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Electrohydrodynamic Instability of Two Superposed Viscous Dielectric Fluids Flowing down an Inclined Plane with Thermal Conductivity Variation

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Abstract
The linear electrohydrodynamic instability of two superposed viscous dielectric fluids flowing down an inclined plane in the presence of thermal conductivity variation and applied electric fields is investigated. Using long-wavelength approximation, a new instability is presented. It is shown that when there is a variation in thermal conductivity in the fluid even in absence of electric fields or when applied electric fields are present even in absence of thermal conductivity variation, instability can occur under a longitudinal gravitational field. The effects of various parameters as Prandtl number, Reynolds number, electric field, inclination angle, and thermal conductivity variation on the stability of the system are discussed analytically and numerically in detail. The presence of electric field is important to prevent the drop out of the analysis when there is no stratification in thermal conductivity.

Key words
Hydrodynamic stability; Viscous fluids; Flows down an inclined plane; Thermal conductivity; Electrohydrodynamics

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INTRODUCTION

The dynamics of thin film waves has received much attentiom from various industries due to its dramatic effect on transport rate of mass, heat, and momentum in designing distillation and adsorption columns, evaporators, condensers, nuclear reactor emergency colling system ...etc. (Bankoff [1], Lin [2], Yih and Seagrave [3], Sanyal and Sanyal [4]). Knowledge of film waves is necessary in connection with the modern precision coating of photographic emulsions, magnetic material, protective paints, flow of molten metal/lava,

...etc. (Yih [5], Joseph and Ranardy [6]). The interfacial instability of single-phase falling films is marked by a finite bandwidth of modes becoming unstable for any nonzero parallel flow, in contrast to the viscosity-stratification instability described above. Smith [7, 8] described the different possible mechanisms for a long-wave falling film instability. One is the "velocity-induced" instability, driven by the velocity gradient normal to the flow of the basic state. The second is a "shear-induced" instability, which is driven by the gradient of the shear stress normal to the flow of the basic state. Up to date works on this fascinating problem can be seen by the review works of Yih [9], Lin [10], De Bruin [11], Nakata [12], Loewenherz and Lawrence [13], Kelly et al. [14], and Hseih [15].

The flow of two immiscible, incompressible fluids in an inclined channel, separated by an interface with surface tension, is driven by the component of gravity along the channel wall, and by an imposed pressure gradient (Tougou [16]). Since the fluids may have distinct physical properties, the interface is susceptible to the viscosity-stratification instability found by Yih [17], the instabilities associated with density stratification, shear-flow instabilities, as well as the interfacial instability found in single-phase falling films. Goussis and Kelly [18, 19] performed an energy analysis of two-layer Couette-Poiseuille flow in an inclined channel for cocurrent flows in the direction of gravity, and found that density stratification manifests its instability through perturbation shear stresses at the deformed interface, while viscosity stratification acts through perturbation velocities across the perturbed interface. For an excellent review about this topic see the work of Prokopiou et al. [20].

Electrohydrodynamics is the study of the relation between the electric field and fluid mechanics. In recent years, some attention has been paid to the engineering application of electrohydrodynamics, in particular the behavior of interface between two dielectric fluids in applied electrostatic fields. One important problem in electrohydrodynamics is the impact of the electric field on the stability of a two-fluid system. The discontinuity of the electrical properties of the fluids across the interface affects the force balance at the fluid-fluid interface, which may either stabilize or destabilize the interface in question. The effect of electric fields on the stability and dynamics of fluid-fluid interface has been an area of extensive research, beginning from the classical work of Taylor and McEwan [21]. Hoburg and Melcher [22] studied the stability of two fluids stressed by a tangential electric field with a conductivity gradient in a diffusive layer. Melcher and his collaborators [23−25] studied the stability of two fluids under the influence of a tangential electric field as well as a normal electric field, both in an unbounded domain. Abdella and Rasmussen [26] analysed Couette flpw in an unbounded domain subjected to a normal electric field. These works and subsequent studies, see for example, the review of Saville [27], have amply demonstrated the role of electrical stresses on fluid interfaces, and the associated electrohydrodynamic instabilities in such systems. One of the basic problems here is to understand the stability of the interface between two fluid layers bounded on the top and bottom by rigid plates, and this has been the subject of many previous studies. These studied have largely considered systems in which gravitational effects are important, and therefore, a critical applied voltage is required to cause the instability: very long waves are stabilized by gravity, and short waves are stabilized by interfacial tension, and waves of intermediate lengths become unstable. For an excellent review about this topic see the works of Shankar and Sharma [28], Ozen et al. [29] and Tomar et al. [30].

In the majority of cases of electrohydrodynamic instability, there is a stratification of either a fluid property or some quantity of flow. The most obvious case of instability is that of two superposed fluids, with the upper fluid heavier than the lower one. A statically stratified fluid can be unstable if the fluid is accelerated downward with an acceleration greater than the gravitational acceleration g. The stratification in density of an incompressible fluid has its counterpart in the stratification of entropy of a compressible fluid, as meteorologists who invented the concept of potential density to account for the effect of compressibility have long recognized. Stratification in density in the presence of longitudinal gravity can be unstable, as is now well known. Less well known is the instability resulting from a stratification in electric conductivity, as shown by Taylor and McEwan [21] for a steady vertical electric field. Instability resulting from viscosity variation in shear flows (Yih [17]) is a subject that, after many years, is now enjoying a period of revival of interest. In porous media, a less viscous fluid pushing a more viscous one can induce instability and produce fingers of penetration. But it does not necessarily need to be a fluid property that, when stratified, can induce instability. If some quantity of the flow of a fluid is stratified, it can be unstable too. A prominent example is the Couette flow, which can be unstable if the square of the circulation decreases outwards, resulting in the formation of Taylor vortices. The electromagnetic counterpart of Taylor vortices is the result of a radial stratification of a circular magnetic field. In two-dimensional flows the stratification of vorticity can induce instability when there is a point of inflection in the velocity profile, a farnous and extreme case of which is the Helmholtz instability, where the density stratification is stabilizing and the instability results from the vortex sheet. Even when there is no point of inflection in the velocity profile of a two-dimensional flow of a viscous fluid, stratification of vorticity is still important for instability, as indicated by the stability of plane Couette flows, which has uniform vorticity. (For axisymmetric flows it is the stratification of the azimuthal vorticity divided by the radial distance that is important. When this quantity is constant, as in Poiseuille flow, the flow is stable against axisymmetric small disturbances).

In this article, We shall show a new instability: the instability resulting from thermal-conductivity stratification. With the other instances of how a fluid or a flow can be unstable when a stratification is present, one could perhaps make the point that hydrodynamic stability is a subject within the field of stratified flows (Yih [5], Kao [31]). We have several practical reasons to consider this problem. Film flows of dielectric fluids serve in elements of nuclear energy equipment, and other technological applications, as in metallurgy. Applications of dielectric and electric films in different cooling systems is also very promising. The generation and development of surface waves can drastically affect the properties of such systems. In some applications, it is desirable to promote surface waves, and in others to suppress them. It is worth noting that the effect of external body forces can accelerate or impede the onset of waves over a flow, as is observed in convection. It is well known that the effect of the Lorentz force causes a loss of energy in electrohydrodynamic flow, and anisotropy of propagation of the waves due to the interaction of the flow and the external electric field. Therefore, we suggest that the Lorentz force may essentially affect the critical conditions for the onset of long-wave instability in dielectric films.

1. PRIMARY TEMPERATURE, VELOCITY AND ELECTRIC FIELDS

Consider two superposed fluids, each of thickness d, between two plane boundaries inclined at an angle β to the horizontal. To show that the instability to be revealed results from conductivity variation alone, we shall assume the two fluids to have the same viscosity and the same dependence of density on temperature, but different thermal conductivities: k (for the upper fluid) and k (for the lower fluid). That two such fluids are not easy to find is not necessarily an objection to this study, since instabilities resulting from density and viscosity variations are known, and the new cause of instability is in addition to those other known causes of instability.

