CSCanada
Progress in Applied Mathematics
Vol. 5, No. 1, 2013, pp. [11–21]
DOI: 10.3968/j.pam.1925252820130501.2271

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
www.cscanada.org

Half-Sweep Quadrature-Difference Schemes with Iterative Method in Solving Linear Fredholm Integro-Differential Equations

E. Aruchunan[a],* and J. Sulaiman[b]

[a]School of Engineering and Science, Curtin University, Malaysia.
[b]School of Science and Technology, University Malaysia Sabah, Malaysia.

* Corresponding author.
Address: School of Engineering and Science, Curtin University, Miri 98009, Sarawak, Malaysia; E-Mail: elayarajah@yahoo.com

Received: September 19, 2012/ Accepted: November 20, 2012/ Published: January 31, 2013

Abstract: In this paper, half-sweep iteration concept applied on quadrature-difference schemes with Gauss-Seidel (GS) iterative method in solving linear Fredholm integro-differential equations. The combinations of discretization schemes of repeated trapezoidal and Simpson's $\frac{1}{3}$ with central difference schemes are analyzed. The formulation and the implementation of the proposed methods are explained in detail. In addition, several numerical experiments and computational complexity analysis were also carried out to validate the presentation of the schemes and methods. The findings show that, the HSGS iteration method is superior to the standard GS method. As well the high order quadrature scheme produced more accurate approximation solution compared to combination of repeated trapezoidal-central difference schemes.

Key words: Linear Fredholm integro-differential equations; Simpson's scheme; Central difference; Half-Sweep Gauss-Seidel

Aruchunan, E., & Sulaiman, J. (2013). Half-Sweep Quadrature-Difference Schemes with Iterative Method in Solving Linear Fredholm Integro-Differential Equations. *Progress in Applied Mathematics, 5*(1), 11–21. Available from http://www.cscanada.net/index.php/pam/article/view/j.pam.1925252820130501.2271 DOI: 10.3968/j.pam.1925252820130501.2271

1. INTRODUCTION

Generally, the first and second order linear Fredholm integro-differential equations (LFIDEs) can be defined as follows

$$y'(x) = p(x)y(x) + g(x) + \lambda \int_a^b K(x,t)y(t)dt, \quad x,t \in [a,b] \tag{1}$$

with the Dirichlet boundary condition, $y(a) = A_1$, and

$$y''(x) = q(x)y'(x) + p(x)y(x) + g(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt, \quad x,t \in [a,b] \tag{2}$$

with the Dirichlet boundary conditions $y(a) = A_1$ and $y(b) = B_1$, where $K(x,t)$, $g(x)$, $q(x)$ and $p(x)$ are defined variables, λ is a real parameter whereas $y(x)$ is the unknown function to be determined. In this paper, we focus on numerical solutions for first and second order linear integro-differential equations of Fredholm types. In many application areas, it is necessary to use the numerical approach to discretize problem (1) to generate system of linear equation then solved by numerical methods such as Lagrange interpolation [1] and Taylor polynomial [2] and rationalized Haar functions [3], Tau [4], Conjugate Gradient [5], GMRES [6] and collocation methods [7]. However in this paper we emphasize quadrature-difference schemes [8] to derive the approximation equation to generate system of linear equations. In addition to that, in this paper, we proposed a new half-sweep quadrature-difference discretization scheme which is combination of half-sweep reduction technique [9] on standard quadrature-difference schemes.

In this paper, two combinations of half-sweep discretization schemes namely half sweep repeated trapezoidal-central difference (HSRT-HSCD) and repeated Simpson -central difference (HSRS-HSCD) schemes will be implemented to discretize problem (1) to generate system of linear equations. Then the generated linear system will be solved iteratively by using half-sweep Gauss-Seidel (HSGS) method. In point of fact, the HSGS represents combination of half-sweep iteration concept on standard Gauss-Seidel (GS) which is also known as Full-Sweep Gauss Seidel (FSGS) method. The concept of the half-sweep iteration has been introduced by Abdullah [9] via Explicit Decoupled Group (EDG) iterative method to solve two-dimensional Poisson equation.Then, the idea of half-sweep iteration concept also identified as the complexity reduction approach [9] extensively studied by many researchers [10–13].

