AN EQUI-AVERAGE MATRIX GAME AND ITS GENERAL SOLUTIONS[1]

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Abstract: In this paper, we study matrix games whose averages of all rows and all columns are the same. A necessary and sufficient condition and a necessary condition for a matrix game being the form are given. A method of solving the form of matrix games by elementary transformations is given as well.
Key words: OR, game theory, equi-average matrix game, average, solutions to a matrix game

INTRODUCTION

As everyone knows, it is very complicated to compute solutions to a general $m \times n$ matrix game [1]. There is no general algorithm except for linear programming method. The paper [2] studied a matrix game with unique Neumann-Shannon game solution that is a refinement of general game solutions. For this form of matrix games, averages of all rows and all columns are equal.

The games appear frequently in intellectual games played by two players. We can consider the following examples

To determine who is required to do the nightly chores, two children first select who will be represented by "same" and who will be represented by "different." Then, each child conceals in her palm a penny either with its face up or face down. Both coins are revealed simultaneously. If they match (both are heads or both are tails), the child "same" wins. If they are different (one heads and one tails), "different" wins. The payoff matrix can be written as

$$\begin{array}{cc} & \text{same} \\ & \begin{array}{cc} \text{heads} & \text{tails} \end{array} \\ \begin{array}{c} \text{different} \\ \\ \end{array} \begin{array}{c} \text{heads} \\ \text{tails} \end{array} & \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \end{array}.$$

The game is quite similar to a three strategy version - rock, paper, scissors.

Two players each make a fist. They count together "1 ... 2 ... Shoot!", "Rock ... Paper ... Scissors ... Shoot!", "Rock ... Paper ... Scissors!", "Scissors... Paper... Stone!", or "Ro ... Sham ... Bo!" while simultaneously bouncing their fists. On "Shoot", "Go", or "Scissors", each player simultaneously changes their fist into one of three hands (or weapons):

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Rock (or Stone): a clenched fist.

Paper (or Cloth): all fingers extended, palm facing downwards, upwards, or sideways (thumb pointing to the sky).

Scissors: forefinger and middle finger extended and separated into a "V" shape.

The objective is to defeat the opponent by selecting a weapon which defeats their choice under the following rules:

1st. Rock smashes (or blunts) Scissors (rock wins)
2nd. Scissors simply cuts Paper (scissors win)
3rd. Paper covers Rock and roughness is covered (paper wins)
If players choose the same weapon, the game is a tie and is played again.

Often the short game is repeated many times so that the person who wins two out of three or three out of five times wins the entire game. The payoff matrix can be written as

$$\begin{array}{c} \\ \mathrm{r} \\ \mathrm{s} \\ \mathrm{p} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{r} & \mathrm{s} & \mathrm{p} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}.$$

The two matrix games have a common property, i.e., averages of all rows and all columns in its payoff matrix are equal.

In ancient China, there was a very famous competition called King Qi's Horse Racing [3] that is a game in the form. By algebraic and liner programming methods, respectively, the papers [4,5] proved that it has infinite number of solutions.

In this paper, we study matrix games, called equi-average matrix games, of which all rows and all columns have the same average. We give a necessary and sufficient condition and a necessary condition for an equi-average matrix game. We also give an elementary transformation method to compute solutions to an equi-average matrix game. The new method is faster and more convenient.

1. AN EQUI-AVERAGE MATRIX AND ITS AVERAGE

Definition 1 If average of every row (resp. column) in $A=(a_{ij})_{m\times n}$ is v , then the matrix is called row (resp. column)- equi-average matrix, and v_r (resp. v_c) is called its row (resp. column)-average.

If a matrix $A=(a_{ij})_{m\times n}$ is row-equi-average and column-equi-average, and $v=v_r=v_c$, then the matrix is called an equi-average matrix with the average v.

Obviously, we have that

Theorem 1 If a row (resp. column)-equi-average matrix is symmetrical, then it is equi-average.

Theorem 2 If a row (resp. column)-equi-average matrix is anti-symmetrical and its average is equal to zero, then it is equi-average.

2. TWO CONDITIONS FOR EQUI-AVERAGE MATRIX GAMES

Let $A=(a_{ij})_{m\times n}$ be the payoff matrix of a matrix game and let

$$I=\{1,2,\cdots,m\},\ J=\{1,2,\cdots,n\}$$

be players 1's and 2's finite sets of pure strategies, respectively. A probability distribution on I (resp. J) is called player 1's (resp. player 2's) mixed strategy. Let X and Y be players 1's and 2's sets of mixed strategies, respectively.

A mixed situation $(x^*,y^*)\in X\times Y$ is called a game solution of a matrix game $A=(a_{ij})_{m\times n}$ if

$$xAy^{*T}\le x^*Ay^{*T}\le x^*Ay^T,\ \forall(x,y)\in X\times Y,$$

where x^* and y^* are called the players 1's and 2's optimal strategies, respectively.

