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A Method to Get the General Solutions of Linear Differential Equation or Equations with Constant Coefficients Based on Laplace Transformation

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Abstract

In management science research, a large number of cases analysis often require from qualitative analysis to quantitative analysis, differential equation and differential equations are often used in quantitative analysis. The core of quantitative analysis of a case is to build a corresponding mathematical model. The differential equation models are widely used mathematical models. In particular, linear differential equation models with constant coefficients are often used mathematical models in the quantitative analysis of cases in management science research. In practice, the particular solution of linear differential equation model with constant coefficients often been found by Laplace transformation. Follow this way, a method to get the general solutions of linear differential equation with constant coefficients based on Laplace transformation be given. In this method, it is easy to find the general solution of homogeneous linear differential equation, of non-homogeneous linear differential equation and of linear differential equations with constant coefficients based on Laplace transformation.

Key words: Management; Mathematical model; Linear differential equation; Laplace transformation

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INTRODUCTION

In management science and engineering, a large number of cases analysis changes from qualitative analysis to quantitative analysis (Drucker, 2008). The core of quantitative analysis is to build a corresponding mathematical model (Jiang, 1993). Differential equation model is a widely used mathematical model. Quantitative analysis of many cases needs to use differential equations (Fulford, 1997). In management science, that quantitative analysis of cases commonly used mathematical models are the linear differential equations with constant coefficients (Du & Wu, 2008).

The linear differential equation with constant coefficients can be divided into two categories: the homogeneous linear differential equation with constant coefficients and the homogeneous linear differential equation with constant coefficients. There are many ways to solve the general solutions of the homogeneous linear differential equation with constant coefficients. Each method has its own advantages (Dept. of Mathematics, Tongji University., 1997; Zhang, 2009; Guo, 1979; Fang, 1998; Spiegel, 1978). There are many ways to solve the general solutions of the non-homogeneous linear differential equation with constant coefficients. First find the general solution $X(t)$ of the corresponding homogeneous linear differential equation with constant coefficients, then find a particular solution $x^*(t)$ of the non-homogeneous linear differential equation with constant coefficients, So to get the general solution $x(t)= X(t)+x^*(t)$ of the non-homogeneous linear differential equation with constant coefficients.

For the management of science workers, above method often failings. Because it is sometimes too tricky or too complicated computation. It sometimes can only be solved through computer (Quinney, 1985).

In practice, we often find the particular solution $x^*(t)$ by Laplace transform method (Zhang, 2009; Guo, 1979; Spiegel, 1978). Follow this way, we give a quick and easy method to solve the general solutions of linear differential equation with constant coefficients. This method will only require knowledge of elementary algebra factorization and Laplace transformation table. In this way, It is easy to find the general solution of non-homogeneous linear differential equation with constant coefficients, the general solution of homogeneous linear differential equation with constant coefficients and the general solutions of linear differential equations with constant coefficients. The method is particularly suitable for management science and engineering workers. The following describes the techniques used in the method.

1. TO GET THE GENERAL SOLUTIONS OF NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

General form of non-homogeneous linear differential equation with constant coefficients:

$$\begin{cases} a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = f(t) \\ x^{(n-1)}(t_0), \cdots x'(t_0), x(t_0) \end{cases} \tag{1}$$

How to quickly find the general solution of the non-homogeneous linear differential equation (1) with constant coefficients and initial conditions? First, Laplace transform be applied to the equation (1), the equation (1) that include an unknown function $x(t)$ be became an algebraic equation to contain an unknown image function $\boldsymbol{X}(s)$ as follow:

$$\boldsymbol{X}(s) = \boldsymbol{F}(s) \tag{2}$$

Second, solve this image function $\boldsymbol{X}(s)$ from the equation (2). Third, Laplace inverse transform be applied to this image function $\boldsymbol{X}(s)$, we obtain the general solution $x(t)$ of the non-homogeneous linear differential equation (1) with constant coefficients and initial conditions.

This method has the advantage as long as using the Laplace transform table and along with some knowledge of elementary algebra factorization, we could easily find the general solution of the non-homogeneous linear differential equation (1) with constant coefficients and initial conditions.

