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# The Coproduct of Unital Quantales

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**Abstract:** In this paper, the definition of the saturated element in quantale is given, Based on the coproduct of monoids, the concrete forms of the coproduct of unital quantales is obatined. Also, some properties of their are discussed.

**Key Words:** Quantale; Monoid; Saturated element; Coproduct; Category

## 1. INTRODUCTION

Quantale was introduced by C.J.Mulvey in 1986 in order to provide a lattice theoretic setting for studying non-commutative C*-algebras[1], as well as a constructive foundations of quantum logic. A quantale-besed (non-commutative logic theoretic) approach to quantum mechanics was developed by Piazza. It is known that quantales are one of the semantics of linear logic. The systematic introduction of quantale theory came from the book [2], which written by K.I.Rosenthal in 1990. Quantale theory provides a powerful tool in studying noncommutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory [3], the ideal theory of commutative ring[4], linear logic [5] and so on. Following C.J.Mulvey, the quantale theory have been studied by many researches [6-21].

Since coproducts is very important concept in many categories, and their coproducts product have been studied systemically. In this paper, the concrete forms of the coproducts of unital quantales is obatined. For notions and concepts concerned, but explained, please refer to [2,22].

## 2. PRELIMINARIES

**Definition 2.1**[2] A *quantale* is a complete lattice $Q$ with an associative binary operation "&" satisfying:

$$a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i) \text{ and } (\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a),$$

for all $a, b_i \in Q$, where $I$ is a set, 0 and 1 denote the smallest element and the greatest element of $Q$, respectively.

A quantale Q is said to be *unital* if there is an element $u \in Q$ such that $u\&a = a\&u = a$ for all $a \in Q$.

**Definition 2.2**[2] Let $Q$ be a quantale and $a \in Q$.

(1) $a$ is $right-sided$ if and only if $a\&1 \leq a$.

(2) $a$ is $left-sided$ if and only if $1\&a \leq a$.

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(3) $a$ is $two - sided$ if and only if $a$ is both right and left side.

(4) $a$ is *idempotent* if and only if $a\&a = a$.

**Definition 2.3**[2] Let $Q$ and $P$ be quantales. A function $f : Q \longrightarrow P$ is a homomorphism of quantale if $f$ preserves arbitrary sups and the operation "&". If $Q$ and $P$ are unital, then $f$ is unital homomorphism if in addition to being a homomorphism, it satisfies $f(u_Q) = u_P$, where $u_Q$ and $u_P$ are units of $Q$ and $P$, respectively.

**Definition 2.4**[2] Let $Q$ be a quantale. A subset $S \subseteq Q$ is a *subquantale* of $Q$ iff the inclusion $S \hookrightarrow Q$ is a quantale homomorphism, i.e., $S$ is closed under sups and "&".

**Definition 2.5**[2] Let $Q$ be a quantale. A quantic nucleus on $Q$ is a closure operator $j$ such that $j(a)\&j(b) \leq j(a\&b)$ for all $a, b \in Q$.

# 3. THE COPRPDUCTS OF MONOIDS

The present section is dedicated to The Coproducts of Monoids. we will show its existence, and some properties are discussed.

Let $\{A_i\}_{i\in I}$ be a family of nonempty monoids with $\bigcap\limits_{i\in I} A_i \neq \emptyset$. A word on $\{A_i\}_{i\in I}$ is a sequence $(a_1 a_2 \cdots\cdots a_k)$ with $a_i \in A_i$, where $k \in N$,and $a_i$,$a_j$ belong to different monoids. Let us $\prod\limits_{i\in I}^{*} A_i$ to denote the set of words on $\{A_i\}_{i\in I}$. Define binary operator " $*$ " : $\forall\ w_1 = a_{i_1} a_{i_2} \cdots\cdots a_{i_s}$, $w_2 = b_{j_1} b_{j_2} \cdots\cdots b_{j_t} \in \Pi^*$, $w_1 * w_2 = a_{i_1} a_{i_2} \cdots\cdots a_{i_s} b_{j_1} b_{j_2} \cdots\cdots b_{j_t}$.