Let the origin of Cartesian coordinates be situated on the interface of the fluids, and let x be measured along the interface down the incline, and y be measured upward in a direction normal to the interface. The temperatures at the lower and upper boundaries will be denoted by $T - \Delta T$ and $T + \Delta T$, respectively. We shall measure x and y in units of d, so that they are dimensionless. The temperature in the lower and upper fluids will be denoted by $T_1(y)$ and $T_2(y)$, respectively. Defining $\bar{h}_1$ and $\bar{h}_2$ by (Yih [32])

$$\bar{h}_1 = \frac{T_1(y) - T_0}{\Delta T}, \quad \bar{h}_2 = \frac{T_2(y) - T_0}{\Delta T} \tag{1}$$

one can readily solve the Laplace equation governing heat conduction, with regards to the boundary and interfacial conditions, and obtain

$$\bar{h}_1 = \frac{(\lambda - 1)}{(\lambda + 1)} + \frac{2\lambda}{(1 + \lambda)} y \tag{2}$$

$$\bar{h}_2 = \frac{(\lambda - 1)}{(\lambda + 1)} + \frac{2}{(1 + \lambda)} y \tag{3}$$

where

$$\lambda = \frac{k_2}{k_1} \tag{4}$$

The interfacial conditions for the temperature field are the continuity of temperature and the continuity of heat flux across the interface.

The variation of density with temperature is assumed the same for both fluids only for the sake of simplicity. This assumption is not at all necessary. We make it here only to isolate the variation of thermal conductivity as the cause of instability. The dependence of the density ρ on temperature is

$$\rho = \rho_0[1 - \bar{\alpha}(T - T_0)] \tag{5}$$

where ρ_0 is the density at temperature T_0, and $\bar{\alpha}$ is the coefficient of thermal expansivity. With $\bar{u}$ denoting the velocity (in the x direction) of the primary flow and μ denoting the viscosity (assumed constant), we have

$$\mu\frac{d^2\bar{u}}{dY^2} + g\rho\sin\beta - K = 0 \tag{6}$$

where g is the gravitational acceleration, and

$$K = \frac{d\overline{\Pi}}{dX}, \quad \overline{\Pi} = \bar{p} - \frac{\varepsilon}{2}\bar{E}^2 \tag{7}$$

In equation (7), $\bar{p}$ is the pressure in the primary flow, and $\bar{E}$ is the electric field, and $X = xd$, x being measured in units of d.

The velocity $\bar{u}$ for the lower and upper fluids are, respectively, denoted by $\bar{u}_1(Y)$ and $\bar{u}_2(Y)$. The boundary conditions are

$$\bar{u}_1(-d) = 0, \quad \bar{u}_2(d) = 0 \tag{8}$$

and the interfacial conditions are

$$\bar{u}_1(0) = \bar{u}_2(0), \quad \frac{d\bar{u}_1}{dY} = \frac{d\bar{u}_2}{dY} \quad at \quad Y = 0 \tag{9}$$

Equation (6) gives two equations, one for $\bar{u}_1$, and one for $\bar{u}_2$. When these are solved with conditions (8) and (9), one obtains

$$\bar{u}_1 = \frac{\widetilde{K}}{2\mu}(d^2 - Y^2) + \frac{V\sin\beta}{6d^2(\lambda+1)}\left\{(\lambda-1)(3Y^2 - 2d^2) - (\lambda+1)dY + \frac{2\lambda}{d}Y^3\right\} \tag{10}$$

$$\bar{u}_2 = \frac{\widetilde{K}}{2\mu}(d^2 - Y^2) + \frac{V\sin\beta}{6d^2(\lambda+1)}\left\{(\lambda-1)(3Y^2 - 2d^2) - (\lambda+1)dY + \frac{2}{d}Y^3\right\} \tag{11}$$

where

$$V = \frac{\bar{\alpha}gd^2\Delta T}{\nu}, \quad \nu = \frac{\mu}{\rho_0} \tag{12}$$

has the dimension of a velocity, and will be used as the velocity scale, and $\widetilde{K} = \rho_0 g\sin\beta - K$.

For our purpose of demonstrating instability resulting from conductivity variation, it is sufficient to take a special $\widetilde{K}$. We shall take

$$\frac{\widetilde{K}d^2}{\mu} = \frac{(\lambda-1)}{(\lambda+1)}V\sin\beta \tag{13}$$

because it will give us the simplest forms of $\bar{u}_1$, and $\bar{u}_2$. Using V as the velocity scale, and adopting equation (13), we have

$$U_1 = \frac{\bar{u}_1}{V} = \frac{1}{6(\lambda+1)}[(\lambda-1) - (\lambda+1)y + 2\lambda y^3]\sin\beta \tag{14}$$

$$U_2 = \frac{\bar{u}_2}{V} = \frac{1}{6(\lambda+1)}[(\lambda-1)-(\lambda+1)y+2y^3]\sin\beta \tag{15}$$

On the other hand, suppose that the lower rigid boundary is raised at a zero electric potential, while the upper one is raised at an electric potential V in order to produce an electric field acting in the negative y-direction. As usual, in electrohydrodynamics, we assume that the quasi-static approximation is valid, then there exists an electric potential Ψ such that $\widetilde{E} = -\nabla\Psi$. In this case, The Maxwell's equations can be written in the form (Lee[33], Kim and Bankoff[34])

$$\nabla\cdot(\varepsilon\widetilde{E}) = 0, \quad \nabla\times(\widetilde{E}) = 0 \tag{16}$$

where $\widetilde{E}$ is the electric field, and ε is the dielectric constant. We shall also assume that there are no surface charges (in the equilibrium state) at the surface of separation, and therefore, the electric displacement is continuous at the interface, i.e. if the condition $\varepsilon_1 E_{01} = \varepsilon_2 E_{02}$ is satisfied.

2. PERTURBATION EQUATIONS

The stability problem is now formulated following the usual small perturbation technique, and with the usual procedure of considering two-dimensional disturbances only, since Squire's result[35], and later extensions by Yih[9] have shown that the stability or instability of a three-dimensional disturbance can be determined from that of a two-dimensional disturbance at a higher Reynolds number.

The Navier-Stokes equations are

$$\frac{\partial\widetilde{u}_j}{\partial\tau} + \widetilde{u}_j\frac{\partial\widetilde{u}_j}{\partial X} + \widetilde{v}_j\frac{\partial\widetilde{u}_j}{\partial Y} = -\frac{1}{\rho_0}\frac{\partial\widetilde{\Pi}_j}{\partial X} + \frac{\mu}{\rho_0}\nabla^{*2}\widetilde{u}_j + [1-\bar{\alpha}(T_j - T_0)]g\sin\beta \tag{17}$$

$$\frac{\partial\widetilde{v}_j}{\partial\tau} + \widetilde{u}_j\frac{\partial\widetilde{v}_j}{\partial X} + \widetilde{v}_j\frac{\partial\widetilde{v}_j}{\partial Y} = -\frac{1}{\rho_0}\frac{\partial\widetilde{\Pi}_j}{\partial Y} + \frac{\mu}{\rho_0}\nabla^{*2}\widetilde{v}_j + [1-\bar{\alpha}(T_j - T_0)]g\cos\beta \tag{18}$$

where $j = 1$ denotes quantaties associated with the lower fluid, and $j = 2$ denotes quantaties associated with the upper fluid, and $\widetilde{u}_j, \widetilde{v}_j$ are the velociy componens in the X, Y directions, respectively, $\widetilde{\Pi}_i$ is the modified pressure, τ is the time, and $\nabla^{*2} = \partial^2/\partial X^2 + \partial^2/\partial Y^2$.

The continuity equation is

$$\frac{\partial\widetilde{u}_j}{\partial X} + \frac{\partial\widetilde{v}_j}{\partial Y} = 0 \tag{19}$$

The above equations are made dimensionless by setting

$$\bar{h}_j = \frac{T_j - T_0}{\Delta T}, (u_j, v_j) = \left(\frac{\widetilde{u}_j}{V}, \frac{\widetilde{v}_j}{V}\right), (x, y) = \left(\frac{X}{d}, \frac{Y}{d}\right), \Pi_j = \frac{\widetilde{\Pi}_j}{\rho_0 V^2}$$
$$t = \frac{\tau V}{d}, E_j = \frac{\widetilde{E}_j}{\sqrt{\rho_0}V} \tag{20}$$

