where

$$\sigma^* = \sigma^2 - \frac{2}{1-c}\sqrt{\frac{2}{\pi\delta t}}k\sigma.$$

2. UMV MODEL UNDER DIVIDEND AND TRANSACTION COST

Assuming that volatility of the stock fluctuates randomly in a fixed interval, it means $0<\sigma^-<\sigma<\sigma^+$. The constant σ^- and σ^+ represent upper and lower bounds on the volatility that should be input in the model. While considering the dividend, tax payment and transaction costs, the volatility adjust to σ^*, according to Equation (2). Obviously, σ^* is bounded, so the adjusted volatility σ^* fluctuates randomly between σ^{*-} and σ^{*+}, where the constant σ^{*-} and σ^{*+} represent upper and lower bounds on the adjusted volatility.

According to Equation (1), the stock price meet the following process

$$\mathrm{d}S = \frac{1-\tau}{1-c}(r-q)S\mathrm{d}t + \sigma^* S\mathrm{d}W,$$

where

$$\sigma^* \in [\sigma^{*-}, \sigma^{*+}].$$

We could establish a stochastic control system as follows:

$$\begin{cases} \mathrm{d}S = \frac{1-\tau}{1-c}(r-q)S\mathrm{d}t + \left(\frac{\sigma^{*+}+\sigma^{*-}}{2} + \frac{\sigma^{*+}-\sigma^{*-}}{2}u\right)S\mathrm{d}W \\ S_{(t_0)} = S_0 \end{cases}, \quad (2)$$

with the cost functional

$$J(\mathrm{u}(\cdot)) = E(h(S(T))),$$

and the control function u belongs to the following control set

$$u(\cdot)\epsilon U[t_0,T] = \{u(\cdot):[t_0,T] \to \Omega | u(\cdot) \text{ is measurable}\}$$

Where Ω=[-1,1] .

For the European call option, we have $h(S(T))=\max(S_T-K,0)$. For the European put option, we have $h(S(T))=\max(K-S_T,0)$.

Now we set up the dynamic programming problem. Let U be a metric space. For any $(s,y)\in[0,T)\times R^n$, consider the state equation as follows:

$$\begin{cases} \mathrm{d}S = rSd + \left(\frac{\sigma^{*+}+\sigma^{*-}}{2} + \frac{\sigma^{*+}-\sigma^{*-}}{2}u\right)S\mathrm{d}W \\ S_{(q)} = y \end{cases}, \quad (3)$$

with the cost function

$$\begin{cases} U(q,y) = inf\, J(q,y;u(\cdot)) \\ U(T,y) = h(y) \end{cases}.$$

To solve such stochastic control system, we need the interval of option price. We apply the dynamic programming, by which we can minimize the cost functional $J(u(\cdot))$ subject to the state Equation (3). In this way, we get the minimum value of the option, which represents the worst condition of the market. Similarly, we could get the maximum value of the option by a simple equivalent transformation, which represents the best condition of the market.

According to the optimal control theory in the reference 5, the Hamilton function G under the worst condition is shown as follows:

$$G(t,s,\mu,p,P) = \frac{1}{2}\left(\frac{\sigma^{*+}+\sigma^{*-}}{2} + \frac{\sigma^{*+}-\sigma^{*-}}{2}u\right)^2 X^2P + rXP,$$

where

$$P = -\frac{\partial^2 U}{\partial^2 S^2}, p = -\frac{\partial U}{\partial S},$$

When $P>0$, he Hamilton function G is monotone increasing in the control function u, thus we could maximize G by letting u=1. In such case

$$\sup_{\mu\in\Omega} G\left(t,s,\mu,\frac{\partial U}{\partial S},-\frac{\partial^2 U}{\partial^2 S}\right) = -\frac{1}{2}(\sigma^{*+})^2 S^2\frac{\partial^2 U}{\partial S^2} - (r-q)\left(\frac{1-\tau}{1-c}\right)\frac{\partial U}{\partial S}S.$$

And the HJB formula turns out to be as follows:

$$\frac{\partial U}{\partial t} + \frac{1}{2}(\sigma^{*+})^2 S^2\frac{\partial^2 U}{\partial S^2} + (r-q)\left(\frac{1-\tau}{1-c}\right)S\frac{\partial U}{\partial S} = 0.$$