The remainder of this work is organized as follows. In Section 2, the derivation of the approximation equation is elaborated. In section 3, the formulation of the FSGS and HSGS iterative methods are shown. Meanwhile, some numerical results are illustrated in Section 4, to assert the effectiveness of the proposed methods and concluding remarks are given in Section 5.

2. HALF-SWEEP ITERATION CONCEPT

Figure 1(a) and 1(b) show distribution of uniform node points for the full- and half-sweep cases respectively. The full- and half-sweep iteration concept will compute approximate values onto node points of type only until the convergence criterion is reached. Then other approximate solutions at the remaining points (points of the different type) can be computed using the direct method [10–13].

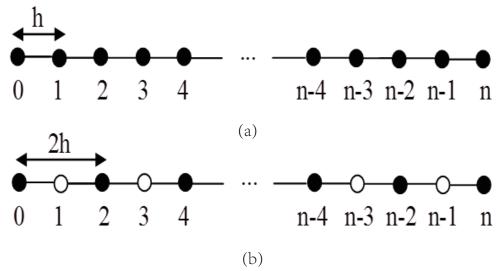

Figure 1
Distribution of Uniform Node Points for the Full and Half-Sweepcases Respectively

A. Derivation of the Half-Sweep Quadrature (HSQ) Schemes
Afore-mentioned, numerical approaches were used widely to solve LFIDEs than the analytical methods [14]. Therefore, quadrature schemes are applied to discretize the LFIDEs to form approximation of system of linear equations. Generally, quadrature formulas can be expressed as follows

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y) \tag{3}$$

where $t_j(j = 0, 1, ..., n)$ are the abscissas of the partition points of the integration interval $[a, b]$. $A_j(j = 0, 1, ..., n)$ are numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of (3). In formulating the full- and half-sweep approximation equations for (1), further discussion will be restricted onto quadrature methods, which is based on interpolation formulas with equally spaced data. Numerical coefficients A_j represented for following relation namely RT and RS schemes respectively.

$$A_j = \begin{cases} \frac{1}{2}ph, & j = 0,\ n \\ ph, & \text{otherwise} \end{cases} \tag{4}$$

$$A_j = \begin{cases} \frac{1}{3}ph, & j = 0, n \\ \frac{4}{3}ph, & j = p,\ 3p,\ 5p,\ \cdots,\ n-p \\ \frac{2}{3}ph, & \text{otherwise} \end{cases} \tag{5}$$

where the constants step size h is defined as

$$h = \frac{b-a}{n} \tag{6}$$

n is the number of step size in the interval $[a, b]$ and then consider the discrete set of points be given as $x_i = a + ih$. The value of p which is corresponds to 1 and 2, represents the full- and half-sweep cases respectively.

B. Derivation of the Half-Sweep Finite Difference (HSFD) Schemes

In solving first order LFIDEs, differential part will be approximated by second order accuracy of central difference scheme given by

$$y'\left(x_i\right) = \frac{y\left(x_{i+1}\right) - y\left(x_{i-1}\right)}{2h} + O\left(h^2\right) \tag{7}$$

for $i = 1, 2, ..., n-1$. However, at the point x_n, second order accuracy of backward difference, which is derived from the Taylor series expansion given as

$$y'\left(x_n\right) = \frac{3y\left(x_n\right) - 4y\left(x_{n-1}\right) + y(x_{n-2})}{2h} + O\left(h^2\right) \tag{8}$$

is considered. For solving second order LFIDEs, the second derivative of central difference scheme is used as follows

