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The Convergence of Filters on Quantales and Its Hausdorffness

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Abstract: In this paper, we introduce the definition of conergence of filters on quantale. Some characterizations of finit completeness and compactness of quantales are studied. At last, the Hausdorff property in quantale using the converence structure is presented.

Key words: Quantale; Point; Ideal; Congerencen of filter; Hausdorff property

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1. INTRODUCTION

Quantale was proposed by C. J. Mulvey in 1986 for studying the foundations of quantum logic and for studying non-commutation C*-algebras. The term quantale was coined as a combination of "quantum logic" and "locale" by C. J. Mulvey in [1]. The systematic introduction of quantale theory came from the book [2], which written by K. I. Rosenthal in 1990. Since quantale theory provides a powerful tool in studying noncommutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory [3], the ideal theory of commutative

ring [4], linear logic [5] and so on. So, the quantale theory has aroused great interests of many scholar and experts, a great deal of new ideas and applications of quantale have been proposed in twenty years [6–32]. The study of the paper [32] was introduced convergence and cauchy structure on locales, and given characterization of Hausdorff property in locale by uniqueness of limit. In paper [33], a new definition of convergence of filters on locale was introduced. Some characteriztions of compactness and description of cauchy completeness are obtained.

Quantale can be regard as the non-commutative generalization of frame. The natural question arising in this context is the following: How to introduce convergence structure, separation Axioms, and another topological properties in quantales? In the paper, we have introduced a new definition of convergence of filters on quantales. We obtained a series of results of topological properties of quantales, which generalize some results of locales.

2. PRELIMINARIES

Definition 2.1 [3] A *quantale* is a complete lattice Q with an associative binary operation "&" satisfying:

$$a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i) \quad and \quad (\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a),$$

for all $a, b_i \in Q$, where I is a set, 0 and 1 denote the smallest element and the greatest element of Q, respectively.

A quantale Q is said to be *unital* if there is an element $u \in Q$ such that $u\&a = a\&u = a$ for all $a \in Q$.

Definition 2.2 [3] Let Q be a quantale and $a \in Q$.

(1) a is *right-sided* if and only if $a\&1 \leq a$.

(2) a is *left-sided* if and only if $1\&a \leq a$.

(3) a is *two-sided* if and only if a is both right and left side.

(4) a is *idempotent* if and only if $a\&a = a$.

Definition 2.3 [3] A quantale Q is *commutative* if and only if $a\&b = b\&a$ for all $a, b \in Q$.

Definition 2.4 [3] Let Q and P be quantales. A function $f : Q \longrightarrow P$ is a homomorphism of quantale if f preserves arbitrary sups and the operation "&". If Q and P are unital, then f is unital homomorphism if in addition to being a homomorphism, it satisfies $f(u_Q) = u_P$, where u_Q and u_P are units of Q and P, respectively.

Definition 2.5 Let Q be a quantales. A non-empty subset I of Q said to be *ideal* if it satisfies the following conditions:

(i) $1 \notin I$;

(ii) $a \vee b \in I$ for all $a, b \in I$;

(iii) $x\&r \in I$ and $r\&x \in I$ for all $x \in Q$, $r \in I$;

(iv) I is a down-set.

The set of all ideals of Q is denoted by $Id(Q)$. Let I be a ideal of Q, then I is said to be prime if $a, b \in I$ and $a\&b \in I$ imply $a \in I$ or $b \in I$. The set of prime ideal of Q is denoted by $PId(Q)$.

Definition 2.6 Let Q be a quantales. A non-empty subset F of Q said to be *filter* if it satisfies the following conditions:

(i) $0 \notin F$;

(ii) $a \in F,\ b \in Q,\ a \leq b$ imply $b \in F$;

(iii) $a, b \in Q$ imply $a\&b \in F$.

The set of all filters of Q is denoted by $Fil(Q)$. The filter F of Q is said to be *prime* if $a \vee b \in F$ imply $a \in F$ or $b \in F$. The set of all prime filters of Q is denoted by $PFil(Q)$.

