sigma_0-\tau_0}} \\
\leq\ & \lambda[\frac{1}{(\sigma_0-\tau_0)\tau_0} \sum_{j=1}^{\sigma_0-\tau_0} \sum_{i=1}^{\tau_0} (d-c-\lambda Q_{m-i-j,n-i})]^{\tau_0} \\
\leq\ & \lambda[d-c-\frac{\lambda}{(\sigma_0-\tau_0)\tau_0} \sum_{j=1}^{\sigma_0-\tau_0} \sum_{i=1}^{\tau_0} (Q_{m-i-j,n-i})]^{\tau_0} \\
\leq\ & \frac{(d-c)^{\tau_0+1}\tau_0^{\tau_0}}{e(\tau_0+1)^{\tau_0+1}} \\
<\ & (\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\tau_0}\theta_1^{\tau_0-\sigma_0}.
\end{aligned}$$

Similarly, we have

$$\lambda[\prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\sigma_0} (d-c-\lambda Q_{m-i,n-i-j})]^{\frac{1}{\tau_0-\sigma_0}} < (\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\sigma_0}\theta_2^{\sigma_0-\tau_0}.$$

By Theorem 2.1, every solutions of (1.1) oscillates. The proof is complete. □

By a similar argument, we have the following results:
Theorem 2.3 Assume that
(i) $\lim_{m,n\to\infty} \sup Q_{m,n} > 0$;

(ii) there exists $M \geq m_0$, $N \geq n_0$ such that if $\sigma_0 = \tau_0 > 0$,

$$\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n-i}) < (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{\sigma_0}$$

Then every solution of (1.1) oscillates.

Corollary 2.2 If the condition of Theorem 2.2 holds, and

$$\lim_{m,n \to \infty} \inf Q_{m,n} = q > \frac{(d-c)^{\sigma_0+1} \sigma_0^{\sigma_0}}{(\sigma_0+1)^{\sigma_0+1}} (\frac{a}{\theta_2} + \frac{b}{\theta_1})^{-\sigma_0},$$

Then every solution of (1.1) oscillates.

Theorem 2.4 Assume that

(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;

(ii) there exists $M \geq m_0$, $N \geq n_0$ such that either

$$\sup_{\lambda \in E, M \geq m, N \geq n} \lambda [\prod_{j=1}^{\tau_0} \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n-j})]^{\frac{1}{\tau_0}} < a^{\sigma_0} \theta_2^{-\tau_0},$$

or

$$\sup_{\lambda \in E, M \geq m, N \geq n} \lambda [\prod_{i=1}^{\sigma_0} \prod_{j=1}^{\tau_0} (d - c - \lambda Q_{m-i,n-j})]^{\frac{1}{\sigma_0}} < b^{\tau_0} \theta_1^{-\sigma_0}.$$

Then every solution of (1.1) oscillates.

Corollary 2.3 In addition to (i) of Theorem 2.3, assume that for $\sigma_0, \tau_0 > 0$, either

$$\lim_{m,n \to \infty} \inf \frac{1}{\sigma_0 \tau_0} \sum_{j=1}^{\tau_0} \sum_{i=1}^{\sigma_0} Q_{m-i,n-j} > a^{-\sigma_0} \theta_2^{\tau_0} \frac{\sigma_0^{\sigma_0}}{(\sigma_0+1)^{\sigma_0+1}},$$

or

$$\lim_{m,n \to \infty} \inf \frac{1}{\sigma_0 \tau_0} \sum_{i=1}^{\sigma_0} \sum_{j=1}^{\tau_0} Q_{m-i,n-j} > b^{-\tau_0} \theta_1^{\sigma_0} \frac{\tau_0^{\tau_0}}{(\tau_0+1)^{\tau_0+1}},$$

Then every solution of (1.1) oscillates.

Theorem 2.5 Assume that

(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;

(ii) there exists $M \geq m_0$, $N \geq n_0$ such that if $\sigma_0 > 0, \tau_0 = 0$,

$$\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n}) < a^{\sigma_0}.$$

Then every solution of (1.1) oscillates.

