CSCanada
Progress in Applied Mathematics
Vol. 2, No. 1, 2011, pp. 56-60
DOI: 10.3968/j.pam.1925252820110201.123

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
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The Application of Least-squares Method in the Group-AHP

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Received 24 June 2011; accepted 22 July 2011

Abstract
Group-AHP plays an important role in the practical use, but there has not been a perfect method to solve it so far. This paper is intended for doing some research about the application of Least-squares in the Group-AHP.

Key words
Group-AHP; Least-squares method AHP

LI Hongyi, & ZHANG Dalu (2011). The Application of Least-squares Method in the Group-AHP. *Progress in Applied Mathematics, 2*(1), 56-60. Available from: URL: http:// www.cscanada.net/index.php/pam/article/view/j.pam.1925252820110201.123
DOI: http://dx.doi.org/10.3968/j.pam.1925252820110201.123

INTRODUCTION

Analytical Hierarchy Process (AHP) is a method of multi-criteria decision-making, which was proposed by Satty [1] in the 1970s. It transfers a complicated question into an orderly hierarchy, and orders the programs by virtue of the people's judgment, then choose the best option. As a method of combining the qualitative and quantitative analysis, AHP has been widely used in many fields. Until now, it has been used in economic analysis and planning, energy and resources policy analysis, research management, human resources forecasting and planning, business management and so on. Most of the researches now are focusing on the single AHP, however, when it is used in practice, in order to make policy decisions more scientific and reliable, it always takes more than one policy-making departments' or relevant experts' opinion, which produces more than one judgment matrix. And because of this, it will result in many problems which are unique to the Group-AHP, such as how to extract useful information from these matrixes to get the final weights. This problem hasn't been resolved completely, and the four methods that were proposed by scholars from home and broad are as follows [4]: the arithmetical method of the judgment matrix, the geometric method of the judgment matrix, the arithmetical method of the ordering vector and the geometrical method of the ordering vector. But there are some drawbacks in these methods. In recent years, some scholars have proposed other new methods. For example, Wang Yingming [5] proposed the generalized-least-deviation priority method in 1994. Zheng Ming [6] proposed the weighted logarithmic least squares in group decision-making. Wang Yingming [7] proposed the geometric least squarepriority method in 1996. Dong Yucheng [8] proposed a method based on the compatibility of the group decision making. Liu Peng [9] proposed an interactive method based on experts' dynamic weights. Lv Yuejin [10] proposed an improved aggregation method based on the theory of the m-th power graph of simple undirected graph. All these methods have their own advantages, but none of them is perfect. In view of the importance of the Group-AHP in reality, based on the achievements of other scholars, this paper concludes a kind of least-squares method to solve the Group-AHP. The practice proves that the arithmetical method of the judgment matrix is most useful among the four methods mentioned above. This paper takes advantage of the least-squares method to overcome the drawbacks of it, and uses the least-squares method again to get the final results.

1. THE METHOD AND THEORY OF THE APPLICATION OF LEAST-SQUARES METHOD IN THE GROUP-AHP

Suppose there are M experts, proposing M judgments $\boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_m$, where $A_k = \left(a_{ij}^{(k)}\right)_{n\times n}$, and $\boldsymbol{A}_k (k = 1, 2, ..., m)$ are positive reciprocal matrixes. Assume that $\alpha_1, ..., \alpha_m$ are the weights of the experts, which are used to measure the experts' level, then

$$\sum_{k=1}^{m} \alpha_k = 1$$

According to the arithmetical method of the judgment matrix, we can get

$$\boldsymbol{A} = (\alpha_1 \boldsymbol{A}_1 + \alpha_2 \boldsymbol{A}_2 + \cdots + \alpha_m \boldsymbol{A}_m) = (a_{ij})_{n\times n}$$

Obviously,

$$a_{ij} = \sum_{k=1}^{m} \alpha_k a_{ij}^{(k)}, \quad i, j = 1, 2, ..., n$$

Matrix A is no longer a positive reciprocal matrix, so it's not a judgment matrix strictly speaking. The way we solve this problem is that taking its upper triangular or the lower triangular matrix to get the final weights, which will lose a lot of useful information. In order to reduce the missing of the useful information and satisfy the condition of a positive reciprocal matrix, we take advantage of the least-squares method to rectify the matrix $\boldsymbol{A}$. And then get a matrix $\boldsymbol{A}^*$, which is close to matrix $\boldsymbol{A}$[3]. Suppose $\boldsymbol{A}^* = (x_{ij})_{n\times n}$, considering a problem of the least-square as follows:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} - a_{ij})^2$$

$$s.t. \begin{cases} x_{ij} \cdot x_{ji} = 1 \\ x_{ij} > 0 \quad (i, j = 1, 2, ..., n) \end{cases}$$