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + O(h^2) \tag{9}$$

where h is step size of interval between nodes as mentioned in (6). Equations (7), (8) and (9) consists the same order of the truncation error where as it mostly under our control to choose number of terms from the expansion of Taylor series. In that, in order to obtain the finite grid work network for formulation of the full- and half-sweep central difference approximation quations over (1), the (7), (8) and (9) can be rewritten in general form as follows:

For $i = 1p, 2p, 3p, ..., n-p$,

$$y'\left(x_i\right) \cong \frac{y\left(x_{i+p}\right) - y\left(x_{i-p}\right)}{2ph}, \tag{10}$$

and at $i = n$,

$$y'\left(x_n\right) \cong \frac{3y\left(x_n\right) - 4y\left(x_{n-p}\right) + y(x_{n-2p})}{2ph}, \tag{11}$$

for discretize differential term in first order LFIDEs. Meanwhile to discretize second order LFIDEs,the second order central difference schemes can be derived as

$$y''(x_i) = \frac{y(x_{i+p}) - 2y(x_i) + y(x_{i-p})}{(ph)^2} + O(h^2) \tag{12}$$

for $i = 1p, 2p, 3p, ..., n-p$. Where the value of p, which corresponds to 1 and 2, represents the full- and half-sweep respectively. In order to generate system of linear equations for first order LFIDEs, Equations (3), (10) and (11) will be substituted into (1). The generated linear system either by the full-or half-sweep approximation equation can be easily shown as

$$Ey_n = f. \tag{13}$$

where,

$$E = \begin{bmatrix} a_{p,p} & b_{p,2p} & d_{p,3p} & \cdots & d_{p,n-2p} & d_{p,n-p} & d_{p,n} \\ c_{2p,p} & a_{2p,2p} & b_{2p,3p} & \cdots & d_{2p,n-2p} & d_{2p,n-p} & d_{2p,n} \\ d_{3p,p} & c_{3p,2p} & a_{3p,3p} & \cdots & d_{3p,n-2p} & d_{3p,n-p} & d_{3p,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{n-2p,p} & d_{n-2p,2p} & d_{n-2p,3p} & \cdots & a_{N-2p,n-2p} & b_{n-2p,n-p} & d_{n-2p,n} \\ d_{n-p,p} & d_{n-p,2p} & d_{n-p,3p} & \cdots & c_{n-p,n-2p} & a_{n-p,n-p} & b_{n-p,n} \\ d_{n,p} & d_{n,2p} & d_{n,3p} & \cdots & b_{n,2p} & e_{n,n-p} & \hbar_{n,n} \end{bmatrix}_{\left(\frac{n}{P}\right)\times\left(\frac{n}{P}\right)}$$

where,

$$\begin{aligned}
a_{i,i} &= -2hP_i - 2hA_iK_{i,i}, \\
b_{i,j} &= 1 - 2hA_jK_{i,j} \\
c_{i,j} &= -1 - 2hA_jK_{i,j} \\
d_{i,j} &= -2hA_jK_{i,j} \\
e_{i,j} &= -4 - 2hA_jK_{i,j} \\
\hbar_{i,i} &= -3 - 2hP_i - 2hA_iK_{i,i}
\end{aligned}$$

$$f = \begin{bmatrix} 2hg_p + (2hA_pK_{p,0} + 1)\ y_0 \\ 2hg_{2p} + (2hA_pK_{2p,0})\ y_0 \\ 2hg_{3p} + (2hA_pK_{3p,0})\ y_0 \\ \vdots \\ 2hg_{n-2p} + (2hA_pK_{n-2p,0})\ y_0 \\ 2hg_{n-p} + (2hA_pK_{n-p,0})\ y_0 \\ 2hg_N + (2hA_pK_{n,0})\ y_0 \end{bmatrix}$$

and

$$y_n = \begin{bmatrix} y_n(x_p) \\ y_n(x_{2p}) \\ y_n(x_{3p}) \\ \vdots \\ y_n(x_{n-2p}) \\ y_n(x_{n-p}) \\ y_n(x_n) \end{bmatrix}$$