The dimensionless form of equations (16)-(19) are then

$$\frac{\partial u_j}{\partial t} + u_j\frac{\partial u_j}{\partial x} + v_j\frac{\partial u_j}{\partial y} = -\frac{\partial\Pi_j}{\partial x} + \frac{1}{R}\nabla^2 u_j + \frac{d}{V^2}[1-\bar{\alpha}\bar{h}_j\Delta T]g\sin\beta \tag{21}$$

$$\frac{\partial v_j}{\partial t} + u_j\frac{\partial v_j}{\partial x} + v_j\frac{\partial v_j}{\partial y} = -\frac{\partial\Pi_j}{\partial y} + \frac{1}{R}\nabla^2 v_j - \frac{d}{V^2}[1-\bar{\alpha}\bar{h}_j\Delta T]g\cos\beta \tag{22}$$

$$\frac{\partial u_j}{\partial x} + \frac{\partial v_j}{\partial y} = 0 \tag{23}$$

in which $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial^2$, and

$$\nabla \cdot (\varepsilon E) = 0, \quad \nabla \times (E) = 0 \tag{24}$$

Assuming small perturbations from the basic flow in the form

$$u_j = U_j + u'_j,\ v_j = v'_j,\ \Pi_j = \Pi_{0j} + \Pi'_j,\ h_j = \bar{h}_j + \Theta_j,\ E_j = E_{0j} + E'_j \tag{25}$$

Let the dimensionless temperature perturbations be expressed by

$$\Theta_1 = \frac{T'_1}{\Delta T}, \quad \Theta_2 = \frac{T'_2}{\Delta T} \tag{26}$$

Neglecting the second-order terms in the primed quantities, and making use of the fact that U_j and Π_{0j} satisfy the basic flow equations, we have upon substitution of equation (25) into equations (21)-(24), the linearized equations governing the disturbance motion, to the first order terms, are

$$\frac{\partial u'_j}{\partial t} + U_j\frac{\partial u'_j}{\partial x} + v'_j\frac{\partial U_j}{\partial y} = -\frac{\partial \Pi'_j}{\partial x} + \frac{1}{R}\Delta u'_j - \Theta_1\left(\frac{\bar{\alpha} g d \Delta t}{V^2}\right)\sin\beta \tag{27}$$

$$\frac{\partial v'_j}{\partial t} + U_j\frac{\partial v'_j}{\partial x} = -\frac{\partial \Pi'_j}{\partial y} + \frac{1}{R}\Delta v'_j + \Theta_1\left(\frac{\bar{\alpha} g d \Delta t}{V^2}\right)\cos\beta \tag{28}$$

$$\frac{\partial u'_j}{\partial x} + \frac{\partial v'_j}{\partial y} = 0 \tag{29}$$

and

$$\nabla \cdot (\varepsilon E') = 0, \quad \nabla \times (E') = 0 \tag{30}$$

From equation (30), the electric field E' can be expressed in terms of a scalar potential Ψ' such that $E' = -\nabla\Psi'$. Hence, Ψ' should satisfy the equation

$$\nabla^2\Psi'_j = 0,\ j = 1, 2 \tag{31}$$

3. FORMULATION OF THE STABILITY PROBLEM

Let $\psi_1(x, y)$ and $\psi_2(x, y)$ be the stream functions for the lower and upper fluids, respectively. Then the velocity perturbations for the two fluid layers, satisfying equations (29), are given by

$$u'_j = \frac{\partial \psi_j}{\partial y}, \quad v'_j = -\frac{\partial \psi_j}{\partial x}, \quad j = 1, 2 \tag{32}$$

We shall assume

$$(\Theta_1, \Theta_2, \psi_1, \psi_2, \Pi'_j) = [h_1(y), h_2(y), \phi(y), \chi(y), f_j(y)]\exp[i\alpha(x - ct)] \tag{33}$$

where the scale for the time t is d/V, the scale for the wave number α is d^1, and c is the dimensionless wave velocity:

$$c = c_r + ic_i \tag{34}$$

The flow is stable or unstable according to whether c_i is negative or positive, respectively.

The linearized heat equations are then, upon use of equations (26), (32), and (33)

$$i\alpha(u_1 - c)h_1 - \frac{2i\lambda\alpha}{\lambda + 1}\phi = \frac{1}{\lambda RP}(h''_1 - \alpha^2 h_1) \tag{35}$$

$$i\alpha(u_2 - c)h_2 - \frac{2i\alpha}{\lambda + 1}\chi = \frac{1}{RP}(h_2'' - \alpha^2 h_2) \tag{36}$$

where, for simplicity, we have assumed the thermal diffusivities of the fluid to have the ratio λ also (in effect ignoring the variation of specific heat, which can be accounted for without difficulty), and

$$R = \frac{Vd}{\nu}, \quad P = \frac{\nu}{\kappa_2} \tag{37}$$

are the Reynolds number and the Prandtl number (for the upper fluid), respectively. The thermal diffusivity of the upper fluid is denoted by κ_2. The boundary conditions are, assuming that the boundaries are thermally much more conductive than the fluid,

$$h_1(-1) = 0 = h_2(1) \tag{38}$$

and the interfacial conditions are

$$h_1'(0) = \lambda h_2'(0) \tag{39}$$

$$h_2(0) - h_1(0) = \frac{2(\lambda - 1)\phi(0)}{(\lambda + 1)}\frac{\phi(0)}{c'} \tag{40}$$

with

$$c' = c - U_1(0) \tag{41}$$

The term on the right-hand side of equation (40) arises from the difference in slope of $\bar{h}_1$ and $\bar{h}_2$ at $y = 0$, which contributes the term when the interface is displaced from its mean position. This term is crucial in the calculation for stability. The ratio $\phi(0)/c'$ multiplied by the exponential factor $\exp[i\alpha(x - ct)]$ is indeed equal to the interfacial displacement ξ, as can be deduced from the kinematic condition at the interface. Therefore

$$\xi = \frac{\phi(0)}{c'}\exp[i\alpha(x - ct)] \tag{42}$$

Using equation (33), the linearized Navier-Stokes equations are, for the lower fluid

$$i\alpha(u_1 - c)\phi' - i\alpha u_1'\phi = -i\alpha f_1 + \frac{1}{R}(\phi''' - \alpha^2\phi') - \frac{\sin\beta}{R}h_1 \tag{43}$$

$$\alpha^2(c - u_1)\phi = f_1' + \frac{i\alpha}{R}(\phi'' - \alpha^2\phi) - \frac{\cos\beta}{R}h_1 \tag{44}$$

The last term in equation (43) arises form the body force term

$$\left(\frac{d}{\rho_0 V^2}\right)(\rho_0 g\bar{\alpha}\Delta T h_1 \sin\beta), \tag{45}$$

the multiplier $d/\rho_0 V^2$ is to make the entire equation dimensionless [similarly for the last term in (44)]. For the upper fluid, the linearized Navier-Stokes equations are

$$i\alpha(u_2 - c)\chi' - i\alpha u_2'\chi = -i\alpha f_2 + \frac{1}{R}(\chi''' - \alpha^2\chi') - \frac{\sin\beta}{R}h_2 \tag{46}$$

$$\alpha^2(c - u_2)\chi = f_2' + \frac{i\alpha}{R}(\chi'' - \alpha^2\chi) - \frac{\cos\beta}{R}h_2 \tag{47}$$

Eliminating f_1 in equations (43) and (44), and f_2 in equations (46) and (47), we obtain the augmented Orr-Sommerfeld equations

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R[(u_1 - c)(\phi'' - \alpha^2\phi) - u_1''\phi] + h_1'\sin\beta + i\alpha h_1\cos\beta \tag{48}$$

$$\chi^{iv} - 2\alpha^2\chi'' + \alpha^4\chi = i\alpha R[(u_2 - c)(\chi'' - \alpha^2\chi) - u_2''\chi] + h_2'\sin\beta + i\alpha h_2\cos\beta \tag{49}$$

The solution of equation (31) can be written in the form

$$\Psi_j' = [a_j\exp(\alpha y) + b_j\exp(-\alpha y)]\exp(i\alpha(x - ct)) \tag{50}$$

where a_j and b_j are arbitrary constants to be determined using the appropriate boundary conditions.