Definition 2 If the payoff matrix of a matrix game is an equi-average matrix with the average v, then the game is called equi-average matrix game with the average v.

Theorem 3 A matrix game $A=(a_{ij})_{m\times n}$ satisfies the logical relation $(1)\Leftrightarrow(2)\Rightarrow(3)$, where

1st. ***A is an equi-average matrix game with the average v.***

2nd. **$(x^0,y^0)=((1/m,\cdots,1/m),\ (1/n,\cdots,1/n))$ and v are game solution and game value, respectively.**

3rd. **(x^*,y^*) is a game solution of A if and only if x^*,y^* satisfies**

$$e_i^mAy^{*T}=x^*Ay^{*T}=v=x^*Ae_j^{nT},i=1,\cdots,m;\ j=1,\cdots,n\ \circ\quad(*)$$

Proof: $(1)\Rightarrow(2)$. See [2].

$(2)\Rightarrow(3)$. Let x^* and y^* be players 1's and 2's optimal mixed strategies, respectively. Then

$$e_i^mAy^{*T}\le x^*Ay^{*T}=v\le x^*Ae_j^{nT},i=1,\cdots,m;\ j=1,\cdots,n.$$

Assume that

$$e_{i_0}^mAy^{*T}<x^*Ay^{*T}=v \text{ for some } i_0(1\le i_0\le m).$$

Since $e_i^mAy^{*T}\le v,i=1,\cdots,m$ and $x^0=(1/m,\cdots,1/m)$ is player 1's optimal strategy,

$$v=x^0Ay^{*T}=(\frac{1}{m},\cdots,\frac{1}{m})Ay^{*T}=(\sum_{i=1}^m\frac{1}{m}e_i^m)Ay^{*T}=\sum_{I=1}^m\frac{1}{m}(e_i^mAy^{*T})<v,$$

a contradiction. Hence $e_i^mAy*^T=v,i=1,\cdots,m$. Similarly $v=x^*Ae_j^{nT},\ j=1,\cdots,n$.

Conversely, let x^*,y^* satisfy (*). It is obvious that x^* and y^* are players 1's and 2's optimal mixed strategies, respectively.

$(2) \Rightarrow (1)$. Let both $x^0 = (1/m, \cdots, 1/m)$ and $y^0 = (1/n, \cdots, 1/n)$ be players 1's and 2's optimal strategies, and v game value. By $(2) \Rightarrow (3)$, we have

$$e_i^m A y^{0T} = x^0 A y^{0T} = v = x^0 A e_j^{nT}, i = 1, \cdots, m; \; j = 1, \cdots, n .$$

Since

$$e_i^m A y^{0T} = (a_{i1}, \cdots, a_{in}) \begin{bmatrix} 1/n \\ \vdots \\ 1/n \end{bmatrix} = \frac{1}{n} \sum_{j=1}^{n} a_{ij}, i = 1, \cdots, m ,$$

$$x^0 A e_j^{nT} = (\frac{1}{m}, \cdots, \frac{1}{m}) \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = \frac{1}{m} \sum_{i=1}^{m} a_{ij}, \; j = 1, \cdots, n , \text{ and}$$

$$v = x^0 A y^{0T} = (\frac{1}{m}, \cdots, \frac{1}{m}) \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1/n \\ \vdots \\ 1/n \end{bmatrix} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} ,$$

we obtain

$$\sum_{j=1}^{n} a_{ij} = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = nv, i = 1, \cdots, m .$$

Therefore we have $v = \frac{1}{n} \sum_{j=1}^{n} a_{ij}$, $i = 1, \cdots, m$. Similarly $v = \frac{1}{m} \sum_{i=1}^{m} a_{ij}, \; j = 1, \cdots, n$.

Note: $(3) \Rightarrow (1)$ is false. For example, consider the game $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. It does not satisfy（1）.

But it is obvious that (x^*, y^*) is a game solution of A, where $x^* = y^* = \frac{1}{3}(2,1)$. Since

$$x^* A = \frac{1}{3}(2,1) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \frac{2}{3}(1,1) \text{ and } A y^{*T} = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we have

$$x A y^{*T} = \frac{2}{3}(x_1, 1 - x_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{3}, \; x^* A y^T = \frac{2}{3}(1,1) \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix} = \frac{2}{3},$$

for any $x = (x_1, 1 - x_1), (0 \le x_1 \le 1)$ and $y = (y_1, 1 - y_1), (0 \le y_1 \le 1)$. We so obtain

$$x^* A y^T = x^* A y^{*T} = x A y^{*T} = \frac{2}{3} .$$

In other words, the game satisfies（3）.