Definition 2.7 Let Q be a quantale, $2 = \{0, 1\}$ is a quantale by taking $x\&y = 0$ with $x = 0$ or $y = 0$ and $1\&1 = 1$. A *point* of Q is a onto homomorphism of quantale from Q to 2. We shall denote the all points of Q by $Pt(Q)$.

Definition 2.8 Let Q be a quantale, $I \in Id(Q)$, $p \in Pt(Q)$.

(1) The point p is called a *cluster point* of I iff $I \subseteq Pt(Q)$.

(2) Ideal I is *converges* to p iff p is a cluster point of I and $x^T \in I$ for all $x \in p^{-1}(1)$.

(3) The point p is a *strongly limit point* of I if p is a cluster point of I and $\forall\ x \in p^{-1}(1)$, there exists $a \in I$ such that $a \vee x = 1$.

3. FILTER-CONVERGENCE IN QUANTALES

Definition 3.1 Let Q be a quantale. A set $A \subseteq Q$ is called cover if $\vee A = 1$.

Definition 3.2 Let Q be a quantale. A filter F in O is said to be weak convergence if $\vee\{x^T \mid x \in F\} \neq 1$. F is called convergence if for each cover A of Q such that $F \cap A \neq \emptyset$.

Example 3.3 (1) Let $Q = \{0, a, b, 1\}$ be a quantale. The order relation and "&" on Q satisfies the following Figure 1 and Diagram 1.

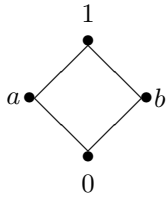

Figure 1

&	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Diagram 1

It is easy to show that $F_1 = \{a, 1\},\ F_2 = \{b, 1\},\ F_3 = \{1\}$ are the filters of Q. since

$$0^T = 1,\ a^T = b,\ b^T = a,\ 1^T = 0,$$

then

$$\vee\{a^T, 1^T\} = b \neq 1,\ \vee\{b^T, 1^T\} = a \neq 1,\ \vee\{1^T\} = 0 \neq 1.$$

Hence $F_1,\ F_2,\ F_3$ are weak convergence filters of Q.

We can easy to prove that

$$A_1 = \{1\},\ A_2 = \{a, 1\},\ A_3 = \{b, 1\},\ A_4 = \{0, a, b, 1\},\ A_5 = \{a, b, 1\}$$

are all the covers of Q. By

$$F_i \cap A_j \neq \emptyset,\ i = 1, 2, 3,\ j = 1, 2, 3, 4, 5.$$

Thus $F_1,\ F_2,\ F_3$ are convergence filters of Q.

(2) Let $Q = ([0,\ 1], \wedge)$ be a quantale, $x \in Q$, but $x \neq 0$. It is easy to show that $F_x = \uparrow x$ is a filter of Q. $\forall\ y \in F_x$. By $y^T = 0$, we know that

$$\vee\{y^T \mid y \in F_x\} = 0 \neq 1.$$

Thus F_x is not only a weal convergence filter, but also a convergence filter.

(3) Let $Q = \{0, a, b, c, d, 1\}$ be a quantale, the relation relation and "&" given by following Figure 2 and Diagram 2.

We can show that $F_1 = \{b, c, 1\}$ and $F_2 = \{a, 1\}$ are the filters of Q, they are not only weak convergence filters, but also convergence filters.

It is easy verify that every convergence filters is weak convergence filters. Is there a weak convergence filter which is a convergence filter? Next, we will discuss this question.

CN: Let X be a nonempty subset of Q, for any nonempty finite subset F of X such that $x_1 \& x_1 \& \cdots\cdots \& x_n \neq 0$, where $x_i \in F$, $i = 1, 2 \cdots, n$.

Lemma 3.4 Let Q be a commutative and idempotent quantale, F be a maximal filter of Q. Then F is the maximal subset satisfies CN.

