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The Multi-Soliton Solutions to The KdV Equation by Hirota Method

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Abstract: The Hirota bilinear method is used to solve the KdV model. As a result, the exact expression of multi-soliton solutions of the KdV equation is obtained.
Key words: Nonlinear partial differential equations; The KdV equation; Hirota bilinear method; Multi-soliton solution

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1. INTRODUCTION

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear partial differential equation [PDE]. It is well known that there are infinitely many solutions for every nonlinear equation, such as the tanh-function method[1-2], the Jacobi elliptic function expansion method[3-4], the F-expansion method[5-8], sin-cosin method[9-10], the homogeneous balance method[11-13], and the Hirota bilinear method[14-17]. Among them, the Hirota bilinear method is one of the most effectively methods for constructing exact solutions to PDEs. In this paper, the Hirota bilinear method is used to solve the KdV equation. Our aim in this paper is to investigate multi-soliton solutions of the KdV equation by the Hirota bilinear method.

We consider the the KdV equation[18]

$$u_t+auu_x+buu_{xxx}=0.$$

2. THE HIROTA BILINEAR METHOD

Definite the Hirota bilinear operator has the form

$$D_t^m D_x^n (g\cdot f)=[(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})^m(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})g(t,x)f(t,x)]\Big|_{t=t',x'=},$$

where m, n are nonnegative integer.

Hence

$$D_t(g\cdot f)=g_t f - g f_t, \quad (1)$$

$$D_x(g\cdot f)=g_z f - g f_x, \quad (2)$$

$$D_x^2(g\cdot f)=g_{xx} f - 2 g_x f_x + f g_{xx}. \quad (3)$$

In particular

$$D_x^2(f\cdot f)=2(f_{xx} f - f_x^2). \quad (4)$$

Let $w=\frac{G}{F}$,

hence

$$w_t=\frac{D_t(G\cdot F)}{F^2}, \quad (5)$$

$$w_x=\frac{D_x(G\cdot F)}{F^2}, \quad (6)$$

$$w_{xx}=\frac{D_x^2(G\cdot F)}{F^2}-\frac{G}{F}\frac{D_x^2(F\cdot F)}{F^2}, \quad (7)$$

$$w_{xxx}=\frac{D_x^3(G\cdot F)}{F^2}-3\frac{D_x(G\cdot F)}{F^2}\frac{D_x^2(F\cdot F)}{F^2}. \quad (8)$$

3. SOLUTION OF THE KDV EQUATION

We consider the generalized KdV Equation (1), let $u=w_x$, we have

$$w_{xt}+aw_x w_{xx}+bw_{xxxx}=0. \quad (9)$$

By integrated two side of the Equation (9) with respect to x, we derive

$$w_t+\frac{a}{2}w_x^2+bw_{xxx}=0. \quad (10)$$

Again, Let $w=\frac{G}{F}$, use Formula (1) to (8) and arrangement, we derive

$$\frac{D_t(G\cdot F)+bD_x^3(G\cdot F)}{F^2}+\frac{\frac{a}{2}D_x(G\cdot F)D_x(G\cdot F)-3bdD_x(G\cdot F)D_x^2(F\cdot F)}{F^4}=0\cdot \quad (11)$$

Let the coefficient of F are zero, we have equations

$$D_t(G\cdot F)+bDx^3(G\cdot F)=0, \quad (12)$$

$$D_x(G\cdot F)[\frac{a}{2}D_x(G\cdot F)-3bD_x^2(F\cdot F)]=0 \quad (13)$$

Again, we let $G=mF_x$, by Equation (13), we derive the conditions

$$a=\frac{12b}{m} \quad (14)$$

Below, by aid of the conditions (14), we consider the special case of Equation (1).

Case 1 when m=1, then a=12b, so the equation (1) become

$$u_t+12buu_x+bu_{xxx}=0. \quad (15)$$

We consider the Equation (15) and give the exact expression of multi-soliton solutions of the Equation (15).

