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On the Distribution of Characteristic Roots of Delay Differential Equations

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Abstract
In this article we analyze the distribution in the complex plane of the roots of the characteristic functions for first and second order linear delay differential equations. We prove necessary and sufficient conditions which are satisfied with respect to the variation of the coefficients in order the characteristic roots have negative real parts.

Key words
Delay differential equations advanced type; Characteristic root

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1. INTRODUCTION

As it is well known the study of stability of solutions of delay differential equations occurs through the analysis of the position in $\mathbb{C}$ of its characteristic roots. Also the expansions in series of exponentials of solutions of these equations are based on the characteristic roots. These equations are widely used to describe many problems that contain time dependence, so the investigation of the distribution in the complex plane of the characteristic roots is great interesting. For delay differential equations there are, in general, infinitely many characteristic roots in contrary to ordinary differential equations, where they are finite, and therefore there are infinitely many exponential solutions. For details on this field the books [1], [2], [4] and [6] are available. In [3] stability charts for first order classical autonomous delay differential equations with real or complex coefficients have derived, while in [7] these problems have studied when the equation has periodic coefficients. Stability for delay differential equations, using the characteristic roots, is also study in [5]. More general classes involving multiple and distributed delays are studied in [8], [9] and first order neutral systems in [10]. In [9] one can find a rich bibliography and some interest aspects.

Here we will study mainly delay differential equations advanced type with constant real coefficients. These equations have an essential difference from classical delay differential equations: its characteristic roots asymptotically distributed along a curve on which the real part of the characteristic roots tend to infinite. In particular we prove necessary and sufficient conditions with respect to the variation of the coefficients. Also for classical first and second order linear delay differential equations we prove conditions in order the characteristic roots have negative real parts.

2. PRELIMINARIES AND RESULTS

The characteristic roots of scalar autonomous delay differential equations with either real or complex coefficients has been studied in [3]. In particular, in [3] the classical equation has been considered

$$\dot{x}(t) = ax(t) + bx(t-1), \tag{1}$$

with coefficients a, b real or complex. The delay has been assumed equal to one without loss of generality, since with an appropriate transform the general case reduce to (1). A general theorem, which connects the sign of the real parts of the characteristic roots with the coefficients of the equation (1) is formulated below and its proof is a simple consequence of Hayes' theorem ([1], p.444).

Theorem 1: All the characteristic roots of (1), where a and b are real, have negative real parts if and only if

(i) $a < 1$, and

(ii) $a < -b < \sqrt{a^2 + a_1^2}$,

where a_1 is the root of the equation $a_1 = a\,tan a_1$ such that $0 < a_1 < \pi$. If $a = 0$, we take $a_1 = \frac{\pi}{2}$.

Proof: The characteristic equation of (1) is $z - a - be^{-z} = 0, z \in \mathbb{C}$, or equivalently

$$ae^z + b - ze^z = 0. \tag{2}$$

In (2) we apply the result of Hayes' p.444 in [1], and the proof is complete. □

Now we consider the advanced type equation

$$u'(t-1) + au(t) + bu(t-1) = 0, \tag{3}$$

with real coefficients a, b and characteristic equation

$$ze^{-z} + a + be^{-z} = 0 \tag{4}$$

or equivalently,

$$b + z + ae^z = 0.$$

Let $z = x + iy, x, y \in \mathbb{R}$ be a solution of (4) . Then it is obvious that $\overline{z} = x - iy$ is also a solution of (4). Therefore we assume $y \geqslant 0$ without loss of generality. From (4) we take

$$\begin{cases} x + b = -ae^x \cos y, \\ y = -ae^x \sin y. \end{cases} \tag{5}$$

From (5) we have

$$\begin{cases} (x+b)^2 + y^2 = a^2 e^{2x}, \\ \dfrac{y}{x+b} = \tan y, \end{cases} \tag{6}$$

for $x \neq -b$ and $y \neq \frac{\pi}{2} + \kappa\pi, \kappa \in \mathbb{Z}$. The case $x = -b$ can be lead to a solution of (5) if and only if $y = \frac{\pi}{2} + \kappa\pi$, $\kappa \in \mathbb{Z}$, but this is possible if and only if $a = \pm e^b(\frac{\pi}{2} + \kappa\pi)$.

We observe from (6), that the point (x, y) in the complex plane, is layed on the curves,

$$(c_1): \ y = \pm\sqrt{a^2 e^{2x} - (x+b)^2}$$
$$(c_2): \ x = -b + \frac{y}{\tan y}.$$

Proposition 1: (A). If $a > 0$ then there exists a unique solution of (4). Moreover the following equivalents are valid.