Proof. Let F' satisfies CN with $F \subseteq F' \subseteq Q$. Put

$$\overline{F} = \uparrow \{x_1 \& x_1 \& \cdots\cdots \& x_n \mid x_i \in F,\ i = 1, 2 \cdots, n,\ n \in N^+\}.$$

Then $\overline{F}$ is filter of Q, and $F' \subseteq \overline{F}$. Thus $F \subseteq \overline{F}$. Since F is the maximal filter of Q, so $F = \overline{F}$, which implies $F = F'$. □

Theorem 3.5 Let Q be a commutative and idempotent quantale, $F \in Fil(Q)$. If Q is the maximal filter, then F is convergence filter iff F is weak convergence filter.

Proof. Let F is a convergence filter of Q, suppose $\vee\{x^T \mid x \in F\} = 1$, i. e., the set $\{x^T \mid x \in F\}$ be a cover of Q. Thus there exists $x \in F$ such that $x^T \in F$. Therefore $x \& x^T \in F$, i.e., $0 \in F$, which is a contradiction. Hence F is a weak convergence filter. □

Conversely, let F be a weak convergence filter, and A is any cover of Q. Suppose

$$F \cap A = \emptyset, i.e., \forall\ a \in A,\ a \in Q \setminus F.$$

Since F is a maximal filter. By Lemma 5.4 we know that for any $a \in A$, there is a $a' \in F$ such that $a' \& a = 0$. Thus $a \leq a'^T$. Therefore, $\vee\{x^T \mid x \in F\} \geq \vee A = 1$, which is a contradiction. Hence F be a convergence filter of Q.

Definition 3.6 Let Q be a quantale, n be a nature number, F be a filter of Q. Filter F is said to n-convergence iff for any cover A of Q with $|A| \leq n$, which implies $F \cap A \neq \emptyset$. Filter F is called finite convergence iff for any finite cover A, we have $F \cap A \neq \emptyset$. The quantale Q is called n-completeness iff every n-convergence filter of Q is convergence filter.

Definition 3.7 Let Q be a quantale. A element $a \in Q$ is compact iff for every $S \subseteq Q$ with $a \leq \vee S$, there is a finite subset $F \subseteq S$ with $a \leq \vee F$. Quantale Q is called compacted iff the greatest element 1 is compact.

Theorem 3.8 Let Q be a quantale. Then the following are true:

(1) Q is 1-completeness iff for any filter of Q is convergence;

(2) If $m \leq n$, then n-completeness quantale are m-completeness quantale;

(3) If Q is two sided and $1\&1 = 1$, then Q is compact iff for any finite convergence filter of Q is convergence.

Proof. (1), (2) are clear.

(3) " $\Leftarrow$ " is clear.

" $\Rightarrow$ ". Let $S \subseteq Q$ with $0 \in S$, and for any finite subset $F \subseteq S$ such that $\vee F \neq 1$. By Zorn lemma we know that there exists a maximal subset S' such that $S \subseteq S'$, for any finite subset $F \subseteq S$, $\vee F \neq 1$. Put $F^* = Q \setminus S'$. Next, we will show that F^* is a finite convergence filter of Q.

Firstly, it is obvious that $0 \in S \subseteq S'$, then $0 \in Q \setminus F^*$.

Secondly, $\forall\ a \in F^*$, $b \in Q$. If $a \leq b$, then there exists finite elements $a_1, a_2, \cdots\cdots, a_n \in S'$ such that

$$a \vee a_1 \vee a_2 \vee \cdots\cdots \vee a_n = 1.$$

Thus

$$b \vee a_1 \vee a_2 \vee \cdots\cdots \vee a_n = 1, i.e., b \in F^*.$$

A last, $\forall\ a, b \in F^*$, then there exists finite elements

$$a_1, a_2, \cdots\cdots, a_n \in S',\ b_1, b_2, \cdots\cdots, b_n \in S'$$

such that

$$a \vee a_1 \vee a_2 \vee \cdots\cdots \vee a_n = 1, b \vee b_1 \vee b_2 \vee \cdots\cdots \vee b_n = 1.$$