If $a_i$,$b_j$ belong to the same set $A_i$, then $a_i$,$b_j$ designated as letter of $A_i$. It is easy to show that $\prod\limits_{i\in I}^{*} A_i$ is a monoid, the empty word is the unit of $\prod\limits_{i\in I}^{*} A_i$ and is denoted $e^*$.

**Theorem 3.1** *Let $\{A_i\}_{i\in I}$ be a family of nonempty monoids with $\bigcap\limits_{i\in I} A_i \neq \emptyset$, the map $\mu_i : A_i \longrightarrow \prod\limits_{i\in I}^{*} A_i$ ł $x \longmapsto \mu_i(x) = x$. If $g_i : A_i \longrightarrow G$is a family of monoid homomorphisms. Then there exists a unique monoid homomorphismh : $\prod\limits_{i\in I}^{*} A_i \longrightarrow G$ such that $h \circ \mu_i = g_i$for all $i \in I$, and this property determines $\prod\limits_{i\in I}^{*} A_i$ uniquely up to ismorphism. In order words,$((\mu_i)_{i\in I}, \prod\limits_{i\in I}^{*} A_i)$ is a coproduct in the category of monoids.*

*Proof.* At first, we define $h : \prod\limits_{i\in I}^{*} A_i \longrightarrow G$ give by $\forall\ w_1 = a_{i_1} a_{i_2} \cdots\cdots a_{i_s} \in \prod\limits_{i\in I}^{*} A_i$,

$$h(a_{i_1} a_{i_2} \cdots a_{i_s}) = \begin{cases} e_G, & w_1 = \emptyset \\ g_{i_1}(a_{i_1}) \cdot g_{i_2}(a_{i_2}) \cdot \cdots \cdot g_{i_s}(a_{i_s}), & otherwise. \end{cases}$$

Then the map $h$ is well defined and it also preserves unit. Next, we will prove that the map $h$ preserves the operator of $\prod\limits_{i\in I}^{*} A_i$.

For all $w_1 = a_{i_1} a_{i_2} \cdots\cdots a_{i_s} w_2 = b_{j_1} b_{j_2} \cdots\cdots b_{j_t} \in \Pi^*$, then

$$\begin{aligned} h(w_1 * w_2) &= h((a_{i_1} a_{i_2} \cdots a_{i_s}) * (b_{j_1} b_{j_2} \cdots b_{j_t})) \\ &= h(a_{i_1} a_{i_2} \cdots a_{i_s} b_{j_1} b_{j_2} \cdots b_{j_t}) \\ &= g_{i_1}(a_{i_1}) g_{i_2}(a_{i_2}) \cdots g_{i_s}(a_{i_s}) g_{j_1}(b_{j_1}) g_{j_2}(b_{j_2}) \cdots g_{j_t}(b_{j_t}) \\ &= (g_{i_1}(a_{i_1}) g_{i_2}(a_{i_2}) \cdots g_{i_s}(a_{i_s})) \cdot (g_{j_1}(b_{j_1}) g_{j_2}(b_{j_2}) \cdots g_{j_t}(b_{j_t})) \\ &= h(a_{i_1} a_{i_2} \cdots a_{i_s}) \cdot h(b_{j_1} b_{j_2} \cdots b_{j_t}). \end{aligned}$$

It is not hard to see that $h \circ \mu_i = g_i$. At last, we will prove the $h$ is an unique monoid homomorphism.