Since $\boldsymbol{A}_k (k = 1, 2, ..., m)$ are all positive reciprocal matrixes, so $a_{ii}^{(k)} = 1 (k = 1, 2, ..., m)$. Then we get

$$a_{ii} = \sum_{k=1}^{m} \alpha_k a_{ii}^{(k)} = \sum_{k=1}^{m} \alpha_k = 1 \quad (i = 1, 2, ..., n)$$

Observe the objective function we can easily get that there must be $x_{ii} = 1(i = 1, 2, ..., n)$ in the optimal solution. Then according to the constraint condition $x_{ij}x_{ji} = 1$, we can get $x_{ji} = \frac{1}{x_{ij}}, (i, j = 1, 2, ..., n)$, substitute it into the objective function, the primal problem can be transformed like this:

$$\min\left\{(x_{12} - a_{12})^2 + \left(\frac{1}{x_{12}} - a_{21}\right)^2 + \cdots + (x_{1n} - a_{1n})^2 + \left(\frac{1}{x_{1n}} - a_{n1}\right)^2 + \cdots + \left(\frac{1}{x_{n-1,n}} - a_{n-1,n}\right)^2 + \left(\frac{1}{x_{n-1,n}} - a_{n,n-1}\right)^2\right\}$$
$$stx_{ij} > 0 \quad (1 \leqslant i \leqslant j \leqslant n)$$

Furthermore, the question can be divided into $\frac{n(n-1)}{2}$ smaller questions:

$$\min\left\{(x_{ij} - a_{ij})^2 + \left(\frac{1}{x_{ij}} - a_{ji}\right)^2\right\}$$
$$st \quad x_{ij} > 0 \quad (1 \leqslant i \leqslant j \leqslant n)$$

Then we construct the function $f(x_{ij}) = (x_{ij} - a_{ij})^2 + \left(\frac{1}{x_{ij}} - a_{ji}\right)^2$.

Because when $x_{ij} \to 0$ or $x_{ij} \to +\infty$, $f(x_{ij}) \to +\infty$, so the minimum of the function must be a stagnation point, derivate the function, we get

$$f(x_{ij}) = 2(x_{ij} - a_{ij}) + 2\left(\frac{1}{x_{ij}} - a_{ji}\right)\left(-\frac{1}{x_{ij}^2}\right)$$

So the minimum must meet the condition

$$2(x_{ij} - a_{ij}) + 2\left(\frac{1}{x_{ij}} - a_{ji}\right)\left(-\frac{1}{x_{ij}^2}\right) = 0$$

Arrange it, we can $x_{ij}^4 - a_{ij}x_{ij}^3 + a_{ji}x_{ij} - 1 = 0$.

Work out all the normal solution of the equation, the x_{ij} that makes $f(x_{ij})$ get the minimum is what we want. Then we can get matrix $\boldsymbol{A}^*$.

After getting matrix $\boldsymbol{A}^*$, we take advantage of the least-squares method again, instead of using the eigenvalue method. The method is as follows:

In practice, the judgment matrix can't meet the condition $a_{ij} = a_{ik}/a_{jk}$ completely, which means $a_{ij} \neq W_i/W_j$. So, we introduce the deviation $\varepsilon_{ij} = a_{ij} - W_i/W_j \quad i, j = 1, 2, ..., n$.

Then we construct the departure function

$$F(W) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - W_i/W_j)^2$$

According to the idea of the least-squares, we should work out the minimum of $F(W)$, that is solve the following problem:

$$\begin{cases} \min F(W) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - W_i/W_j)^2 \\ s.t. \sum_{i=1}^{n} W_i = 1 \end{cases}$$

Obviously, the problem is a nonlinear least squares problem with constraint conditions. It's hard to solve such a problem. We use the iteration process from document[2]:

It can be proved that departure function $F(W)$ has at least one minimum W^*, but it's not unique. And the minimum is the solution of the following equations:

$$\sum_{j=1}^{n}\left[\left(a_{ij}-\frac{W_i}{W_j}\right)\frac{W_i}{W_j}-\left(a_{ji}-\frac{W_j}{W_i}\right)\frac{W_j}{W_i}\right]=0 \quad i=1,2,...,n$$

Please refer to document [2] for the details of the proving process.

After we get the conclusion mentioned above, the specific iteration procedures are as follows:

(1) Get the initial ordering vector $W(0)=(W_1(0),W_2(0),...,W_n(0))^T$, set the iterated accuracy as ε, $k=0$. As usual, we get

$$W_i(0)=\sum_{i=1}^{n}a_{ij}\Big/\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij} \quad i=1,2,...,n$$

(2) Calculate

$$\rho_i(W(k))=\sum_{j=1}^{n}\left[\left(a_{ij}-\frac{W_i(k)}{W_j(k)}\right)\frac{W_i(k)}{W_j(k)}-\left(a_{ji}-\frac{W_j(k)}{W_i(k)}\right)\frac{W_j(k)}{W_i(k)}\right] \quad i=1,2,...,n$$

If $\forall i\in\{1,2,...,n\}$, $\rho_i(W(k))\leqslant\varepsilon$ is true, then we stop the iteration process. $W^*=W(k)$. Otherwise, turn to the next step.