Thus

$$(a \vee a_1 \vee a_2 \vee \cdots\cdots \vee a_n)\&(b \vee b_1 \vee b_2 \vee \cdots\cdots \vee b_n) = 1\&1 = 1.$$

Since Q is two sides quantale, so

$$\begin{aligned}&(a \vee a_1 \vee a_2 \vee \cdots\cdots \vee a_n)\&(b \vee b_1 \vee b_2 \vee \cdots\cdots \vee b_n)\\ \leq&(a\&b) \vee a_1 \vee a_2 \vee \cdots\cdots \vee b_1 \vee b_2 \vee \cdots\cdots \vee b_n.\end{aligned}$$

Hence

$$(a\&b) \vee a_1 \vee a_2 \vee \cdots\cdots \vee b_1 \vee b_2 \vee \cdots\cdots \vee b_n = 1,$$

which implies $a\&b \in F^*$. Therefore, $F^*$is a filter of Q.

Let A is a cover of Q, then $\vee A = 1$. By S' is a maximal subset, then there exists $a \in A$ such that $a \in F^*$. Thus F^* is a finite convergence filter. Hence F^* is a convergence filter, i. e., $\vee S' \neq 1$. Since $S \subseteq S'$, $\vee S \neq 1$. Therefore Q is compact. □

Definition 3.9 Let Q be a quantale, $j : Q \longrightarrow Q$ is a quantale nuclei. The quotient quantale Q_j is called *retract quotient* of Q if for any cover A of Q_j implies $j^{-1}(A)$ is a cover of Q.

Theorem 3.10 Let Q be a n-completeness quantale, $j : Q \longrightarrow Q$ is a quantale nuclei. Then the following are true:

(1) If F is a n-convergence filter of Q_j, then $j^{-1}(F)$ is a convergence filter of Q;

(2) If Q_j is a retract quotient of Q, then Q_j is a n-completenss quotient;

Proof. (1) Let A be a cover of Q with $|A| \leq n$. We will show that $j^{-1}(F)$ is a filter of Q.

Firstly, $0 \in Q \setminus j^{-1}(F)$ is clear. Otherwise, if $0 \in j^{-1}(F)$, then $j(0) \in F$, which is a contradiction with F is a n-convergence filter. Secondly, if $a \in j^{-1}(F)$ and $a \leq b$, then $j(b) \in F$ by F is a up-set. Hence $b \in j^{-1}(F)$. Therefore, $j^{-1}(F)$ is a up-set. At last, $\forall\ x, y \in j^{-1}(F)$, then $j(x), j(y) \in F$, and $j(x\&y) = j(j(x)\&j(y)) = j(x)\&_j j(y) \in F$. Hence, $x\&y \in j^{-1}(F)$.

We shall show that $j^{-1}(F)$ is a convergence filter of Q. Since $j(A_1) \subseteq Q_j = j(Q)$, and $j : Q \longrightarrow Q_j$ be a surjective homomorphism of quantale, then

$$j(\bigvee^{Q} A_1) = j(1) = 1_{Q_j}.$$

Thus

$$\bigvee^{Q_j} \{j(x) \mid x \in A_1\} = j(\bigvee^{Q} A_1) = 1_{Q_j}.$$

Therefore $\{j(x) \mid x \in A_1\}$ is a cover of Q_j with $\mid j(A_1) \mid \leq n$. By F be a n-convergence filter of Q_j. We know $j(A_1) \cap F \neq \emptyset$, then there exists $x_1 \in A_1$, such that $j(x_1) \in F$, i.e., $x_1 \in j^{-1}(F)$. Thus $j^{-1}(F) \cap A_1 \neq \emptyset$. Hence $j^{-1}(F)$ be a n-convergence filter of Q. Since Q be a n-completeness quantale, then $j^{-1}(F)$ is a convergence filter of Q.