Now, let $h' : \prod\limits_{i\in I}^{*} A_i \longrightarrow G$ be another monoid homomorphism with $h \circ \mu_i = g_i$. For all $w_1 = a_{i_1} a_{i_2} \cdots\cdots a_{i_s} \in \Pi^*$,we have

$$\begin{aligned}
h'(w_1) &= h'(a_{i_1}a_{i_2}\cdots a_{i_s}) = h'(\mu_{i_1}(a_{i_1})\mu_{i_2}(a_{i_2})\cdots\mu_{i_s}(a_{i_s}))\\
&= h'(\mu_{i_1}(a_{i_1}))\cdot h'(\mu_{i_2}(a_{i_2}))\cdots h'(\mu_{i_s}(a_{i_s}))\\
&= g_{i_1}(a_{i_1})\cdot g_{i_2}(a_{i_2})\cdots g_{i_s}(a_{i_s})\\
&= h(\mu_{i_1}(a_{i_1}))\cdot h(\mu_{i_2}(a_{i_2}))\cdots h(\mu_{i_s}(a_{i_s}))\\
&= h(\mu_{i_1}(a_{i_1})\mu_{i_2}(a_{i_2})\cdots\mu_{i_s}(a_{i_s}))\\
&= h(a_{i_1}a_{i_2}\cdots a_{i_s}) = h(w_1).
\end{aligned}$$

Therefore, $((\mu_i)_{i\in I}, \prod\limits_{i\in I}^{*} A_i)$ is a coproduct of $\{A_i\}_{i\in I}$ in the category of monoids.

# 4. THE SATURATED ELEMENT OF QUANTALES

Let $Q$ is a quantale, $R \subseteq Q\times Q$ is a relation, we want to construt a new quantale with $R$.

An element $s \in Q$ is saturated if $\forall\, a,b,c,d \in Q$,with $(a,b)\in R$, then $a\&c\&d \le s \Longleftrightarrow b\&c\&d \le s$, $c\&a\&d \le s \Longleftrightarrow c\&b\&d \le s$.

Let us $S_R(Q)$ denote the set of all saturated element of $Q$. Obviously, any meet of saturated sets is saturated.

**Theorem 4.1** *Let $Q$ be a quantale with $e$, $R \subseteq Q\times Q$ is a relation on $Q$, then $S_R(Q)$ is a quotient quantale of $Q$.*

*Proof.* Obviously, $S_R(Q)$ is nonempty set, and meet of $S_R(Q)$ is closed.

For all $x, y, z, a, b \in Q,\ s \in S_R(Q)$ with $(a,b)\in R$, we have

$$\begin{aligned}
a\&x\&y \le z \longrightarrow_l s &\Longleftrightarrow a\&x\&y\&z \le s \Longleftrightarrow b\&x\&y\&z \le s \Longleftrightarrow a\&x\&y \le z \longrightarrow_l s,\\
x\&a\&y \le z \longrightarrow_l s &\Longleftrightarrow x\&a\&y\&z \le s \Longleftrightarrow x\&b\&y\&z \le s \Longleftrightarrow x\&b\&y \le z \longrightarrow_l s,\\
a\&x\&y \le z \longrightarrow_r s &\Longleftrightarrow z\&a\&x\&y \le s \Longleftrightarrow z\&b\&x\&y \le s \Longleftrightarrow b\&x\&y \le z \longrightarrow_r s,\\
x\&a\&y \le z \longrightarrow_r s &\Longleftrightarrow z\&x\&a\&y \le s \Longleftrightarrow z\&x\&b\&y \le s \Longleftrightarrow x\&b\&y \le z \longrightarrow_r s.
\end{aligned}$$

Thus$z \longrightarrow_l s,\ z \longrightarrow_r s \in S_R(Q)$.

Therefor $S_R(Q)$ is a quotient quantale of $Q$.

**Theorem 4.2** *Let $Q$ be a quantale with $e$, $R \subseteq Q\times Q$ is a relation on $Q$. Define $v_R : Q \longrightarrow Q$such that $v_R(x) = \bigwedge\{s \in S_R(Q) \mid x \le s\}$ for all $x \in Q$. Then is a quantale nucleus on $Q$.*

**Theorem 4.3** *Let $Q$ be a quantale with $e$, $R \subseteq Q\times Q$ is a relation on $Q$. We have*

(1) $v_R(Q) = \{x \mid v_R(x) = x\} = S_R(Q)$;

(2) *The map $v_R : Q \longrightarrow S_R(Q)$is a quantale surjective homomorphism*;

(3) $v_R(e)$*is a unit of* $S_R(Q)$.