4. THE BOUNDARY CONDITIONS

Here, we used the following boundary conditions (Mohamed et al.[36]):

(1) The velocity vanished at the rigid boundaries, i.e.

$$\phi(-1) = 0 = \phi'(-1), \quad \chi(1) = 0 = \chi'(1) \tag{51}$$

expressing the no-slip condition

(2) The velocity must be continuous at the interface, i.e.

$$\phi(0) = \chi(0), \quad \phi'(0) = \chi'(0) \tag{52}$$

(3) The tangential component of the electric field should be continuous at the interface, i.e.

$$\boldsymbol{n} \times \boldsymbol{E}_1 = \boldsymbol{n} \times \boldsymbol{E}_2 \quad \text{at} \quad y = 0 \tag{53}$$

where the total electric fields $\boldsymbol{E}_j (j = 1, 2)$ for the two layers are given by

$$\boldsymbol{E}_j = -\frac{\partial \Psi'_j}{\partial x}\boldsymbol{i} - \left(E_{0j} + \frac{\partial \Psi'_j}{\partial y}\right)\boldsymbol{j}, \quad (j = 1, 2) \tag{54}$$

and the unit normal vector $\boldsymbol{n}$ to the interface $y = \xi$ between the two layers, to first order terms is

$$\boldsymbol{n} = -i\alpha\xi\boldsymbol{i} + \boldsymbol{j} \tag{55}$$

substitute from equations (50), (54), and (55) into condition (53), we obtain

$$a_1 + b_1 = a_2 + b_2 + \frac{\phi(0)}{c'}(E_{02} - E_{01}) \tag{56}$$

(4) The normal component of the electric displacement is continuous at the interface, i.e.

$$\boldsymbol{n} \cdot (\varepsilon_1 \boldsymbol{E}_1) = \boldsymbol{n} \cdot (\varepsilon_2 \boldsymbol{E}_2) \quad at \quad y = 0 \tag{57}$$

which, using equations (50), (54), and (55), yields

$$\varepsilon_1(a_1 - b_1) = \varepsilon_2(a_2 - b_2) \tag{58}$$

(5) Some other conditions should be satisfied at the rigid planes, since the lower plane is earthed and the upper plane is raised to a potential V_0. Then, we have the following conditions

$$\frac{\partial \Psi'_1}{\partial x} = 0 \quad \text{at} \quad y = -1 \quad \text{and} \quad \frac{\partial \Psi'_2}{\partial x} = 0 \quad \text{at} \quad y = 1 \tag{59}$$

Then, using equation (50), we get

$$a_1 \exp(-\alpha) + b_1 \exp(\alpha) = 0 \tag{60}$$

$$a_2 \exp(\alpha) + b_2 \exp(-\alpha) = 0 \tag{61}$$

We solve the equations (56), (58), (60), and (61) to get the following constants

$$a_{1,2} = -\frac{(E_{01}, E_{02})(\varepsilon_2 - \varepsilon_1)}{(\varepsilon_1 + \varepsilon_2)} \frac{\phi(0)}{c'} \frac{\exp(\pm\alpha)}{2\sinh\alpha}$$

$$b_{1,2} = \frac{(E_{01}, E_{02})(\varepsilon_2 - \varepsilon_1)}{(\varepsilon_1 + \varepsilon_2)} \frac{\phi(0)}{c'} \frac{\exp(\mp\alpha)}{2\sinh\alpha} \tag{62}$$

Hence, equations (50) can be written in the form

$$\Psi'_{1,2} = -\frac{(E_{01}, E_{02})(\varepsilon_2 - \varepsilon_1)}{(\varepsilon_1 + \varepsilon_2)} \frac{\phi(0)}{c'} \frac{\sinh[\alpha(y \pm 1)]}{\sinh \alpha} \exp[i\alpha(x - ct)] \tag{63}$$

(6) The tangent component of the stress tensor is continuous at the interface, i.e.

$$\phi''(0) = \chi''(0) \tag{64}$$

(7) The normal component of the stress tensor is discontinuous at the interface by the effective interfacial tension $\widehat{S}$, i.e.

$$\phi''' - 3\alpha^2\phi' - \chi''' + 3\alpha^2\chi' = \left\{ i\alpha^3 SR - \frac{2(\lambda - 1)}{(\lambda + 1)} \sin\beta - i\alpha R \frac{E_{01}E_{02}(\varepsilon_2 - \varepsilon_1)^2}{(\varepsilon_1 + \varepsilon_2)} \right\} \frac{\phi(0)}{c'} \quad \text{at} \quad y = 0 \tag{65}$$

where $S = \widehat{S}/(\rho_0 V^2 d)$, and $\widehat{S}$ being the surface tension. Note that in deriving equation (65) we have used the approximation $\alpha \coth(\alpha) \simeq 1$.

The stability problem is thus governed by four simultaneous differential equations, two of which are of the second order and the other two of the fourth order, and sixteen boundary or interfacial conditions. Given the parameters R, P, λ, α, β, and S, one seeks to determine c. Note that, in the corresponding case studied by Yih[32], which is a limiting case of our work in absence of the applied electric field. He missed (or neglected) the second-term on the right-hand side of equation (65). Therefore, his calculations after this stage are not correct. Hence, we have obtained here the correct calculations in the general case including the effect of electric field, and discussed the stability analysis in view of the correct results.

5. SOLUTIONS AND LONG-WAVE INSTABILITY

We consider long waves, and adopt the method of solution given by Yih[32]. First, we expand the unknowns in power series of α as follows

$$\begin{aligned} h_1 &= H_0 + \alpha H_1 + \alpha^2 H_2 + \cdots, \\ h_2 &= G_0 + \alpha G_1 + \alpha^2 G_2 + \cdots, \\ \phi &= \phi_0 + \alpha\phi_1 + \alpha^2\phi_2 + \cdots, \\ \chi &= \chi_0 + \alpha\chi_1 + \alpha^2\chi_2 + \cdots, \\ c &= c_0 + \alpha c_1 + \alpha^2 c_2 + \cdots, \end{aligned} \tag{66}$$

Substituting these into the governing differential system (35), (36), and collecting terms of order α^0 only, we obtain

$$H_0'' = 0, \quad G_0'' = 0 \tag{67}$$

with the boundary conditions (38)-(40) as

$$H_0(-1) = 0 = G_0(1), \quad H_0'(0) = \lambda G_0'(0) \tag{68}$$

and

$$G_0(0) - H_0(0) = \frac{2(\lambda - 1)}{(\lambda + 1)} \frac{\phi_0(0)}{c_0'} \tag{69}$$

Leaving equation (69) alone for the moment, one solves equations (67), (68), and obtains

$$H_0 = 1 + y, \quad G_0 = \frac{1}{\lambda}(-1 + y) \tag{70}$$

The equations (48) and (49) yield, upon use of equation (70)

$$\phi_0'''' = \sin\beta, \quad \chi_0'''' = \frac{\sin\beta}{\lambda} \tag{71}$$

for which the boundary conditions are

$$\phi_0(-1) = 0 = \phi_0'(-1), \quad \chi_0(1) = 0 = \chi_0'(1) \tag{72}$$

$$(\phi_0, \phi_0', \phi_0'') = (\chi_0, \chi_0', \chi_0'') \quad at \quad y = 0 \tag{73}$$

$$\phi_0''' - \chi_0''' = -\frac{2(\lambda - 1)}{(\lambda + 1)} \sin\beta \frac{\phi_0(0)}{c_0'} \quad at \quad y = 0 \tag{74}$$

Solution of equations (71)-(73) gives

$$\phi_0 = \frac{\sin\beta}{24} y^4 + A_1 y^3 + By^2 + Cy + D \tag{75}$$

$$\chi_0 = \frac{\sin\beta}{24\lambda} y^4 + A_2 y^3 + By^2 + Cy + D \tag{76}$$

where

$$A_1 = \frac{(11\lambda + 5)}{96\lambda} \sin\beta,$$
$$A_2 = -\frac{(5\lambda + 11)}{96\lambda} \sin\beta,$$
$$B = \frac{(\lambda + 1)}{12\lambda} \sin\beta,$$
$$C = -\frac{(\lambda - 1)}{96\lambda} \sin\beta,$$
$$D = -\frac{(\lambda + 1)}{48\lambda} \sin\beta$$

Substitute from equations (75) and (76) into equations (69) and (74), and then solving the resulting equations to obtain

$$c_0' = \frac{(\lambda - 1)}{24(\lambda + 1)} \sin\beta \tag{77}$$

or

$$c_0 = \frac{5(\lambda - 1)}{24(\lambda + 1)} \sin\beta \tag{78}$$

We now proceed to the next approximation. Collecting terms of order α in equations (35) and (36), we have

$$H_1'' = i\lambda RP \left[(u_1 - c_0) H_0 - \frac{2\lambda}{(\lambda + 1)} \phi_0 \right] \tag{79}$$

$$G_1'' = iRP \left[(u_2 - c_0) G_0 - \frac{2}{(\lambda + 1)} \chi_0 \right] \tag{80}$$

The conditions (38) and (39) give

$$H_1(-1) = 0 = G_1(1), \quad H_1'(0) = \lambda G_1'(0) \tag{81}$$

and condition (40) gives

$$G_1(0) - H_1(0) = \frac{2(\lambda - 1)}{(\lambda + 1)} \left(\frac{\phi_1(0)}{c_0'} - \frac{\phi_0(0) c_1}{(c_0')^2} \right) \tag{82}$$