(2) Let F_1 is a n-convergence filter of Q_j, A is a cover of Q_j, i.e., $\bigvee^{Q_j} A = 1_{Q_j}$. Since Q_j be a retract quotient of Q, then $j^{-1}(A) = \{a \in Q \mid j(a) \in A\}$ is a cover of Q. By (1) we know $j^{-1}(F_1)$ is a n-convergence filter of Q. Thus $j^{-1}(A) \cap j^{-1}(F_1) \neq \emptyset$, i.e., there is $x \in Q$ such that $j(x) \in A \cap F_1$. Hence $A \cap F_1 \neq \emptyset$. Thus F_1 is a convergence filter of Q_j. Therefore, Q_j is n-completeness. □

Theorem 3.11 Let Q and P are n-completeness quantales. Then be $Q \times P$ a n-completeness quantales.

Proof. Let F be a n-convergent filter of $Q \times P$ and A be a cover of $Q \times P$, p_1, p_2 are projective from $Q \times P$ to Q and P, respectively. We shall prove that $p_1(F)$ and $p_2(F)$ are n-convergence filters of Q and P, respectively.

It is easy prove that $p_1(F)$ and $p_2(F)$ are filters. Next, we will check that $p_1(F)$ and $p_2(F)$ n-convergence filters.

Let $A_1 = \{a_1,\ a_2, \cdots\cdots,\ a_n\}$ is a finite cover of Q. Define $q_1 : Q \longrightarrow Q \times P$ such that $x \longmapsto (x, 1_P)$. Then $q_1(A_1)$ is a cover of $Q \times P$, and $\mid q_1(A_1) \mid \leq n$ is obvious. Since F is a n-convergence filter of $Q \times P$, then $F \cap q_1(A_1) \neq \emptyset$. Thus there exists $(x_1, x_2) \in F \cap q_1(A_1)$. Hence $x_1 \in A_1 \cap p_1(F) \neq \emptyset$. Therefore $p_1(F)$ is a n-convergent filter of Q. Similarly, we can prove that $p_2(F)$ is a n-convergent filter of P.

Next, we shall prove that F is a convergent filter of $Q \times P$.

Firstly, It is easy prove that $p_1(F)$ and $p_2(F)$ are cover of Q and P, respectively. $p_1(F)$ and $p_2(F)$ are n-convergent filters by proof of above. Since Q and P are n-completeness quantales, then $p_1(F) \cap p_1(A) \neq \emptyset$, $p_2(F) \cap p_2(A) \neq \emptyset$, i. e., there exists $x_0 \in p_1(F) \cap p_1(A)$, $y_0 \in p_2(F) \cap p_2(A)$. Thus $(x_0, y_0) \in F \cap A$. Therefore $Q \times P$ is a completeness quantale. □

Theorem 3.12 Let $\{Q_i\}_{i \in I}$ be a family n-completeness quantales. Then $\prod_{i \in I} Q_i$ is n-completeness quantale.

4. HAUSDORFF PROPERTIES OF QUANTALE

Definition 4.1 A quantale Q is called Hausdorff quantale or T_2 quantale if for any ideal of Q has one limit point at most.

Remark 4.2 Q is a Huasdorff quantale iff for any ideal I of Q, there exist unique prime element r of Q such that $I \subseteq\downarrow r$.

Definition 4.3 Let Q be a quantale, $a, b \in Q$, b is said to be well inside of a if there exist $c \in Q$ with $b\&c = 0$ and $c \vee a = 1$. We shall denote this by $a \preceq b$.

Definition 4.4 A quantale Q is called T_2^* quantale if for any $r \in Pr(Q)$, we have $r = \vee\{x \in Q \mid x \preceq r\}$.

It is easy show that every regular quantale is T_2^* quantale.

Definition 4.5 Let Q be a quantale, Q is said to be T_2^{**} quantale if for any $r_1,\ r_2 \in Pr(Q)$ with $r_1 \neq r_2$, there exists $a, b \in Q$ such that $a \not\leq r_1,\ b \not\leq r_2$ and $a\&b = 0$.