**Theorem 4.4** *Let $Q$ be a quantale with $e$, $a, b \in Q$. We have*

(i) *If $(a,b)\in R$, then $v_R(a) = v_R(b)$*;

(ii) *Let $h : Q \longrightarrow P$ be a unital quantale homomorphism, such that $(a,b)\in R \Longrightarrow h(a) = h(b)$, there exist an unital quantale homomorphism $\overline{h} : S_R(Q) \longrightarrow P$ such that $\overline{h}\circ v_R = h$ and for all $x \in S_R(Q)$.*

*Proof.* (i) If $(a,b)\in R$, then $a = a\&e\&e \le v_R(a)$. since $v_R(a)$ is saturated with $R$, we have $v_R(b) \le v_R(a)$, and by symmetry $v_R(a) = v_R(b)$.

(ii) Let $h : Q \longrightarrow P$ be a untial quantale homomorphism such that $(a, b) \in R \Longrightarrow h(a) = h(b)$. Define $\sigma(x) = \bigvee\{y \in Q \mid h(y) \leq h(x)\}$, obviously $x \leq \sigma(x)$, and$h \circ \sigma(x) = h(x)$.

Let $(a, b) \in R$, $c, d \in Q$, and $a\&c\&d \leq \sigma(x)$, then $h(b\&c\&d) = h(b)\&h(c)\&h(d) = h(a)\&h(c)\&h(d) = h(a\&c\&d) \leq h(\sigma(x)) = h(x)$, hence $b\&c\&d \leq \sigma(x)$. similarly, if $b\&c\&d \leq \sigma(x)$, then $a\&c\&d \leq \sigma(x)$, we have $a\&c\&d \leq \sigma(x) \Longleftrightarrow b\&c\&d \leq \sigma(x)$.

If $c\&a\&d \leq \sigma(x)$, then $h(c\&b\&d) = h(c)\&h(b)\&h(d) = h(c)\&h(a)\&h(d) = h(c\&a\&d) \leq h(\sigma(x)) = h(x)$. Hence $c\&b\&d \leq \sigma(x)$. similarly, if $c\&b\&d \leq \sigma(x)$, then $c\&a\&d \leq \sigma(x)$. Hence $c\&a\&d \leq \sigma(x) \Longleftrightarrow c\&b\&d \leq \sigma(x)$. Now, we can see that $\sigma(x)$ is saturated with $R$.

Since $x \leq \nu_R(x) \leq \sigma(x)$ for all $x \in Q$, then $h(x) \leq (h \circ \nu_R)(x) \leq (h \circ \sigma)(x) = h(x)$, hence$(h \circ \nu_R)(x) = h(x)$. Define $\overline{h} = h \mid^{S_R(Q)}$, we can see that $\overline{h}$ is a unital quantale homomorphism such that $\overline{h} \circ \nu_R = h$. Obviously, $\overline{h}(s) = h(s)$ for all $x \in S_R(Q)$.

# 5. THE COPRODUCT OF THE CATEGORY OF UNITAL QUANTALES

Let **Quant** denote the category of quantale and homomorphism, **Quant** be the category of unital quantales and unital quantale homomorphism,

In [], if $M$ is a monoid, the the power set $P(M)$ is a quantake with a operator &.

**Theorem 5.1** *Let $M$ be a monoid with unit $e$. Define: $A\&B = \{a \cdot b \mid a \in A,\ b \in B\}$ for all $A, B \in P(M)$, then $(P(M), \&, \{e\})$ is a unital quantale.*

Let Mon denote the category of monoids with monoid homomorphism.