(3) Find m, it makes

$$|\rho_m(W(k))|=\max_i\{|\rho_i(W(k))|\}$$

And set

$$\begin{cases} X_i(k)=\begin{cases} T(k)W_m(k) & i=m \\ W_i(k) & i\neq m \end{cases} \\ W_i(k+1)=X_i(k)\Big/\sum_{j=1}^{n}X_j(k) \end{cases}$$

$T(k)$ is the minimum of departure function $F(W(k))$. It can be proved in theory that $T(k)$ is the nonnegative real root of the following equation:

$$t^4+bt^3+dt+c=0$$

Where

$$\begin{cases} b=-\sum_{j\neq m}a_{mj}\frac{W_m(k)}{W_j(k)}\Big/\sum_{j\neq m}\left(\frac{W_m(k)}{W_j(k)}\right)^2 \\ d=\sum_{j\neq m}a_{jm}\frac{W_j(k)}{W_m(k)}\Big/\sum_{j\neq m}\left(\frac{W_m(k)}{W_j(k)}\right)^2 \\ c=-\sum_{j\neq m}\left(\frac{W_j(k)}{W_m(k)}\right)^2\Big/\sum_{j\neq m}\left(\frac{W_m(k)}{W_j(k)}\right)^2 \end{cases}$$

Work out the nonnegative real root of the equation, the t_i that makes the departure function $F(W(k))$ get the minimum is $T(k)$.

(4) Set $k=k+1$, turn to (2).

Since the judgment matrix may not answer the consistency demand, so we must make a consistency check on the result. As for the consistency check for the Group-AHP, there hasn't been a perfect method so far. This paper takes the method that making the consistency check on the comprehensive matrix, instead of on the single matrix. Since we take the least-squares ordering method, we didn't get the max characteristic root, so the traditional consistency check method that $CI = \dfrac{\lambda_{max} - n}{n-1}$ can't be used directly, so we must transform it to the following form:

$$CI = \frac{1}{n(n-1)} \sum_{1 \leqslant i \leqslant j \leqslant n} \left(a_{ij} \frac{W_j^*}{W_i^*} + a_{ji} \frac{W_i^*}{W_j^*} - 2 \right)$$

Obviously, this method don't need the characteristic root, instead, it uses the final ordering vector. So, it applies to all the ordering methods, which makes it can be used widely.

CONCLUSION

Based on the methods that were proposed by former scholars, this paper uses the least-squares method comprehensively to solve the Group-AHP. The accuracy of the arithmetic has been improved deeply, making full use of the primary information. Though it's more complicated than the methods mentioned in the Introduction, it can be implemented by the software easily, so it has practical meaning.

REFERENCES

[1] XU Shubo. (1988). *Practical Decision-making Methods-analytical Hierarchy Process Theory*. Tianjin: Tianjin University Press.

[2] WANG Yingming. (1997). The Least-squares Ordering Method of AHP and the Algorithm Research. *Journal of Applied Mathematics and Computational Mathematics, 7*.

[3] HUANG Chongchao, LI Yaping, & LIU Weizheng. A Group-AHP Method Based on the Least-squares Method. *Practice and Understanding of Mathematics, 3*.

[4] JIA Hongxun. (1989). The Application of Group Judgments in Analytical Hierarchy Process. *Decision-making and Analytical Hierarchy Process, 9*.

[5] WANG Yingming. (1996). A Geometric Least Square Priority Method of Group-AHP. *Chinese Management Science, 6*.

[6] ZHENG Ming, & HUANG Zhibin. (2001). Weighted Logarithmic Least Squares in Group Decision-making. *Journal of Fudan University, 21*.

[7] WANG Yingming, & FU Guowei. (1994). A New Generalized-least-deviation Priority Method of Group Comparison Matrices. *System Theory and Practice, 19*.

[8] DONG Yucheng, CHEN Yihua, & WANG Shuang. (2004). Algorithm of Solving Weights for Group Decision Making by Improving Compatibility. *System Theory and Practice, 14*.

[9] LIU Peng, ZHANG Yuanlin, YAN Xiangtao, & KUANG Xinghua. (2007). The Method of Interactive Group Decision-making for AHP Based on Experts' Dynamic Weights. *Mathematics in Prctice and Theory, 14*.

[10] LV Yuejin, & GUO Xinrong. (2007). An Effective Aggregation Method for the AHP Judgement Matrix in Group Decision-Making. *System Theory and Practice, 11*.