By Equation (15), we obtain the Hirota equation

$$D_t (G \cdot F)+bD_x^3(G \cdot F)=0.$$

i.e.

$$D_x (D_t+bD_x^3) (F \cdot F)=0. \tag{16}$$

Firstly, we suppose F_0=const, $F_1=e^{k1x+\omega_1 t+\delta_1}$ are the solutions, substitute in to Equation (15), we have

$$D_x (D_t+bD_x^3) (F_0 \cdot F_1)=F_0 D_x (D_t+bD_x^3) (F_1)$$
$$=F_0 D_x (D_t+bD_x^3) (e^{k1x+\omega_1 t+\delta_1})$$
$$=F_0 k_1(\omega_1+bk_1^3) (e^{k1x+\omega_1 t+\delta_1})=0.$$

Then the dispersion relation of the Equations (15)

$$\omega_1=-bk_1^3.$$

Secondly, we let

$$F = \sum_{j=1}^{n} e^{\theta_j}, \theta_j = k_j x + \omega_j t + \delta_j,\ \omega_1 = -k_1^5.$$

where k_j,δ_j are arbitrary constant. Also let

$$F(x,t) = \sum_{n=1}^{\infty} \varepsilon^n F_n(x,t).$$

Substitute in to Equation (16), we have

$$D_x(D_t + bD_x^3)(F \cdot F) = D_x(D_t + bD_x^3)[\sum_{n=0} \varepsilon^n F_n(x,t) \sum_{n=0} \varepsilon^n F_n(x,t)]$$
$$= D_x(D_t + bD_x^3)[\sum_{n=0}^{\infty} \varepsilon^n \sum_{m+l=n}^{\infty} F_m(x,t)F_l(x,t)] = 0$$
$$= \sum_{n=0}^{\infty} \varepsilon^n \sum_{m+l=n}^{\infty} D_x(D_t + bD_x^3)(F_m(x,t)F_l(x,t)) = 0.$$

Let the coefficient of ε^n is zero, then

$$D_x(D_t + bD_x^3)(\sum_{m+l=n}^{\infty} F_m F_l) = 0\,, n > 0\,.$$

i.e.

$$2D_x(D_t + bD_x^3)(F_n \cdot F_o) + D_x(D_t + bD_x^3)\sum_{m=0}^{n} F_{n-m}(x,t)F_m = 0. \tag{17}$$

When n=1, the reduction Equation (17) is

$$2D_x(D_t+bD_x^3) (F_1 \cdot F_0)=0. \tag{18}$$

Suppose that $F=F_0+F_1$ where $F_0=1$, $F_1=\mathrm{e}^{k1x+\omega_1 t+\delta_1}$, $\omega_1=-dk_1^5$, we have the single soliton solution to the equations (15) is

$$u = w_x = \frac{\partial^2 (\ln F)}{\partial x^2} = \frac{\partial(\frac{F_x}{F})}{\partial x} = \frac{FF_{xx} - F_x^2}{F^2} = \frac{k_1^2 e^{k_1 x+\omega_1 t+\delta_1}}{(1+e^{k_1 x+\omega_1 t+\delta_1})^2}. \tag{19}$$

When n=2, the reduction Equation (17) is

$$2D_x (D_t+bD_x^3) (F_2 \cdot F_0)+D_x(D_t+bD_x^3) (F_1 \cdot F_1)=0.$$

i.e.

$$2D_x(D_t+bD_x{}^3)(F_2\cdot F_o)=D_x(D_t+bD_x{}^3)(F_1\cdot F_1)$$

$$=-D_x(D_t+bD_x{}^3)(\sum_{i=1}^{N}e^{k_ix+\omega_it+\delta_i}\sum_{j=1}^{N}e^{k_jx+\omega_jt+\delta_j})$$

$$=-D_x(D_t+bD_x{}^3)(\sum_{i=1}^{N}\sum_{j=1}^{N}e^{k_ix+\omega t_it+\delta_i}e^{k_1x+\omega_1t+\delta_{ij}})$$

$$=-\sum_{i=1}^{N}\sum_{j=1}^{N}D_x(D_t+bD_x{}^3)(e^{k_ix+\omega_jt+\delta_i}e^{k_jx+\omega_jt+\delta_j})$$

$$=-\sum_{i=1}^{N}\sum_{j=1}^{N}(k_i-k_j)[(\omega_j-\omega_i)+b(k_i-k_j)^3](e^{k_ix+\omega_jt+\delta_i}e^{k_jx+\omega_jt+\delta_j})$$

$$=-2\sum_{1\le i<j\le N}[(k_j-k_i)[(w_i-w_j)+b(k_i-k_j)^3](e^{k_ix+\omega_jt+\delta_i}e^{k_jx+\omega_jt+\delta_j}). \tag{20}$$