Define

$$\begin{aligned} P : \mathbf{Mon} &\longrightarrow \mathbf{UnQuant} \\ M &\longmapsto P(M) \\ f : M \longrightarrow N &\longmapsto P(f) : P(M) \longrightarrow P(N) \\ &\qquad\qquad A \longmapsto \{f(a) \mid a \in A\} \end{aligned}$$

It is easy to prove that $P : \mathbf{Mon} \longrightarrow \mathbf{UnQuant}$ is a functor.

**Theorem 5.2**[] *Functor $P$ : **Mon** $\longrightarrow$ **UnQuant** is left adjoint to the forgetful functor $U$ : **UnQuant** $\longrightarrow$ **Mon**.*

For the convenience of the following statements, the proof of Theorem 5.2 is simply described as follows.

Let $M$ is a monoid, $\mu : M \longrightarrow P(M)$ such that for all $x \in M$. Obviously, the map $\mu$ is a monoid homomorphism.

Assume that $f : M \longrightarrow Q$ is a monoid homomorphism. Defined $\overline{f} : P(M) \longrightarrow Q$ such that $\overline{f}(A) = \bigvee\{f(a) \mid a \in A\}$ for all $A \in P(M)$. It is easy to verify that $f = \overline{f} \circ \mu$ , i. e. the triangle commutes. The uniqueness of $\overline{f}$ is immediate.

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[dr, "f"'] & P(M) \arrow[d, "\overline{f}"] \\ & Q \end{tikzcd}$$

Next, we shall give coproduct of the category unital quantales based on the above discussions.

Let $\{Q_i\}_{i\in I}$ be a family of nonempty unital quantales with $\bigcap\limits_{i\in I} Q_i \neq \emptyset$, by Theorem 4.1 and 4.2, we can see that $((\mu_i)_{i\in I}, \prod\limits_{i\in I}^* Q_i)$ is the coproduct of the category of monid, and $(P(\prod\limits_{i\in I}^* Q_i), \&)$ is a unital quantale. We define a mapping $R = \{((\mu\circ\mu_i)(\bigvee\limits_{j\in J} x_j), \bigvee\limits_{j\in J}(\mu\circ\mu_i(x_j))) \mid i\in I, \{x_j\}_{j\in J}\subseteq Q_i\} \subseteq P(\prod\limits_{i\in I}^* Q_i)\times P(\prod\limits_{i\in I}^* Q_i)$ as follows:

$$\begin{aligned} \nu_R : P(\prod_{i\in I}^* Q_i) &\longrightarrow S_R(P(\prod_{i\in I}^* Q_i)) \\ x &\longmapsto \bigwedge\{s\in S_R(P(\prod_{i\in I}^* Q_i) \mid x\leq s\}. \end{aligned}$$

By theorem 3.2, we have that the map $\nu_R$ is a quantic nucleus, we use $S_R(P(\prod\limits_{i\in I}^* Q_i))$ to denote the class of all saturated elements of$P(\prod\limits_{i\in I}^* Q_i)$ with $R$, and $S_R(P(\prod\limits_{i\in I}^* Q_i))$ is a quantic quotient of $P(\prod\limits_{i\in I}^* Q_i)$ by the above discussions.

The following theorem gives the concrete forms of the coproduct in **UnQuant**.

**Theorem 5.3** *Let $\{Q_i\}_{i\in I}$ be a family of nonempty unital quantales with $\bigcap\limits_{i\in I} Q_i\neq\emptyset$, then $(l_i, S_R(P(\prod\limits_{i\in I}^* Q_i)))$ is a coproduct of $\{Q_i\}_{i\in I}$ in* **UnQuant**, *where $l_i = \nu_R\circ\mu\circ\mu_i$, $\nu_R, \mu, \mu_i$ are some unital quantale homomorphism, $h$ is the monoid homomorphism obtained by theorem* 2.1, *$h'$ is the unital quantale homomrophism form theroem* 4.2.