where E is a dense nonsymmetric coefficient matrix, f is given function and y_n is unknown function to be determined. Nevertheless, in solving first order LFIDEs, the combination of discretization schemes of RT-CD and RS-CD leads to the non-positive definite coefficient matrices. Therefore, for GS iterative methods, the generated linear systems will be modified by multiplying the coefficient matrices with its transpose in order to strengthen the diagonal elements. Thus, the new linear system (13) can be simplified as

$$E^* y_n = f^* \tag{14}$$

where $E^* = E^T E$ and $f^* = E^T f$

Now the linear system (14) can be solved iteratively via FSGS and HSGS iterative methods. For second order LFIDEs, Equations (3) and (12) will be substituted into (1) to generate linear system either by the full-or half-sweep approximation equation easily shown as follows

$$Gy_n = \ell \tag{15}$$

where

$$G = \begin{bmatrix} \sigma_{p,p} & \varsigma_{p,2p} & \tau_{p,3p} & \cdots & \tau_{p,n-3p} & \tau_{p,n-2p} & \tau_{p,n-p} \\ \varsigma_{2p,p} & \sigma_{2p,2p} & \varsigma_{2p,3p} & \cdots & \tau_{2p,n-3p} & \tau_{2p,n-2p} & \tau_{2p,n-p} \\ \tau_{3p,p} & \varsigma_{3p,2p} & \sigma_{3p,3p} & \cdots & \tau_{3p,n-3p} & \tau_{3p,n-2p} & \tau_{3p,n-p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \tau_{n-3p,p} & \tau_{n-3p,2p} & \tau_{n-3p,3p} & \cdots & \sigma_{n-3p,n-3p} & \varsigma_{n-3p,n-2p} & \tau_{n-2p,n-p} \\ \tau_{n-2p,p} & \tau_{n-2p,2p} & \tau_{n-2p,3p} & \cdots & \varsigma_{n-2p,n-3p} & \sigma_{n-2p,n-2p} & \varsigma_{n-p,n} \\ \tau_{n-p,p} & \tau_{n-p,2p} & \tau_{n-p,3p} & \cdots & \tau_{n-p,n-3p} & \varsigma_{n-p,n-2p} & \sigma_{n-p,n-p} \end{bmatrix}_{(\frac{n}{P}-1)\times(\frac{n}{P}-1)}$$

where,

$$\sigma_{i,i} = -2 - h^2 P_i - h^2 A_i K_{i,i}$$
$$\varsigma_{i,j} = 1 - h^2 A_j K_{i,j}$$
$$\tau_{i,j} = -h^2 A_j K_{i,j}$$

$$\ell = \begin{bmatrix} h^2 g_p + \left(1 + h^2 A_p K_{p,0}\right) y_0 + \left(h^2 A_n K_{p,n}\right) y_n \\ h^2 g_{2p} + \left(h^2 A_p K_{2p,0}\right) y_0 + \left(h^2 A_n K_{2p,n}\right) y_n \\ h^2 g_{3p} + \left(h^2 A_p K_{3p,0}\right) y_0 + \left(h^2 A_n K_{3p,n}\right) y_n \\ \vdots \\ h^2 g_{n-3p} + \left(h^2 A_p K_{n-3,0}\right) y_0 + \left(h^2 A_n K_{n-3p,n}\right) y_n \\ h^2 g_{n-2p} + \left(h^2 A_p K_{n-2,0}\right) y_0 + \left(h^2 A_n K_{n-2p,0}\right) y_n \\ h^2 g_{n-p} + \left(h^2 A_p K_{n-p,0}\right) y_0 + \left(-1 + h^2 A_n K_{n-p,0}\right) y_n \end{bmatrix}$$

and

$$y_n = \begin{bmatrix} y_n(x_p) \\ y_n(x_{2p}) \\ y_n(x_{3p}) \\ \vdots \\ y_n(x_{n-2p}) \\ y_n(x_{n-p}) \\ y_n(x_n) \end{bmatrix}$$

where G is a positive definite, nonsymmetric coefficient matrix, ℓ is given function, and y_n is unknown function to be determined.