Setting equation (82) aside for the moment and solving equations (79)-(81), we have

$$H_1 = \frac{i\lambda RP \sin\beta}{48(\lambda + 1)} \left\{ \frac{(125 + 19\lambda)}{60} y + 2y^2 - \frac{(9\lambda + 7)}{6} y^3 - \frac{4(\lambda + 1)}{3} y^4 + \frac{(\lambda - 1)}{4} y^5 + \frac{2\lambda}{5} y^6 \right\} \tag{83}$$

The term of zeroth power in y is deliberately dropped to keep $h_1(0) = 1$, since the amplitude of the disturbance is immaterial and already $H_0(0) = 1$. The result for G_1 is

$$G_1 = \frac{iRP\sin\beta}{48\lambda(\lambda+1)}\left\{\frac{-(19\lambda^2+250\lambda+19)}{60} + \frac{(125\lambda+19\lambda^2)}{60}y + 2\lambda y^2 + \frac{(7\lambda+9)}{6}y^3 - \frac{4(\lambda+1)}{3}y^4 + \frac{(\lambda-1)}{4}y^5 + \frac{2}{5}y^6\right\} \tag{84}$$

Equations (48) and (49) give

$$\phi_1'''' = iR[(u_1 - c_0)\phi_0'' - u_1''\phi_0] + H_1'\sin\beta + iH_0\cos\beta \tag{85}$$

$$\chi_1'''' = iR[(u_2 - c_0)\chi_0'' - u_2''\chi_0] + G_1'\sin\beta + iG_0\cos\beta \tag{86}$$

The boundary conditions are

$$\phi_1(-1) = 0 = \phi_1'(-1), \quad \chi_1(1) = 0 = \chi_1'(1) \tag{87}$$

and the four interfacial conditions are obtained from the continuity of ϕ_1 and χ_1, at $y = 0$. are

$$(\phi_1, \phi_1', \phi_1'') = (\chi_1, \chi_1', \chi_1'') \quad \text{at} \quad y = 0 \tag{88}$$

$$\phi_1''' - \chi_1''' = \frac{iRE_{01}E_{02}(\varepsilon_2-\varepsilon_1)^2}{2(\varepsilon_2+\varepsilon_1)}\frac{(\lambda+1)^2}{\lambda(\lambda-1)} + \frac{iRP\sin^2\beta}{48\lambda(\lambda+1)}\left(\frac{19\lambda^2+250\lambda+19}{60}\right) \quad \text{at} \quad y = 0 \tag{89}$$

A straightforward solutions of equations (85) and (86) satisfying the boundary conditions (87)-(89) give

$$\begin{aligned}\phi_1 = {} & \frac{iR\sin^2\beta}{(2880)\lambda(\lambda+1)}\left\{\left[\frac{\lambda^2 P}{24}(19\lambda+125) - \frac{5}{6}(\lambda^2-1)\right]y^4 + \left[2\lambda^2 P - \frac{1}{48}(17\lambda^2 - 2\lambda + 17)\right]y^5 - \frac{1}{12}[\lambda^2 P(9\lambda+7)\right. \\ & +(11\lambda^2+16\lambda+5)]y^6 - \frac{2\lambda(\lambda+1)}{3}\left(1+\frac{4}{7}\lambda P\right)y^7 + \frac{5\lambda^2 P}{112}(\lambda-1)y^8 \\ & \left.+\frac{\lambda^2}{21}\left(\lambda P + \frac{5}{3}\right)y^9\right\} + i\left(\frac{y^5}{120} + \frac{y^4}{24}\right)\cos\beta + \widehat{C}_1 y^3 + \widehat{C}_2 y^2 + \widehat{C}_3 y + \widehat{C}_4\end{aligned} \tag{90}$$

and

$$\begin{aligned}\chi_1 = {} & \frac{iR\sin^2\beta}{(2880)\lambda(\lambda+1)}\left\{\left[\frac{\lambda P}{24}(19\lambda+125) - \frac{5}{6}(\lambda^2-1)\right]y^4 + \left[2\lambda P - \frac{1}{48}(17\lambda^2 - 2\lambda + 17)\right]y^5 - \frac{1}{12}[P(7\lambda+9)\right. \\ & +(5\lambda^2+16\lambda+11)]y^6 - \frac{2(\lambda+1)}{3}\left(1+\frac{4}{7}P\right)y^7 + \frac{5P}{112}(\lambda-1)y^8 \\ & \left.+\frac{1}{21}\left(P + \frac{5}{3}\right)y^9\right\} + \frac{i}{\lambda}\left(\frac{y^5}{120} - \frac{y^4}{24}\right)\cos\beta + \widehat{D}_1 y^3 + \widehat{C}_2 y^2 + \widehat{C}_3 y + \widehat{C}_4\end{aligned} \tag{91}$$

where

$$\widehat{C}_1 = \frac{iR\sin^2\beta}{(483840)\lambda(\lambda+1)}\left\{\frac{P}{8}(283\lambda^3 + 3575\lambda^2 + 19745\lambda + 1613)\right.$$

$$-\frac{1}{3}(647\lambda^2 + 714\lambda - 529)\Big\} + \frac{11i}{480\lambda}(\lambda + 1)\cos\beta$$
$$+\frac{iRE^{*2}(\varepsilon - 1)^2}{24(\varepsilon + 1)}\frac{(\lambda + 1)^2}{\lambda(\lambda - 1)}$$

$$\widehat{D}_1 = \frac{iR\sin^2\beta}{(483840)\lambda(\lambda + 1)}\Big\{\frac{P}{8}(283\lambda^3 - 681\lambda^2 - 36255\lambda - 2643)$$
$$-\frac{1}{3}(647\lambda^2 + 714\lambda - 529)\Big\} + \frac{11i}{480\lambda}(\lambda + 1)\cos\beta$$
$$+\frac{iRE^{*2}(\varepsilon - 1)^2}{24(\varepsilon + 1)}\frac{(\lambda + 1)^2}{\lambda(\lambda - 1)}$$

$$\widehat{C}_2 = \frac{iR\sin^2\beta}{(5760)\lambda(\lambda + 1)}\Big\{\frac{109}{21}(\lambda^2 - 1) - \frac{P}{84}(53\lambda^3 + 139\lambda^2 - 3905\lambda - 319)\Big\}$$
$$-\frac{i}{32\lambda}(\lambda - 1)\cos\beta + \frac{iRE^{*2}(\varepsilon - 1)^2}{16(\varepsilon + 1)}\frac{(\lambda + 1)^2}{\lambda(\lambda - 1)}$$

$$\widehat{C}_3 = \frac{iR\sin^2\beta}{(2880)(336)\lambda(\lambda + 1)}\Big\{\frac{P}{4}(-\lambda^3 + 11\lambda^2 + 3245\lambda + 265)$$
$$+(425\lambda^2 + 238\lambda - 359)\Big\} - \frac{i}{160\lambda}(\lambda + 1)\cos\beta$$

$$\widehat{C}_4 = \frac{iR\sin^2\beta}{(2880)(12)\lambda(\lambda + 1)}\Big\{\frac{P}{56}(71\lambda^3 + 359\lambda^2 - 4125\lambda - 337)$$
$$-\frac{65}{3}(\lambda^2 - 1)\Big\} + \frac{7i}{480\lambda}(\lambda - 1)\cos\beta - \frac{iRE^{*2}(\varepsilon - 1)^2}{48(\varepsilon + 1)}\frac{(\lambda + 1)^2}{\lambda(\lambda - 1)}$$

Substitute from equations (75), (77), (83), (84), and (90) into the condition (82) yields

$$c_1 = iJ \tag{92}$$

where

$$J = \frac{(1 - \lambda)}{(240)(\lambda + 1)^3}\Big[\frac{PR\sin^2\beta}{12096}\Big\{3P(71\lambda^3 + 625\lambda^2 - 625\lambda + 71)$$
$$-\frac{3640(\lambda^2 - 1)}{P}\Big\} + 7(\lambda^2 - 1)\cos\beta\Big] + \frac{RE^{*2}(\varepsilon - 1)^2}{24(\varepsilon + 1)} \tag{93}$$

in which $E^{*2} = \varepsilon_1 E_{01} E_{02}$ and $\varepsilon = \varepsilon_2/\varepsilon_1$. One can proceed further with the systematic procedure of approximation, but equations (92) and (93) are sufficient as a criterion for instability against long waves.