3. SOLVING AN EQUI-AVERAGE GAME BY LINEAR EQUATIONS

Theorem 4 Let $A = (a_{ij})_{m \times n}$ be an equi-average game with the average v. Then v is its game

value. And a situation $(x, y) = ((x_1, \cdots, x_m), (y_1, \cdots, y_n)) \in X \times Y$ is a game solution if and only if x and y satisfy the two systems of linear equations

$$\begin{cases} \sum_{i=1}^{m} (a_{ij} - v)x_i = 0, j = 1, \cdots, n \\ \sum_{i=1}^{m} x_i = 1 \\ x_i \geq 0, i = 1, \cdots, m \end{cases} \quad and \quad \begin{cases} \sum_{j=1}^{n} (a_{ij} - v)y_j = 0, i = 1, \cdots, m \\ \sum_{j=1}^{n} y_j = 1 \\ y_j \geq 0, j = 1, \cdots, n \end{cases},$$

respectively.

Proof: The games $A = (a_{ij})_{m \times n}$ and $A' = (a_{ij} - v)_{m \times n}$ are equivalent.

Theorem 5 *Let* $\mathrm{Rank}A' = r$ *and let, without lose of generality, Hermite form of the matrix* $A' = (a_{ij} - v)_{m \times n}$ *be*

$$\begin{bmatrix} 1 & \cdots & 0 & \xi_{1,r+1} & \cdots & \xi_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \xi_{r,r+1} & \cdots & \xi_{rn} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $\alpha_k = 1 - \sum_{j=1}^{r} \xi_{jk}, k = r+1, \cdots, n$. Then player 2's set of optimal strategies is

$$\{(-\sum_{k=r+1}^{n} \xi_{1k} c_{k-r}, \cdots, -\sum_{k=r+1}^{n} \xi_{rk} c_{k-r}, c_1, \cdots, c_{n-r}) \mid$$

$$\sum_{k=r+1}^{n} \alpha_k c_{k-r} = 1; \sum_{k=r+1}^{n} c_{k-r} \xi_{jr} \leq 0, j = 1, \cdots, r; c_j \geq 0, j = 1, \cdots, n-r\}.$$

Player 1 has the dual result.

Proof: The simplest form of the second given systems of linear equations is

$$\begin{cases} y_1 = -\xi_{1,r+1} y_{r+1} - \cdots - \xi_{1n} y_n \\ \cdots\cdots\cdots \\ y_r = -\xi_{r,r+1} y_{r+1} - \cdots - \xi_{rn} y_n \end{cases}.$$

Its general solution vectors are

$$\begin{pmatrix} y_1 \\ \vdots \\ y_r \\ y_{r+1} \\ \vdots \\ y_n \end{pmatrix} = c_1 \begin{pmatrix} -\xi_{1,r+1} \\ \vdots \\ -\xi_{r,r+1} \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_{n-r} \begin{pmatrix} -\xi_{1n} \\ \vdots \\ -\xi_{rn} \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The nonnegative solution is

$$y_j = \begin{cases} -\sum_{k=r+1}^{n} c_{k-r}\xi_{jk} \geq 0, & j = 1,\cdots,r \\ c_{j-r} \geq 0, & j = r+1,\cdots,n \end{cases}.$$

When $\sum_{j=1}^{n} y_j = 1$, we have

$$1 = \sum_{j=1}^{n} y_j = \sum_{j=1}^{r} y_j + \sum_{j=r+1}^{n} y_j = -\sum_{j=1}^{r}\sum_{k=r+1}^{n} c_{k-r}\xi_{jk} + \sum_{j=r+1}^{n} c_{j-r}$$

$$= -\sum_{k=r+1}^{n} c_{k-r}\sum_{j=1}^{r}\xi_{jk} + \sum_{k=r+1}^{n} c_{k-r} = \sum_{k=r+1}^{n}(1-\sum_{j=1}^{r}\xi_{jk})c_{k-r} = \sum_{k=r+1}^{n}\alpha_k c_{k-r}.$$

Example 1 Compute the general solution to rock, scissors, paper games [2, 6].

Solution:

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{v^*=0} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $[1-(-1)-(-1)]c = 1$ or $c = 1/3$. Hence $y^* = (c,c,c) = (1/3,1/3,1/3)$. Similarly $x^* = (1/3,1/3,1/3)$. The general solution is $((1/3,1/3,1/3),(1/3,1/3,1/3))$.

Example 2 Solve the game [2, 3]

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{v^*=1} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$[1-(-1)]c + (1-0)d = 1, c \geq 0, 1-2c = d \geq 0$. So $y^* = (c,c,1-2c), 0 \leq c \leq 1/2$.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \xrightarrow{v^*=1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$2c = [1-(-1)]c = 1 \Rightarrow c = 1/2$. Hence $x^* = (c, c) = (1/2, 1/2)$. The general solution is $((1/2, 1/2), (c, c, 1-2c)), 0 \le c \le 1/2$.

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