3. FORMULATION OF FSGS AND HSGS ITERATIVE METHODS

In this section, generated system of linear equation of first order and second order LFIDEs as shown in (14) and (15) will be solved by using FSGS and HSGS iterative methods. For first order LFIDEs, the coefficient matrix, E^* be decomposed into

$$E^* = D - L - U \tag{16}$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. In fact, the both iterative methods attempt to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the vector y_n. Iterations for both methods continue until the solution is within a predetermined acceptable bound on the error.

By determining values of matrices D, $-L$ and $-U$ as stated in (16), the proposed algorithm for FSGS and HSGS iterative methods to solve (1) generally can be described in Algorithm 1.

Algorithm 1: FSGS and HSGS algorithms

(i) Initializing all the parameters. Set $k = 0$.

(ii) For $i = 1p, 2p, ..., n-p$ and $j = 1, p, 2p, ..., n-p, n$, calculate

$$y_i^{(k+1)} = \frac{1}{E^*_{i,i}} \left(f_i^* - \sum_{j=p,2p}^{i-p} E^*_{i,j} y_j^{(k+1)} - \sum_{j=i+p,i+2p}^{n} E^*_{i,j} {y_j}^{(k)} \right)$$

(iii) Convergence test. If there error of tolerance $\left\|y_i^{(k+1)} - y_i^{(k)}\right\| \leq \varepsilon = 10^{-10}$ is satisfied, then algorithms stop.

(iv) Else, set $k = k + 1$ and go to step (ii).

For second order LFIDEs, the general algorithm for FSGS and HSGS iterative methods to solve (1) commonly can be described in Algorithm 2. In second order LFIDEs, the coefficient matrix, G be decomposed into

$$G = D - L - U \tag{17}$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively.

Algorithm 2: FSGS and HSGS algorithms

(i) Initializing all the parameters. Set $k = 0$.

(ii) For $i = 1p, 2p, ..., n - p$ and $j = 1, p, 2p, ..., n - p$, calculate

$$y_i^{(k+1)} = \frac{1}{G_{i,i}} \left(\ell_i - \sum_{j=p,2p}^{i-p} G_{i,j} y_j^{(k+1)} - \sum_{j=i+p,i+2p}^{n-1} G_{i,j} {y_j}^{(k)} \right)$$

(iii) Convergence test. If there error of tolerance $\left\|y_i^{(k+1)} - y_i^{(k)}\right\| \leq \varepsilon = 10^{-10}$ is satisfied, then algorithms stop.

(iv) Else, set $k = k + 1$ and go to step (ii).

4. NUMERICAL EXPERIMENTS

In order to evaluate the performances of the HSGS iterative methods described in the previous section, several numerical experiments were carried out. In this paper, we will only consider well posed equations and the case where $a = 0$ and $b = 1$.

Problem 1 [15]. Consider the first order LFIDE

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xy(t)dt \quad 0 < x \leq 1 \tag{18}$$

with boundary condition $y(0) = 0$ and exact solution is $y(x) = x$.

Problem 2 [16]. Consider the second order LFIDE

$$y''(x) = x - 2 + 60 \int_0^1 (x - t)y(t)dt \quad 0 < x < 1 \tag{19}$$

with boundary conditions $y(0) = 0$ and $y(1) = 0$, with exact solution given as $y(x) = x$.