Corollary 2.4 In addition to (i) of Theorem 2.4, assume that $\sigma_0 > 0, \tau_0 = 0$, and

$$\lim_{m,n \to \infty} \inf Q_{m,n} > \frac{(d-c)^{\sigma_0+1} \sigma_0^{\sigma_0}}{a^{\sigma_0} (\sigma_0+1)^{\sigma_0+1}}.$$

Then every solution of (1.1) oscillates.

Theorem 2.6 Assume that

(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;

(ii) there exists $M \geq m_0$, $N \geq n_0$ such that if $\sigma_0 = 0, \tau_0 > 0$,

$$\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{j=1}^{\tau_0} (d - c - \lambda Q_{m,n-j}) < b^{\tau_0}.$$

Then every solution of (1.1) oscillates.

Corollary 2.5 In addition to (i) of Theorem 2.5, assume that $\sigma_0 = 0, \tau_0 > 0$, and

$$\lim_{m,n \to \infty} \inf Q_{m,n} > \frac{(d-c)^{\tau_0+1} \tau_0^{\tau_0}}{b^{\tau_0} (\tau_0 + 1)^{\tau_0+1}}.$$

Then every solution of (1.1) oscillates.

Theorem 2.7 Assume that

(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;

(ii) for $\sigma_0, \tau_0 > 0$,

$$\lim_{m,n \to \infty} \inf Q_{m,n} = q > 0, \tag{2.18}$$

and

$$\lim_{m,n \to \infty} Q_{m,n} > (d-c)\theta_1^{\sigma_0} \theta_2^{\tau_0} - \frac{a\theta_1 + b\theta_2}{(d-c)} q > 0. \tag{2.19}$$

Then every solution of (1.1) oscillates.

Proof. Suppose, to the contrary, $A_{m,n}$ is an eventually positive solution. From (2.3) and (2.18), for any $\epsilon > 0$, we have $Q_{m,n} > q - \epsilon$ for $m \geq M, n \geq N$. From (2.3), Lemma 2.1 and above inequality, we obtain

$$A_{m,n} \geq \frac{(q-\epsilon)}{(d-c)} A_{m-\sigma_0, n-\tau_0} \geq \frac{(q-\epsilon)}{(d-c)} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} A_{m-1,n-1},$$

$A_{m,n} \geq \frac{(q-\epsilon)}{(d-c)} \theta_1^{1-\sigma_0} \theta_2^{-\tau_0} A_{m-1,n}$, and $A_{m,n} \geq \frac{(q-\epsilon)}{(d-c)} \theta_1^{-\sigma_0} \theta_2^{1-\tau_0} A_{m,n-1}$.

Substituting above inequalities into (2.3), we get

$$\Big[\frac{a\theta_1^{1-\sigma_0} \theta_2^{-\tau_0} + b\theta_1^{-\sigma_0} \theta_2^{1-\tau_0}}{d-c} (q - \epsilon) - (d-c) + Q_{m,n} \theta_1^{-\sigma_0} \theta_2^{-\tau_0} \Big] A_{m,n} < 0,$$

which implies

$$\lim_{m,n \to \infty} Q_{m,n} \leq (d-c)\theta_1^{\sigma_0} \theta_2^{\tau_0} - \frac{a\theta_1 + b\theta_2}{(d-c)} q > 0.$$

This contradicts (2.19). The proof is complete. □

Theorem 2.8 Assume that

(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;

(ii) $\sigma_0 = \tau_0 = 0$, and

$$\lim_{m,n \to \infty} \sup Q_{m,n} > d - c. \tag{2.20}$$

Then every solution of (1.1) oscillates.

Proof. Let $u \in S$. Then from (2.3) and Lemma 2.1, we have $-(d-c)+Q_{m,n}A_{m,n} < 0$, which implies $\lim_{m,n\to\infty} \sup Q_{m,n} \leq d-c$.

This contradicts (2.20). The proof is complete. □

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