6. STABILITY DISCUSSION AND CONCLUSIONS

In this work, the stability discussion can be obtained in view of equations (92) and (93), and it is clear that the system will be unstable if $J > 0$, otherwise, it is stable. Examination of equation (93) shows that the term containing $\cos\beta$ arises from gravity normal to the boundaries, and its stabilizing effect is well recognized for $\lambda \lessgtr 1$, respectively. The term containing P^{-1} in the bracket of equation (93) arises from the convective terms in equations (85) and (86), so that these convective are destabilizing for $\lambda \lessgtr 1$, respectively. This isolates the longitudinal body-force terms in equations (85) and (86) as the cause of instability. But this instability would not have a chance to manifest itself without the conductivity discontinuity at the interface, which gives rise to the term on the right-hand side of equation (40). This term in absence of electric fields is crucial, so without it the calculation could not be started, and the long-wave instability would not exist. We note also, from equation (93), that the presence of electric fields makes the calculations possible for all values of λ (inclusing the case $\lambda = 1$), since $J \neq 0$ in this case, and this shows the destabilizing effect of electric fields in the absence or presence of thermal conductivity variation.

Now, to see the effects of various parameters (E^*, P, R, β, and ε) included in the analysis, we draw the quantity J given by equation (93) as a function of the thermal conductivities ratio λ, and the obtained results are illustrated in Figs. (1)-(10). Fig. (1) shows the variation of J versus λ for various values of the inclination angle β in the case of absence of the applied electric fields, i.e. when $E^* = 0$ with small values of P and R. It is clear from this figure that when $\beta = 0$ (horizontal planes), J increases by increasing λ and reaches its maximum value at $\lambda = 1$, and it decreases afterwords by increasing λ. Thus the system is unstable for $0 < \lambda < 1$, and it is stable for $\lambda > 1$. For any other values of $0 < \beta < \pi/2$, the quantity J behaves as the case when $\beta = 0$, but it has higher values for $0 < \lambda < 1$ and $\lambda > 1$, respectively. Therefore, the angle of inclination β has a destabilizing effect in the two regions of λ. When $\beta = \pi/2$ (vertical planes), we found that J decreases by increasing λ; hence the system is stable for all values of $\lambda > 0$; while $J = 0$ usually at $\lambda = 1$, and $E^* = 0$, and this can be observed from equation (93).

Figs. (2) and (3) are drawn for the same system considered in Fig. (1), for small values of P and R when $E^* = 5$, in the cases when $\varepsilon = 0.5$ and $\varepsilon = 2$, respectively. Comparing the obtained figures with Fig. (1), we found that the curves behave in the same manner, but with higher values of J . Therefore, from Fig. (2), it is clear that the electric field parameter E^* has a destabilizing effect; and Fig. (3) indicates that the dielectric constant ratio ε has also a destabilizing effect on the system. The obtained result from Fig. (2) can be confirmed using equation (93), since the electric field term in this equation is positive, then J increases by increasing E^* values.

Figs. (4) and (5) show the variation of J versus λ for various values of the inclination angle β in the presence of electric fields with high values of P, R and E^*, when $\varepsilon = 0.5$ and $\varepsilon = 2$, respectively. It is clear from Fig. (4) that the system is unstable as well as stable when $\beta = 0$ (horizontal planes), as discused above in Figs. (1)-(3), but for any other value of β (including the case of vertical planes) the system is stable for small values of λ ($0 < \lambda \ll 1$), and then stable for other values less than 1 at which the system is neutrally stable after which the system is stable for $\lambda > 1$ values. we observe from Fig. (4) also that the angle of inclination β has a destabilizing effect for $0 < \lambda < 1$, and a stabilizing effect for $\lambda > 1$.

Fig. (5) shows that J has higher values than the corresponding values obtained in Fig. (4) by increasing the value of ε. Therefore, the dielectric constant ratio has a destabilizing effect on the system, which confirms the result obtained from Fig. (3). Fig. (6) shows the variation of J with λ for various values of Prandtl number P with small values of R and E^*. It is clear that, for $0 < \lambda < 0.2$, the Prandtl number P has a destabilizing effect as J increases by increasing P, while in the ranges $0 < \lambda < 1$ and $\lambda > 1$, we found that the Prandtl number P has a stabilizing effect at any value of λ within these intervals, and that the obtained curves coincide at the neutral stability point at which $\lambda = 1$. Fig. (7) is drawn for the system considered in Fig. (6) but with high values of R and E^*, and it shows similar behavior of P to that obtained in Fig. (6), and the only difference that for values of $0 < \lambda < 0.1$ the curves decreases by increasing λ, and the resulting destabilizing effect is more faster than its effect in the previous case; also the stabilizing effect afterwards for values of $\lambda < 1$ is also more faster than that obtained in Fig. (6).

Figs. (8) and (9) show the variation of J with λ for different values of the Reynolds number R in absence and prence of electric fields. It is clear from Fig. (8), as explained before, that Reynolds number R, when $E^* = 0$ has a destabilizing effect in the region $0 < \lambda < 1$, while it has a stabilizing effect in the other region $\lambda > 1$. Similar behavior of Reynolds number R when small electric values are taken into account, but for higher values of electric field parameter E^*, we found that Reynolds number R has a destabilizing effect. Finally, Fig. (10) shows that the electric field parameter E^* has a destabilizing effect at any value of λ since

J increases by increasing E^*, in accordance with the previous results obtained in Fig. (2).

In conclusion, we can summerize the main results in view of equations (92), (93), and the obtained figures in the case of absence of electric fields as follows:

(i) For vertical boundaries, the term containing $\cos\beta$ drops out, and if the Prandtl number P is not extremely small, the flow is unstable for λ small. In this case, the lower fluid is the colder fluid.

(ii) For $\lambda < 1$ and $(1 - \lambda)$ small, the flow is unstable, if P and $\sin\beta$ are not very small.

(iii) For $\lambda > 1$, the flow is stable.

(iv) There is a range of λ within $1 < \lambda < \infty$, for which the flow is stable.

(v) For given values of P and β, and a given λ less than one, if the multiplier of R in equations (92) and (93) is positive, the critical R is obtained by setting the quantity within the brackets in equation (93) equal to zero, which gives

$$R_c = 7(1-\lambda^2)\cos\beta\left[\frac{P\sin^2\beta}{12096}\left\{3(71\lambda^3+625\lambda^2-625\lambda+71) + \frac{3640(1-\lambda^2)}{P}\right\} + \frac{10RE^{*2}(\varepsilon-1)^2(\lambda+1)^2}{(\varepsilon+1)(1-\lambda)}\right]^{-1} \tag{94}$$

(vi) The inclination angle β has a destabilizing effect on the considered system. The system is unstable for $\lambda < 1$, and stable for $\lambda > 1$, whereas when $\lambda = 1$, all the curves coincide at $J = 0$.

(vii) The effects of thermal conductivities on the stability of superposed horizontal layers of immiscible fluids studied by Renardy[37], indicate that the stratification in thermal conductivities never causes any instability as it does here. Thus, the present study brings to light an entirely new cause of hydrodynamic instability, and this instability is more effective in the presence of the applied vertical electric fields.

Observations (ii) and (iii) show that for small $|\lambda - 1|$, the flow is unstable if the less conducting fluid is on top, and stable if it is at the bottom. This rather intriguing point, together with observations (i) and (iv), indicates the rather complex effect of conductivity variation on the stability of the flow.

While in the case of presence of electric fields, we conclude that

(1) For small values of P and R, both the electric field and dielectric constants ratio have dstabilizing effects for any value of λ and β. The system behaves in the same manner as the case of absence of electric fields, except that $J > 0$ in this case at $\lambda = 1$.

(2) For high values of P and R, the inclination angle β has a destabilizing effect for $\lambda < 1$, and a stabilizing effect for $\lambda > 1$. For horizontal boundaries ($\beta = 0$), the system is unstable for $\lambda < 1$, and it is stable for $\lambda > 1$, while for any other value of $\beta > 0$, the system is stable and then unstable for $\lambda < 1$, and it is usually stable for $\lambda > 1$.

(3) For any electric field and/or dielectric constants ratio values, the Prandtl number P has a destabilizing and then stabilizing effects for $\lambda < 1$, whereas it has a stabilizing effect for $\lambda > 1$. In this case, the system is unstable as well as stable for $\lambda < 1$, while it is only stable for $\lambda > 1$, and the Prandtl number P has no effect on the stability of the considered system when $\lambda = 1$.