There are three parameters considered in numerical comparison that number of iterations, execution time and maximum absolute error. As a benchmark, the standard or full-sweep Gauss-Seidel (FSGS) method acts as the control of comparison of numerical results. Throughout the simulations, the convergence test considered the tolerance error of $\varepsilon = 10^{-10}$. All the experimental results have been recorded in Table 3 and 4. Based on the results, noticed that the number of iterations and execution time for each mesh size significantly reduced by implementing half-sweep iteration concept. The percentage reduction analysis of number of iterations and execution time from Table 3 and Table 4 are explained in numerically in conclusion. The computational complexity for iterative methods is measured by analysing number of arithmetic operation involved per iteration. Therefore, an estimation total of computational work was determined for FSGS and HSGS iterative methods. Based on Algorithms 1 and 2, it can be calculated that there are $\left(\frac{n}{p}-1\right)$ additions/subtractions (ADD/SUB) and $\left(\frac{n}{p}+1\right)$ multiplications/divisions (MUL/DIV) involved in computing a value for each node point in the solution domain for LFIDEs. The total numbers of arithmetic operations per iteration for the FSGS, and HSGS iterative methods for solving Equations (14) and (15) have been summarized in Table 1 and 2.

5. CONCLUSIONS

In this work, we implemented half-sweep iterative concept on quadrature-difference schemes and GS iterative methodto solve LFIDEs. Based on the numerical results in Table 3 and Table 4, the half-sweep RT-CD and RS-CD with HSGS iterative method have decreased the number of iterations and execution time approximately 62.81%-74.23% and 85.56%-96.93% respectively for problem 1 and 73.21%-76.25% and 46.71%-83.05% respectively for problem 2. Based on Table 1 and Table 2 the accuracy of numerical solutions for RS-CD combination is more accurate than the RT-CD scheme. Overall, the numerical results have shown that the HSGS method is more superior in term of number of iterations and the execution time than standard method.

REFERENCES

[1] Rashed, M. T. (2003). Lagrange interpolation to compute the numerical solutions differential and integro-differential equations. *Applied Mathematics and Computation, 151*, 869-878.

[2] Yalcinbas, S. (2002). Taylor polynomial solution of nonlinear VolterraCFredholm integral equations. *Applied Mathematics and Computation, 127*, 195-206.

[3] Maleknejad, K., & Mirzaee, F. (2006). Numerical solution of integro-differential equations by using rationalized Haar functions method. *Kybernetes Int. J. Syst. Math., 35*, 735-1744.

[4] Hosseini, S. M., & Shahmorad, S. (2003). Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases. *Appl. Math. Model, 27*, 145-154.

[5] Aruchunan, E., & Sulaiman, J. (2011). Half-sweep conjugate gradient method for solving first order linear fredholm integro-differential equations. *Australian Journal of Basic and Applied Sciences, 5*(3), 38-43.

[6] Aruchunan, E., & Sulaiman, J. (2010). Numerical solution of second order linear fredholm integro-differential equation using generalized minimal residual (GMRES) method. *American Journal of Applied Sciences, 7*(6), 780-783.

[7] Hanggelbroek, R. J., Kaper, H. K., & Leaf, G. K. (1977). Collocation methods for integro-differential equations. *SIAM Journal of Numerical Analysis, 14*(3), 277-390.

[8] Vainikko, G. M. (1971). Convergence of quadrature-difference methods for

linear integro-differential equations. *USSR Computational Mathematics and Mathematical Physics, 11*(3), 292-301.

[9] Abdullah, A. R. (1991). The four point explicit decoupled group (EDG) method: a fast poisson solver. *International Journal of Computer Mathematics, 38*, 61-70.

[10] Hasan, M. K., Othman, M., Abbas, Z., Sulaiman, J., & Ahmad, F. (2007). *Parallel solution of high speed low order FDTD on 2D free space wave propagation.* Lecture Notes in Computer Science LNCS 4706, 13-24.

[11] Sulaiman, J., Hasan, M. K., & Othman, M. (2004). *The half-sweep iterative alternating decomposition explicit (HSIADE) method for diffusion equation.* Lectures Notes in Computer Science LNCS 3314, 57-63.

[12] Aruchunan, E., & Sulaiman, J. (2012). Application of the central-difference scheme with half-sweep gauss-seidel method for solving first order linear fredholm integro-differential equations. *International Journal of Engineering and Applied Sciences, 6*, 296-300.