*Proof.* (1) For all $x, y\in Q$, then $l_i(x\&_i y) = (\nu_R\circ\mu\circ\mu_i)(x\&_i y) = (\nu_R\circ\mu)(x\&_i y) = \nu_R(\{x\&_i y\}) = \nu_R(\{x\}\&\{y\}) = \nu_R(\{x\})\&\nu_R(\{y\}) = (\nu_R\circ\mu)(x)\&(\nu_R\circ\mu)(y) = (\nu_R\circ\mu\circ\mu_i)(x)\&(\nu_R\circ\mu\circ\mu_i)(y) = l_i(x)\&l_i(y)$.

For all $\{x_k\}_{k\in K}\subseteq Q_i$, Since $((\mu\circ\mu_i)(\bigvee\limits_{k\in K} x_k), \bigvee\limits_{k\in K}(\mu\circ\mu_i(x_k)))\in R$, by theorem 3.4, we can see that

$$\nu_R(\mu\circ\mu_i)(\bigvee_{k\in K} x_k) = \nu_R((\bigvee_{k\in K}\mu\circ\mu_i)(x_k)).$$

Hence $l_i(\bigvee\limits_{k\in K} x_k) = (\nu_R\circ\mu\circ\mu_i)(\bigvee\limits_{k\in K} x_k) = \nu_R((\bigvee\limits_{k\in K}\mu\circ\mu_i)(x_k)) = \bigvee\limits_{k\in K}((\nu_R\circ\mu\circ\mu_i)(x_k)) = \bigvee\limits_{k\in K} l_i(x_k)$.

(3) Let $e_i$ be the unit of $Q_i$, then $l_i(e_i) = \nu_R\circ\mu\circ\mu_i(e_i) = \nu_R(\mu(e^*)) = \nu_R(e^*)$. By theorem 3.3(iii), we can show that $\nu_R(e^*)$ be a unit on $S_R(P(\prod_{i\in I}^* Q_i)$. Hence $l_i$ preserves the unit element. By (1),(2),(3), we can see that the mapping $l_i$ is a unital quantale homomorphism.

By theorem 4.2, there is a unital quantale homomorphism $h'$ such that $h'\circ\mu_i = h$ for all $i\in I$.

Since $h'(\bigvee\limits_{k\in K}\mu\circ\mu_i(x_k)) = \bigvee\limits_{k\in K}((h'\circ\mu\circ\mu_i)(x_k)) = \bigvee\limits_{k\in K}((h\circ\mu_i)(x_k)) = \bigvee\limits_{k\in K} g_i(x_k) = g_i(\bigvee\limits_{k\in K} x_k) = h\circ\mu_i(\bigvee\limits_{k\in K} x_k) = h'\circ\mu\circ\mu_i(\bigvee\limits_{k\in K} x_k)$, and be theorem 3.4(i), we can see that there exist a unique unital quantale homomorphism $h''$ satisfy $h' = h''\circ\nu_R$ and $h''\circ l_i = h''\circ(\nu_R\circ\mu\circ\mu_i)$
$= h'\circ\mu\circ\mu_i = h\circ\mu_i = g_i$.

Therefore $(l_i, S_R(P(\prod\limits_{i\in I}^* Q_i)))$ is a coproduct of $\{Q_i\}_{i\in I}$ in **UnQuant**.

$$\begin{tikzcd} Q_i \arrow[r, "\mu_i"] \arrow[d, "g_i"'] & \prod\limits_{i\in I}^* Q_i \arrow[r, "\mu"] \arrow[d, "h"] & P(\prod\limits_{i\in I}^* Q_i) \arrow[r, "\nu_R"] \arrow[d, "h'"'] & S_R(P(\prod\limits_{i\in I}^* Q_i) \arrow[d, "h''"'] \\ Q \arrow[r, equal] & Q \arrow[r, equal] & Q \arrow[r, equal] & Q \end{tikzcd}$$

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