(4) The Reynolds number R, when $E^* = 0$, has a destabilizing effect for $\lambda < 1$, and a stabilizing effect for $\lambda > 1$. and it has no effect on the stability of the system when $\lambda = 1$. In this case, the system is stable as well as unstable for $\lambda < 1$, and it is only stable for $\lambda > 1$. In the case when $E^* \neq 0$, The Reynolds number R has usually a destabilizing effect on the considered system.

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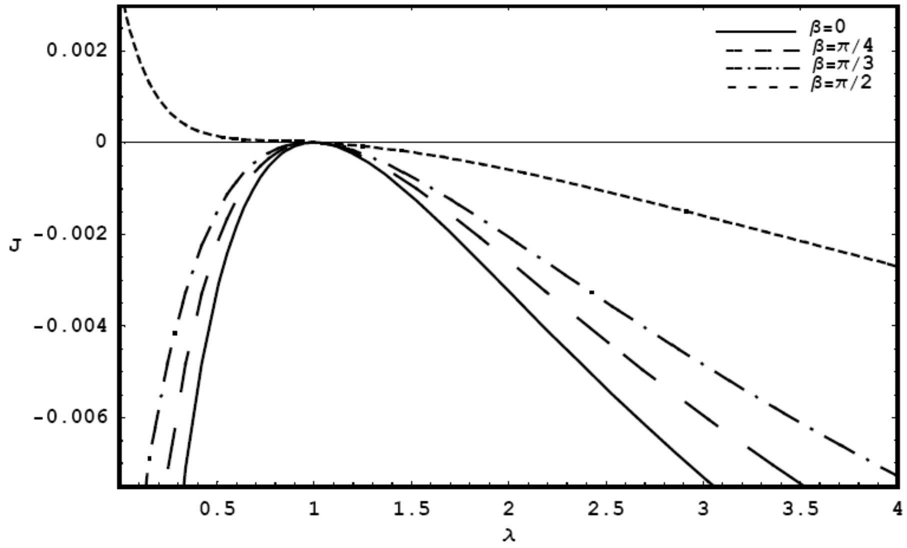

Figure 1
Variation of J with λ for Various Values of β, if, $\varepsilon = 2$, $R = 2$, $P = 6$, $E^* = 0$, when $\beta = 0$, $\pi/4$, $\beta = \pi/3$, and $\pi/2$, Respectively

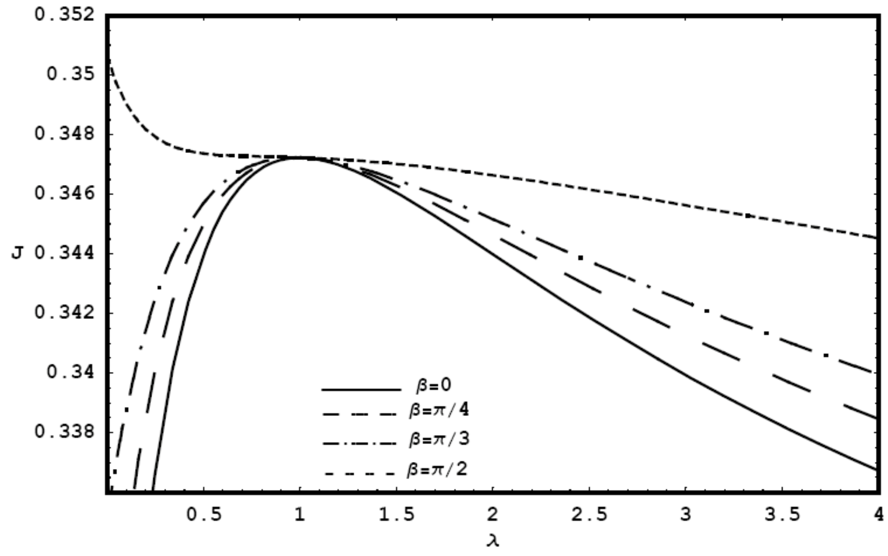

Figure 2
Variation of J with λ for Various Values of β, if, $\varepsilon = 0.5$, $R = 2$, $P = 6$, $E^* = 5$, when $\beta = 0$, $\pi/4$, $\beta = \pi/3$, and $\pi/2$, Respectively

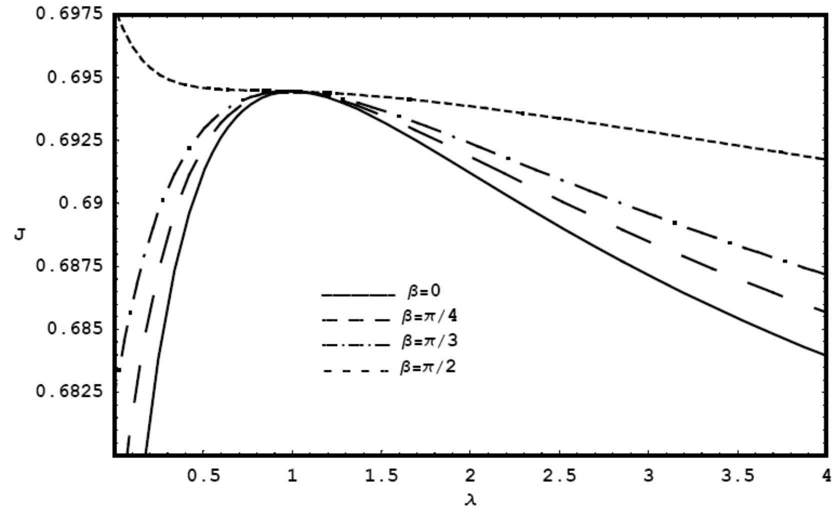

Figure 3
Variation of J with λ for Various Values of β, for the Same System Considered in Fig. 2, but with $\varepsilon = 2$

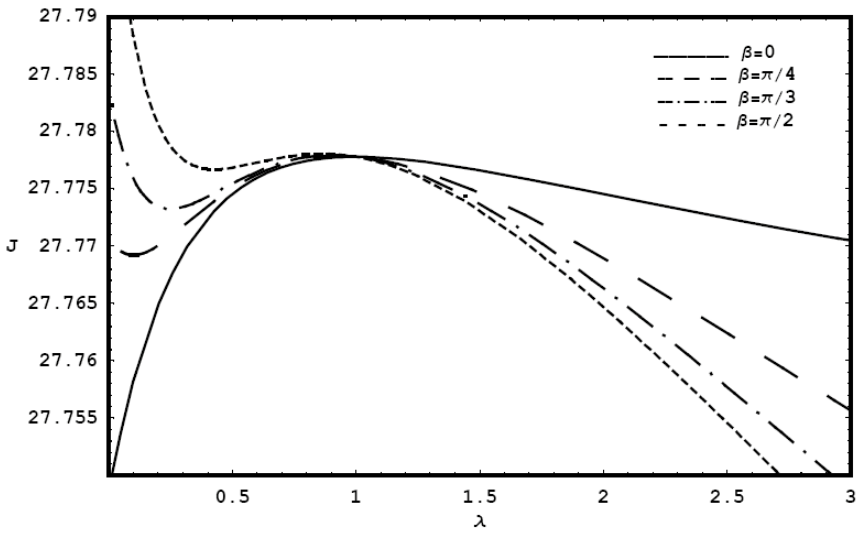

Figure 4
Variation of J with λ for Various Values of β, if, $\varepsilon = 0.5$, $R = 10$, $P = 20$, $E^{*} = 20$, **when** $\beta = 0, \pi/4, \pi/3$ **and** $\pi/2$, **Respectively**

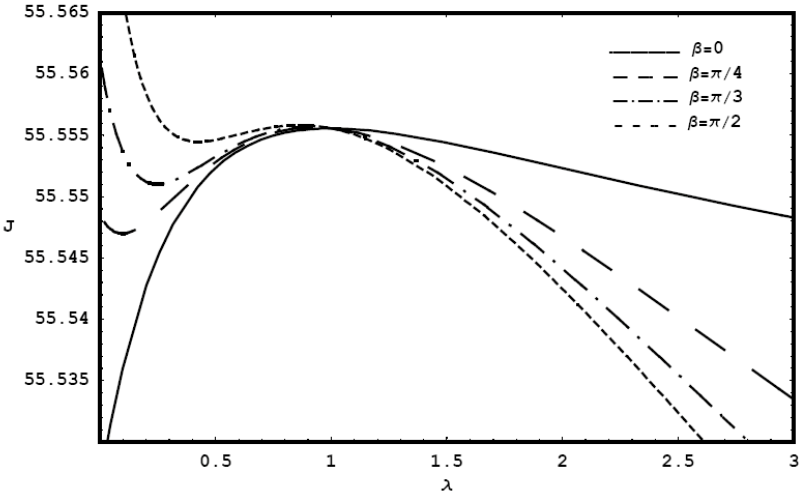

Figure 5
Variation of J with λ for Various Values of β, for the Same System Considered in Fig. 4, but with $\varepsilon = 2$