Example 1.1. Find the general solution of the second-order non-homogeneous linear differential equation (3) with constant coefficients and initial conditions.

$$x''(t) - x(t) = 3t - 2 + 4e^t,\ x'(0),\ x(0) \tag{3}$$

Solution. We take the Laplace transform in equation (3) as follow:

$$L\,[x''(t) - x(t)] = L\,[3t - 2 + 4e^t]$$

The equation (3) to contain an unknown function $x(t)$ be became an algebraic equation to contain an unknown image function $\boldsymbol{X}(s)$ as follow:

$$s^2X(s) - sx(0) - x'(0) - X(s) = 3/s^2 - 2/s + 4/(s - 1)$$

Simplification introduced the image function $\boldsymbol{X}(s)$ as follow:

$$X(s) = -\frac{3}{s^2} + \frac{1}{s-1}\left[\frac{x(0)}{2} + \frac{x'(0)}{2} - \frac{1}{2}\right] + \frac{1}{s+1}\left[\frac{x(0)}{2} - \frac{x'(0)}{2} - \frac{3}{2}\right] + \frac{2}{s} + \frac{2}{(s-1)^2}$$

Laplace inverse transform be applied to this image function $\boldsymbol{X}(s)$, we obtain $x(t)$ as follow:

$$x(t) = -3t + e^t[x(0)/2 + x'(0)/2 - 1/2] + e^{-t}[x(0)/2 - x'(0)/2 - 3/2] + 2 + 2te^t$$

We obtain the general solution $x(t)$ of the non-homogeneous linear differential equation (3) with constant coefficients as follow: $x(t)=C_1 e^t + C_2 e^{-t} + 2te^t - 3t + 2$.

2. TO GET THE GENERAL SOLUTIONS OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

General form of homogeneous linear differential equation with constant coefficients as follow:

$$\begin{cases} a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = 0 \\ x^{(n-1)}(t_0), \cdots x'(t_0), x(t_0) \end{cases} \tag{4}$$

How to quickly find the general solution of the homogeneous linear differential equation (4) with constant coefficients and initial conditions? First, Laplace transform be applied to the equation (4), the equation (4) that include an unknown function $x(t)$ be became an algebraic equation to contain an unknown image function $\boldsymbol{X}(s)$ as follow:

$$\boldsymbol{X}(s) = F(s) \tag{5}$$

Second, solve this image function $\boldsymbol{X}(s)$ from the equation (5). Third, Laplace inverse transform be applied to this image function $\boldsymbol{X}(s)$, we obtain the general solution $x(t)$ of the homogeneous linear differential equation (4) with constant coefficients and initial conditions. See the following example.

Example 2.1. Find the general solution of the second-order homogeneous linear differential equation (6) with constant coefficients and initial conditions.

$$x''(t) + px'(t) + qx'(t) = 0,\ x'(t),\ x'(0) \tag{6}$$

Solution. We take the Laplace transform in equation (6) as follow: $\boldsymbol{L}[x''(t)+ px'(t)+qx(t)]=0$.The equation (6) to contain an unknown function $x\ (t)$ became an algebraic equation to contain an unknown image function $\boldsymbol{X}(s)$ as follow: $(s^2+ps+q)\boldsymbol{X}(s)= sx'(0)+(p+1)x(0)$.Three cases are discussed below.

(1) Let $a,b\in R$ and a,b are non-zero constants and $a\neq b$, if $(s^2+ps+q)=(s-a)(s-b)$,the equation (6) to contain function $x(t)$ be became an algebraic equation to contain an unknown image function $\boldsymbol{X}(s)$ as follow:

$$X(s)=\frac{x'(0)}{s-a}+\frac{bx'(0)+(p+1)x(0)}{(s-a)(s-b)}$$

Laplace inverse transform be applied to this image function $\boldsymbol{X}(s)$, we obtain $x(t)$ as follow:

$$x(t)=x'(0)e^{at}+(e^{at}-e^{bt})\frac{bx'(0)+(p+1)x(0)}{a-b}$$

We obtain the general solution $x(t)$ of the homogeneous linear differential equation (6) with constant coefficients as follow: $x(t)= C_1 e^{at} +C_2 e^{bt}$.

(2) Let $a\in R$ and a is a non-zero constant, if $(s^2+ps+q) = (s-a)^2$, the equation (6) to contain an unknown function $x(t)$ be became an algebraic equation to contain an unknown image function $\boldsymbol{X}(s)$ as follow:

$$X(s)=\frac{x'(0)}{s-a}+\frac{ax'(0)+(p+1)x(0)}{(s-a)^2}$$

Laplace inverse transform be applied to this image function $\boldsymbol{X}(s)$, we obtain $x(t)$ as follow:

$$x(t)=x'(0)e^{at}+te^{at}(ax'(0)+(p+1)x(0))$$

Let $C_1=x'(0)$, $C_2=ax'(0)+(p+1)x(0)$. We obtain the general solution $x(t)$ of the homogeneous linear differential equation (6) with constant coefficients as follow: $x(t)=e^{at}(C_1+C_2 t)$.