Example 4.6 (1) Let Q be a quantale, the order relation and binary operation & on Q as following Figure 2 and Diagram 2.

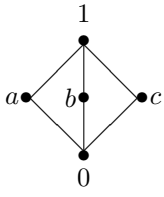

Figure 2

&	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	0	1	1	1	1

Diagram 2

Define $p : Q \longrightarrow 2$, such that

$$p(x) = \begin{cases} 1, & x \in \{a, b, c, 1\}, \\ 0, & x = 0. \end{cases}$$

for all $x \in Q$. We shall show that p is the unique limit point of $I = \{0\}$. $I = \{0\}$ is the only ideal of Q. Thus Q is a T_2 quantale. Since $Pr(Q) = \{0\}$, by Definition 6.5, we know that Q be a T_2^{**} quantale.

(2) Let $Q = \{0, a, b, 1\}$ with $0 \leq a, b \leq 1$, a and b are non-comparability. Then $(Q, \wedge)$ is a quantale.

It is easy check that $Pr(Q) = \{a, b\}$ and

$$0 \preceq 1, a \preceq 1, b \preceq 1, 1 \preceq 1, 0 \preceq a, a \preceq a, 0 \preceq b, b \preceq b, 0 \preceq 0.$$

Hence

$$\vee \Downarrow a = \vee\{0, a\} = a,\ \vee \Downarrow b = \vee\{0, b\} = b.$$

Therefore, Q is T_2^* quantale.

Theorem 4.7 Let Q be a quantale. Then Q is T_2 iff Q is T_2^{**}.

Proof. Let Q is a T_2 quantale. Suppose $r_1, r_2 \in Pr(Q)$ with $r_1 \neq r_2$. $\forall\ a, b \in Q$ such that $a \leq r_1$ or $b \leq r_2$ or $a\&b \neq 0$. Put $I =\downarrow (r_1 \wedge r_2)$. It is easy show that I is a ideal of Q.

Let p_{r_1} and p_{r_2} are points correspond with r_1 and r_2, respectively. Next, we shall prove p_{r_1} and p_{r_2} are limit points of I.

Since

$$r_1 = \vee p_{r_1}^{-1}(0),\ r_2 = \vee p_{r_2}^{-1}(0),\ \forall\ x \in I,\ x \leq r_1 \wedge r_2,$$

then $p_{r_1}(x) \leq p_{r_1}(r_1) = 0$, which implies that $x \in p_{r_1}^{-1}(0)$. Similarly, $x \in p_{r_2}^{-1}(0)$. Thus $I \subseteq p_{r_1}^{-1}(0) \cap p_{r_2}^{-1}(0)$. Therefore, p_{r_1} and p_{r_2} are the cluster points of I.

For any $y \in p_{r_1}^{-1}(1)$. Since $y\&y^T = 0$, but $y \not\leq r_1$, then $y^T \leq r_1$. Similarly, $y^T \leq r_2$. Thus $y^T \leq r_1 \wedge r_2$. Hence, $y^T \in I$. Therefore p_{r_1} is a limit point of I. Similarly, p_{r_2} is a limit point of I. By Q is T_2 quantale and remark 6.2. we know $r_1 = r_2$, which is a contradiction.

Conversely, suppose I is a ideal of Q, p_{r_1} and p_{r_2} are the limit points of I with $p_{r_1} \neq p_{r_2}$, let r_1 and r_2 are prime elements of Q corresspend with p_{r_1} and p_{r_2}, respectively. Then $I \subseteq p_{r_1}^{-1}(0) \cap p_{r_2}^{-1}(0)$. Since $r_1 \neq r_2$, Q is a T^{**} quantale, then there exist $a, b \in Q$ such that

$$a_1 \not\leq r_1, b_1 \not\leq r_2,\ a_1\&b_1 = 0.$$

By $a_1 \not\leq r_1$, we know that $a_1 \in p_{r_1}^{-1}(1)$, but p_{r_1} be a limit point of I. Thus $a_1^T \in I$. Since $a_1\&b_1 = 0$, then $b_1 \leq a_1^T$. Therefore, $b_1 \in I$, which is a contradiction with $b_1 \not\leq r_2$. □

Theorem 4.8 Let Q is a communicative quantale. If Q be a T_2^* quantale, then Q is a T_2^{**} quantale.