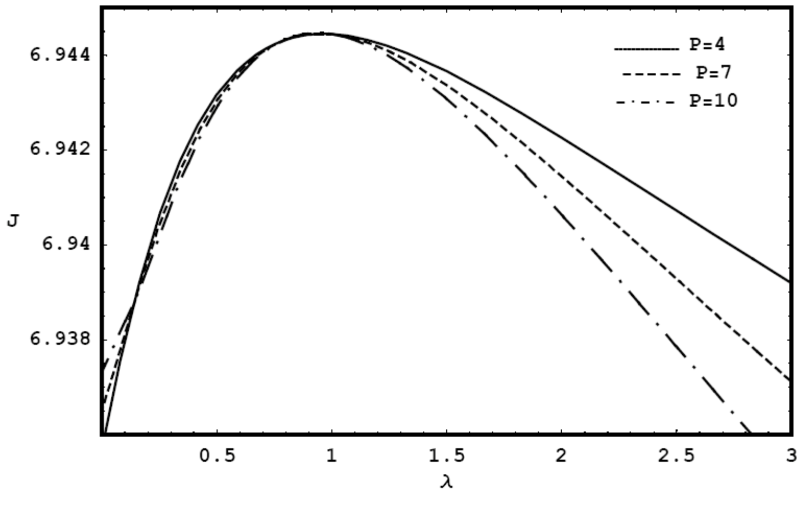

Figure 6
Variation of J with λ for Various Values of P, if, $\varepsilon = 2, \beta = \pi/3$, $E^* = 10$, $R = 5$, when $P = 4$, 7 and 10, Respectively

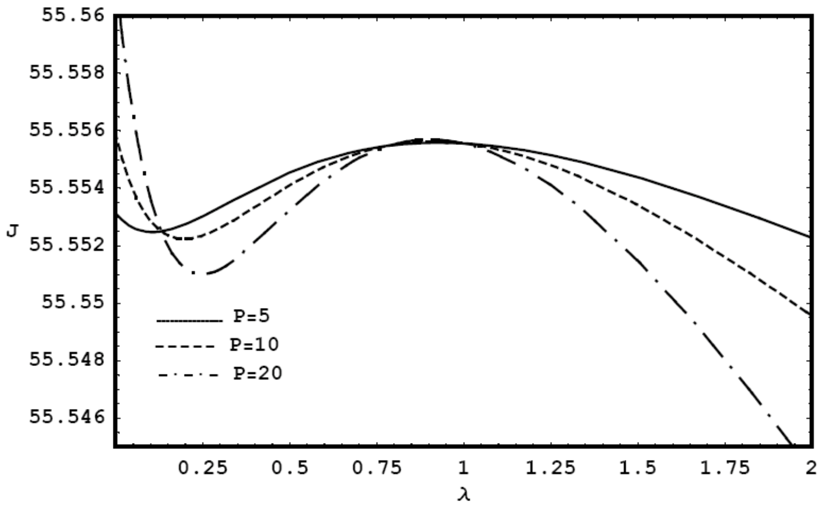

Figure 7
Variation of J **with** λ **for Various Values of** P**, if,** $\varepsilon = 2, \beta = \pi/3$, $E^* = 20$, $R = 10$**, when** $P = 5$, 10 **and** 20**, Respectively**

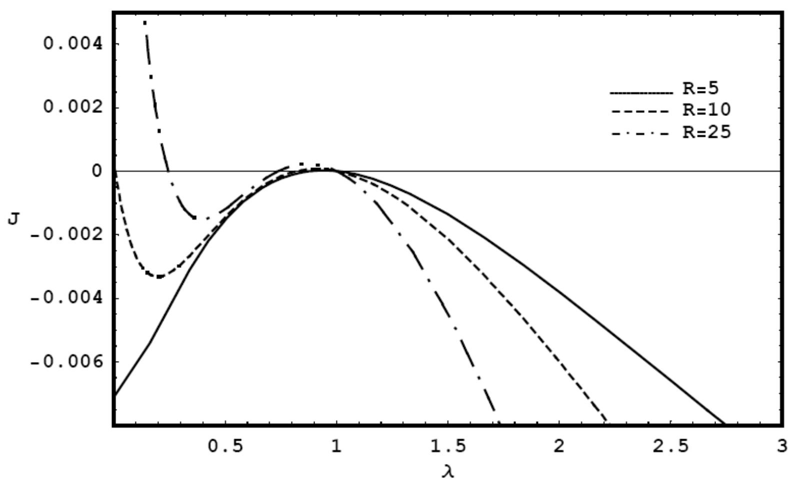

Figure 8
Variation of J **with** λ **for Various Values of** R**, if,** $\varepsilon = 2, \beta = \pi/3$, $P = 10$, $E^* = 0$**, when** $R = 5$, 10 **and** 25**, Respectively**

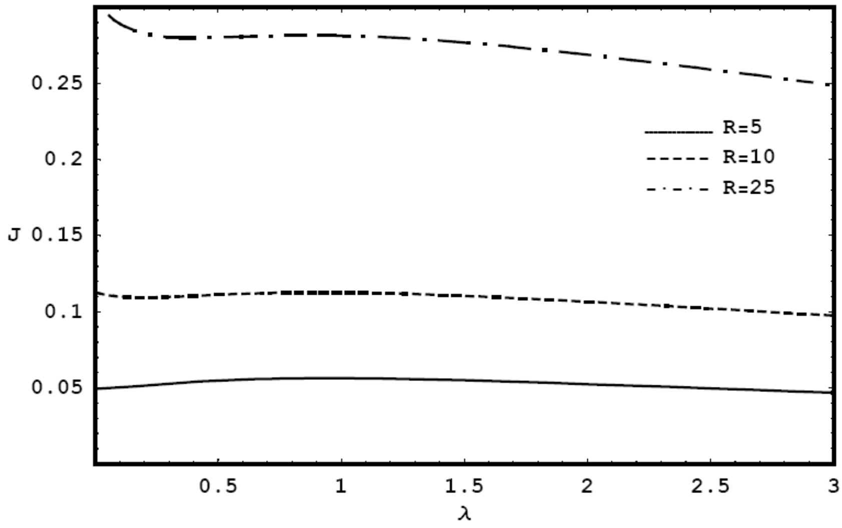

Figure 9
Variation of J with λ for Various Values of R, for the Same System Considered in Fig. 8, but with $E^* = 0.9$

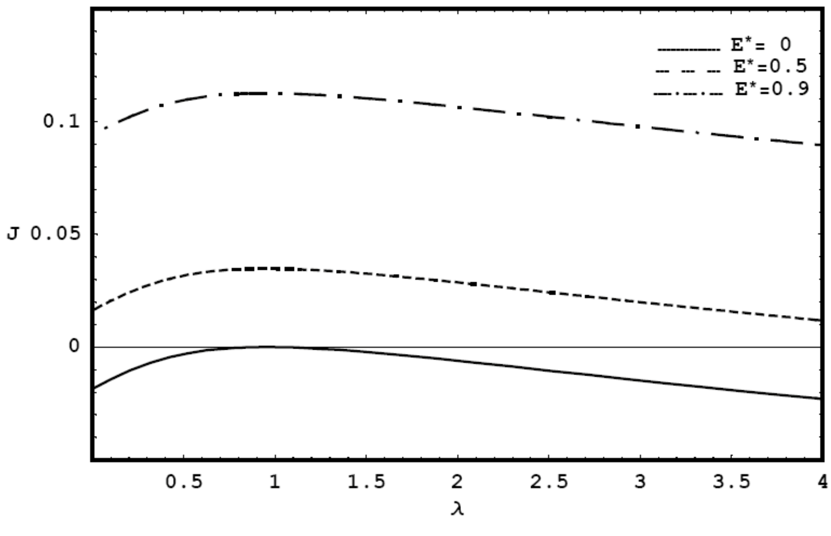

Figure 10
Variation of J with λ for Various Values of E^*, if, $\varepsilon = 2, \beta = \pi/6, P = 20, R = 10$**, when** $E^* = 0, 0.5$ **and** 0.9**, Respectively**