(3) Let $a,k\in R$ and a, k are non-zero constants, if $(s^2+ps+q)=(s+a)^2+k^2$, the equation (6) to contain an unknown function $x(t)$ became an algebraic equation to contain an unknown image function $\boldsymbol{X}(s)$ as follow:

$$X(s)=\frac{(s+a)x'(0)}{(s+a)^2+k^2}+\frac{(p+1)x(0)-ax'(0)}{(s+a)^2+k^2}$$

Laplace inverse transform be applied to this image function $\boldsymbol{X}(s)$, we obtain $x(t)$ as follow:

$$x(t)=x'(0)e^{-at}\cos kt+[(p+1)x(0)-ax'(0)]e^{-at}\sin kt/k$$

We obtain the general solution $x(t)$ of the homogeneous linear differential equation (6) with constant coefficients as follow: $x(t)=e^{-at}(C_1 coskt +C_2 sinkt)$.

3. TO GET THE GENERAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

General form of one order linear differential equations with constant coefficients as follow:

$$\begin{cases}\dot{x}(t)=a_1x(t)+b_1y(t)+f(t)\\ \dot{y}(t)=a_2x(t)+b_2y(t)+g(t)\end{cases}\tag{7}$$

How to quickly find the general solution of the one order linear differential equation (7) with constant coefficients? First, Laplace transform be applied to the equation (7), the equation (7) that include unknown function $x(t)$ and $y(t)$ were became algebraic equations to contain unknown image function $\boldsymbol{X}(s)$ and $\boldsymbol{Y}(s)$ as follow:

$$\begin{cases}X(s)=F(s)\\ Y(s)=G(s)\end{cases}\tag{8}$$

Second, solve there image function $\boldsymbol{X}(s)$ and $\boldsymbol{Y}(s)$ from the equation (8). Third, Laplace inverse transform be applied to there image function $\boldsymbol{X}(s)$ and $\boldsymbol{Y}(s)$, we obtain the general solutions $x(t)$ and $y(t)$ of the one order linear differential equation (7) with constant coefficients. See the following example.

Example 3.1. Find the general solution of the one-order linear differential equations (9).

$$\begin{cases}\dot{x}(t)=2x(t)-y(t)+e^t\\ \dot{y}(t)=4x(t)-3y(t)+t\end{cases}\tag{9}$$

Solution We take the Laplace transform in equations (9), the two functions $x(t)$ and $y(t)$ became algebraic equations to contain two unknown image functions X(s) and Y(s) as follow:

$$\begin{cases}sX(s)-x(0)=2X(s)-Y(s)+1/(s-1)\\ sY(s)-y(0)=4X(s)-3Y(s)+1/s^2\end{cases}$$

Simplification introduced the image function $X(s)$ and $Y(s)$ as follow:

$$X(s)=\frac{s+3}{(s+2)(s-1)}x(0)-\frac{y(0)}{(s+2)(s-1)}+$$
$$\frac{s+3}{(s+2)(s-1)^2}-\frac{1}{(s+2)s^2(s-1)}$$

$=[y(0)/3 - x(0)/3 + 7/36]/(s + 2) + [4x(0)/3 - y(0)/3 - 4/9]/(s - 1) + .25/s + .5/s^2 + 4/3/(s - 1)^2$

$Y(s) = 4[y(0)/3 - x(0)/3 + 7/36]/(s + 2) + [4x(0)/3 - y(0)/3 - 7/9]/(s - 1) + 1/s^2 + 4/3/(s - 1)^2$

Laplace inverse transform be applied to image functions $\boldsymbol{X}(s)$ and $\boldsymbol{Y}(s)$, let $C_1= y(0)/3-x(0)/3+7/36$, $C_2= 4x(0)/3- y(0)/3-4/9$, we obtain $x(t)$ and $y(t)$ as follow:

$$\begin{cases}x(t)=c_1e^{-2t}+c_2e^t+4te^t/3+t/2+1/4\\ y(t)=4c_1e^{-2t}+c_2e^t+4te^t/3-e^t/3+t\end{cases}$$

CONCLUSION

With economic development, a large number of problems and cases in management science and engineering changes from qualitative analysis to quantitative analysis. That quantitative analysis of problems or cases commonly used mathematical model is the linear differential equation with constant coefficients. By Laplace transform method, we give a quick and easy method to solve the general solutions of linear differential equation with constant coefficients. This method will only require knowledge of elementary algebra factorization and Laplace transformation table. In this way, it is easy to find the general solution of non-homogeneous linear differential equation with constant coefficients, the general solution of homogeneous linear differential equation with constant coefficients and the general solutions of linear differential equations with constant coefficients. The method is particularly suitable for management science and engineering workers.

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