Suppose that $F=F_0+F_1+F_2+\frac{1}{F_0}a_{12}F_1F_2$ where $F_0=1, F_1=e^{k_1x+\omega_1t+\delta_1}$, $F_2=e^{k_2x+\omega_2t+\delta_2}$,

$$\omega_1=-bk_1{}^3,\quad a_{12}=-\frac{(k_i-k_j)[(w_i-w_j)+b(k_i-k_j)^3]}{(k_i+k_j)[(w_i+w_j)+b(k_i+k_j)^3]},$$

we have the double soliton solution to the Equations (15) is

$$u=w_x=\frac{\partial^2(\ln F)}{\partial x^2}=\frac{\partial(\frac{F_x}{F})}{\partial x}=\frac{FF_{xx}-F_x{}^2}{F^2}$$

$$=\frac{k_1{}^2e^{k_1x+\omega_1t+\delta_1}+k_2{}^2e^{k_2x+\omega_2t+\delta_2}+a_{12}(k_1+k_2)^2e^{(k_1+k_2)x+(\omega_1+\omega_2)t+(\delta_1+\delta_2)}}{1+e^{k_1x+\omega_1t+\delta_1}+e^{k_2x+\omega_2t+\delta_2}+a_{12}e^{(k_1+k_2)x+(\omega_1+\omega_2)t+(\delta_1+\delta_2)}}$$

$$=\frac{(k_1e^{k_1x+\omega_1t+\delta_1}+k_2e^{k_2x+\omega_2t+\delta_2}+a_{12}(k_1+k_2)e^{(k_1+k_2)x+(\omega_1+\omega_2)t+(\delta_1+\delta_2)})^2}{(1+e^{k_1x+\omega_1t+\delta_1}+e^{k_2x+\omega_2t+\delta_2}+a_{12}e^{(k_1+k_2)x+(\omega_1+\omega_2)t+(\delta_1+\delta_2)})^2}. \tag{21}$$

We suppose the N- soliton solution to the Equations (15) is

$$F=\sum_{\substack{\mu_i=0,1\\1\le i\le n}}e^{\sum_{1\le i\le j\le n}a_{ij}\mu_i\mu_j+\sum_{i=1}\mu_i\eta_i}.$$

Case 2 when m=2, then a=6b, so the Equation (1) become

$$u_t+6buu_x+buu_{xxx}=0. \tag{22}$$

Remark: when b=1, the Equation (22) is the usual KdV equation

$$u_t+6uu_x+uu_{xx}=0. \tag{23}$$

We consider the Equation (22) and give the exact expression of multi-soliton solutions of the Equation (22).

By Equation (16), we obtain the Hirota equation

$$D_t(G\cdot F)+bD_x^3(G\cdot F)=0.$$

i.e.

$$D_x(D_t+bD_x^3)(F\cdot F)=0.$$

So we can repeat case 1 and derive the single soliton solution, the double soliton solution and the N- soliton solution to the Equations (22).

Case 3 when m=3, then a=4b, so the equation (1) become

$$u_t+4buu_x+buu_{xxx}=0. \quad (24)$$

Case 4 when m=4, then a=3b, so the equation (1) become

$$u_t+3buu_x+buu_{xxx}=0. \quad (25)$$

Case 5 when m=6, then a=2b, so the Equation (1) become

$$u_t+2buu_x+buu_{xxx}=0. \quad (26)$$

Case 6 when m=12, then a=b, so the equation (1) become

$$u_t+buu_x+buu_{xxx}=0. \quad (27)$$

Remark: when b=1, the Equation (27) is the usual KdV equation

$$u_t+uu_x+uu_{xxx}=0. \quad (28)$$

Case 7 when m=12, then a=b, so the Equation (1) become

$$u_t+buu_x+buu_{xxx}=0. \quad (29)$$

Remark: when b=1, the Equation (29) is the usual KdV equation

$$u_t+uu_x+uu_{xxx}=0. \quad (30)$$

CONCLUSION

In this paper, we have used the Hirota bilinear method to solve the KdV model. It is significant to observe the condition (14) that the Hirota bilinear method can be used to obtain solutions.

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