(i) $x < 0 \Leftrightarrow a + b > 0$

(ii) $x = 0 \Leftrightarrow a + b = 0$
(iii) $x > 0 \Leftrightarrow a + b < 0$
(B). If $a = 0$, then $z = -b$ is the unique real solution of (4).
(C). If $a < 0$, then
(i) There exist no real roots of (4), if and only if $a < -e^{b-1}$.
(ii) There exist a double real root of (4) if and only is $a = -e^{b-1}$.
(iii)There exist two distinct real roots of (4) if and only if
Proof.: We assume $y = 0$ and hence $z = x \in \mathbb{R}$. Then (5) reduces to

$$-ae^x = x + b \tag{7}$$

All possible solutions of (7) belongs to the set of intersections of the graphs of

$$A(x) = -ae^x,\ B(x) = x + b$$

(A) For $a > 0$, we have $A'(x) = -ae^x < 0$ and $B'(x) = 1 > 0$, which imply uniqueness. The cases (i), (ii), (iii) are trivial.
(B) Straightforward from (7).
(C) For $a < 0$, we have

$$A'(x) = -ae^x > 0, B'(x) = 1 > 0.$$

i.e. the graphs have positive slopes. The point of touch of curves is the solution of the system and hence $x = 1 - b$. So if $A(1 - b) > B(1 - b)$ or $-ae^{1-b} > 1$ that is $a < -e^{b-1}$, then there exist no real characteristic roots and inversely. The other cases are proved similarly. □

Concerning with the imaginary characteristic roots we have the following.
Proposition 2: The imaginary number $z = iy, y \in \mathbb{R}$ is a root of (4) if and only if there exist $\kappa = 0, 1, 2, \ldots$ such that

$$a = -\frac{y}{\sin y}, \quad b = \frac{y}{\cot y}, \quad y \in I_\kappa = (\kappa\pi, (\kappa + 1)\pi) \tag{8}$$

Proof: From (4), for $x = 0$, we take

$$b = -a\cos y, \quad y = a\sin y$$

and hence the relations (8).□

Now we consider the linear delay differential equation second order

$$u''(t) + au'(t) + bu(t) + cu(t - 1) = 0. \tag{9}$$

The characteristic equation of (9) is

$$z^2 + az + b + ce^{-z} = 0. \tag{10}$$

We prove a necessary and sufficient condition that all the characteristic roots of (9) have negative real parts.
Theorem 2: Let $a > 0, b \geq 0, c \in \mathbb{R}$. Also let $a_\kappa, k \geq 0$, be the unique root of equation

$$\cot a_\kappa = (a_k^2 - b)/a,$$

which lies on the interval $(k\pi, k\pi + \pi)$. We define the number n as follows:
(i) $n = 1$, if $c \geq 0$ and $a^2 \geq 2b$,
(ii) $n = odd$ k for which a_k lies closest to $\sqrt{b - a^2/2}$, if $c \geq 0$ and $a^2 < 2b$,
(iii) $n = 2$, if $c < 0$ and $a^2 \geq 2b$,
(iv) $n = even$ k for which a_k lies closest to $\sqrt{b - a^2/2}$, if $c < 0$ and $a^2 < 2b$.

Then a necessary and sufficient condition that all the roots of (10) lie to the left of the imaginary axis is that

$$c \geq 0 \text{ and } (c \sin a_n)/(aa_n) < 1 \text{ or}$$

$$-b < c < 0 \text{ and } (c \sin a_n)/(aa_n) < 1.$$

Proof: The characteristic equation (10) is equivalent to

$$(z^2 + az + b)e^z + c = 0 \tag{11}$$

Then for (11) we apply theorem 13.9 of [1] and the proof is finished. □

Proposition 3: For the equation (9) we have

(i) if $a^2 - 4b \leq 0$ and $c > 0$, then there exist no real characteristic roots,

(ii) if $a^2 - 4b > 0$ and $c < 0$, then there exist two distinct real roots.

Proof: We assume $y = 0$. Then (10) reduces to

$$x^2 + ax + b = -ce^{-x}, \tag{12}$$

being the second equation identically satisfied. All possible solutions of (13) belongs to the set of intersections of the graphs of

$$A(x) = x^2 + ax + b, \quad B(x) = -ce^{-x}$$

(i) if $a^2 - 4b \leq 0$ and $c > 0$, $A(x) \geq 0$ and $B(x) < 0$, hence there exist no intersections.

(ii) if $a^2 - 4b > 0$ and $c < 0$, then $A(-\frac{a}{2}) < 0$ and $B(x) > 0$ and hence there exist two distinct intersections.□

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