[13] Sulaiman, J., Hasan, M. K., & Othman, M. (2007). *Red-black half-sweep iterative method using triangle finite element approximation for 2D poisson equations.* Lectures Notes in Computer Science LNCS 4487, 326-333.

[14] Sweilam, N. H. (2007). Fourth order integro-differential equations using variational iteration method. *Comput. Math. Appl., 54*, 1086-1091.

[15] Darania, P., & Ebadia, A. (2007). A method for numerical solution of tintegro-differetial equations. *Applied Mathematics and Computation, 188*, 657-668.

[16] Delves, L. M., & Mohamed, J. L. (1985). *Computational methods for integral equations.* London: Clarendon Press, Oxford.

APPENDIX

Table 1
Number of Arithmetic Operations per Iterations Involved in a Node Point Based on FSGS and HSGS Method for First Order Linear FIDE

	Arithmetic operations per node	
	ADD/SUB	MUL/DIV
FSGS	$n(n-1)$	$n(n+1)$
HSGS	$\frac{n}{2}(\frac{n}{2}-1)$	$\frac{n}{2}(\frac{n}{2}+1)$

Table 2
Number of Arithmetic Operations per Iterations Involved in a Node Point Based on FSGS and HSGS Method for Second Order Linear FIDE

	Arithmetic operations per node	
	ADD/SUB	MUL/DIV
FSGS	$(n-1)^2$	n^2-1
HSGS	$(\frac{n}{2}-1)^2$	$\frac{n^2}{4}-1$

Table 3
Comparison of Number of Iterations, Execution Time (Seconds) and Maximum Absolute Error by Using RT-CD and RS-CD Discretization Schemes with Iterative Methods for Problem 1

Mesh size	**Schemes & methods**	**Number of iteration**		**Execution time**		**Maximum absolute error**	
		FSGS	**HSGS**	**FSGS**	**HSGS**	**FSGS**	**HSGS**
24	CD-RT	7814	2907	5.93	0.89	1.653e-4	6.620e-4
	CD-RS	7964	2962	6.53	0.96	4.767e-8	1.654e-8
48	CD-RT	23006	7814	108.77	7.40	4.119e-5	1.653e-4
	CD-RS	23428	7964	120.52	6.77	1.518e-8	4.767e-8
72	CD-RT	45002	14536	684.15	40.01	1.807e-5	4.119e-5
	CD-RS	45756	14810	730.15	35.73	3.122e-8	9.269e-8
96	CD-RT	73430	23006	2469.69	142.92	9.828e-6	4.119e-5
	CD-RS	74614	23428	2753.81	124.61	5.291e-8	1.518e-8
120	CD-RT	107988	33174	10347.03	429.21	3.506e-6	2.623e-5
	CD-RS	109685	33759	10460.84	328.75	1.233e-8	2.249e-8

Table 4
Comparison of Number of Iterations, Execution Time (Seconds) and Maximum Absolute Error by USIng RT-CD and RS-CD Discretization Schemes with Iterative Methods for Problem 2

Mesh size	Schemes & methods	Number of iteration		Execution time		Maximum absolute error	
		FSGS	HSGS	FSGS	HSGS	FSGS	HSGS
24	CD-RT	502	130	0.22	0.05	4.656e-4	2.885e-3
	CD-RS	497	134	0.29	0.08	2.414e-6	3.246e-6
48	CD-RT	2101	502	0.49	0.26	1.164e-4	7.912e-4
	CD-RS	2097	497	0.50	0.34	1.389e-8	2.414e-7
72	CD-RT	4628	1183	1.17	0.34	5.172e-5	3.627e-4
	CD-RS	4625	1179	1.18	0.36	8.892e-8	3.777e-8
96	CD-RT	8034	2101	2.30	0.49	2.905e-5	2.072e-4
	CD-RS	8032	2097	2.34	0.50	4.345e-8	2.136e-8
120	CD-RT	12278	3251	4.37	0.72	1.854e-5	1.338e-4
	CD-RS	12276	3249	4.10	0.81	7.967e-8	1.371e-8