Proof. Suppose is not T_2^{**} quantale, then there exists $r_1, r_2 \in pr(Q)$ with $r_1 \neq r_2$, $\forall\ a, b \in Q$, such that $a \leq r_1$ or $b \leq r_2$ or $a\&b \neq 0$. $\forall\ x \in Q$, if $x \preceq r_1$ and $x \not\leq r_2$, then for any $y \in Q$ with $x\&y = 0$, we kown $y \leq r_1$ by the hypothesis. Thus $x^T \vee r_1 \leq r_1 \neq 1$, which is a contradiction with $x \preceq r_1$. This implies that if $x \preceq r_1$, then $x \leq r_2$. Therefore

$$r_1 = \vee\{x \in Q \mid x \preceq r_1\} \leq r_2.$$

Similarly, we know $r_2 \leq r_1$. Thus $r_1 = r_2$, which is a contradiction. □

Definition 4.9 Let Q be a quanatel, $a, b \in Q$ with $a \neq 1$. Define $b \preceq_1 a$ iff $b \leq a$ and $b^T \not\leq a$.

Definition 4.10 A quantale Q is called T_2' quantale if for any $x \in Q$, $x = \vee\{y \in Q \mid y \preceq_1 x\}$.

Example 4.11 (1) Let Q be a quantale, with a binary operation "&" defined by $\forall\ x, y \in Q$, $x\&y = 0$. Let $a, b \in Q$ such that $a \neq 1$ and $b \leq a$. Since

$$b^T = \vee\{c \in Q \mid b\&c = 0\} = 1 \not\leq a,$$

then $b \preceq_1 a$. Hence $\forall\ x \in Q$. If $x \neq 1$, we have

$$\vee\{y \in Q \mid y \preceq_1 x\} = \vee \downarrow x = x.$$

Therefore Q is a T_2' quantale.

(2) Let X be a non-empty set, $P(X)$ is the powerset of X. It is easy check that $(P(X), \cap)$ be a quantale. Let $A, B \in P(X)$ such that $A \neq 1$ and $B \leq A$. Since $B^T = \vee\{C \in P(Q) \mid B \cap C = \emptyset\} = B' \not\subseteq A$, then $(P(X), \cap)$ is a T_2' quantale.

Theorem 4.12 Let Q is a idempotent and right-sided spatial quantale, Q is a T_2' quantale. Then Q is a T_2^* quantale.

Proof. Let Q is a T_2^* quantale, then for any $r \in Pr(Q)$, we have $r = \vee\{a \in Q \mid a \preceq_1 r\}$. $\forall\ a \in Q$, if $a \preceq_1 r$, then $r \vee a^T = 1$. Assume $b = r \vee a^T \neq 1$. Since Q is a T_2^* quantale, then $b = \vee\{d \in Q \mid d \preceq_1 b\}$. Let $d \in Q$ with $d \preceq_1 b$. If $d \not\leq r$, then

$$d^T \leq r \leq bbyd\&d^T = 0 < randr \in Pr(Q),$$

which is a contradiction with $d \preceq_1 b$. Hence $d \leq r$. Therefore

$$\vee\{d \in Q \mid d \preceq_1 b\} \leq r \neq b,$$

which is a contradiction. Hence $r \vee a^T = 1$. Thus

$$a\&a^T = 0,\ a^T \vee r = 1, i.e., a \preceq r.$$

Hence $r = \vee\{x \in Q \mid x \preceq r\}$. This means that Q is a T_2^* quantale. □

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