On the contrary, When $P<0$, the Hamilton function G is monotone decreasing in the control function u. Thus we could maximize G by letting u=-1. In such case

$$\sup_{\mu\in\Omega} G\left(t,s,\mu,\frac{\partial U}{\partial S},-\frac{\partial^2 U}{\partial^2 S}\right) = -\frac{1}{2}(\sigma^{*-})^2 S^2\frac{\partial^2 U}{\partial^2 S} - (r-q)\left(\frac{1-\tau}{1-c}\right)\frac{\partial U}{\partial S}S.$$

And the HJB formula turns out to be as follows:

$$\frac{\partial U}{\partial t}+\frac{1}{2}(\sigma^{*-})^2S^2\frac{\partial^2 U}{\partial S^2}+(r-q)S\left(\frac{1-\tau}{1-c}\right)\frac{\partial U}{\partial S}=0\ .$$

The cost functional U of the Equation (4) under the worst condition satisfy the following partial differential equations:

$$\begin{cases}\dfrac{\partial U}{\partial t}+\dfrac{1}{2}\sigma\left(\dfrac{\partial^2 U}{\partial S^2}\right)^2S^2\dfrac{\partial^2 U}{\partial S^2}+rS\dfrac{\partial U}{\partial S}=0\ , \\ \sigma\left(\dfrac{\partial^2 U}{\partial S^2}\right)=\begin{cases}\sigma^{*+}\ \dfrac{\partial^2 U}{\partial S^2}<0 \\ \sigma^{*-}\ \dfrac{\partial^2 U}{\partial S^2}>0\end{cases}\ , \\ U(x)|_{t=T}=e^{-(T-t)}h(S(T))\ .\end{cases} \tag{4}$$

When it comes to the best condition, we have such equivalent transformation as follows:

$$U_{best}=-(-U)_{worst}$$

Let $U_{best}=-(-U)_{worst}$ substitute into Equation (4), thus we have the cost functional of the Equation (4) under the best condition:

$$\begin{cases}\dfrac{\partial U}{\partial t}+\dfrac{1}{2}\sigma\left(\dfrac{\partial^2 U}{\partial S^2}\right)^2S^2\dfrac{\partial^2 U}{\partial S^2}+(r-q)\left(\dfrac{1-\tau}{1-c}\right)S\dfrac{\partial U}{\partial S}=0\ , \\ \sigma\left(\dfrac{\partial^2 U}{\partial S^2}\right)=\begin{cases}\sigma^{*-}\ \dfrac{\partial^2 U}{\partial S^2}<0 \\ \sigma^{*+}\ \dfrac{\partial^2 U}{\partial S^2}>0\end{cases}\ , \\ U(x)|_{t=T}=e^{-(T-t)}h(S(T))\ .\end{cases} \tag{5}$$

According to the relationship between the cost functional U and the option value function V and the rule of the partial derivatives:

$$U(t,x)=e^{r(T-t)}V(t,x)\ ,$$

we have the following equation,

$$\begin{cases}\dfrac{\partial U}{\partial t}=-re^{r(T-t)}V+e^{r(T-t)}\dfrac{\partial V}{\partial t} \\ \dfrac{\partial U}{\partial S}=e^{r(T-t)}\dfrac{\partial V}{\partial S} \\ \dfrac{\partial^2 U}{\partial S^2}=e^{r(T-t)}\dfrac{\partial^2 V}{\partial S^2}\end{cases}\ . \tag{6}$$

When considering the worst condition Equation (4), we have the minimum option value $V^-(t,x)$ satisfy the PDE formula by applying the transformation Equation (6)

$$\begin{cases}\dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma\left(\dfrac{\partial^2 V}{\partial S^2}\right)^2S^2\dfrac{\partial^2 V}{\partial S^2}+(r-q)\left(\dfrac{1-\tau}{1-c}\right)S\dfrac{\partial V}{\partial S}-rV=0\ , \\ \sigma\left(\dfrac{\partial^2 U}{\partial S^2}\right)=\begin{cases}\sigma^{*+}\ \dfrac{\partial^2 V}{\partial S^2}<0 \\ \sigma^{*-}\ \dfrac{\partial^2 V}{\partial S^2}>0\end{cases}\ , \\ U(x)|_{t=T}=e^{-(T-t)}h(S(T))\ .\end{cases} \tag{7}$$

On the contrary, when considering the best condition Equation (5), we have the maximum option value $V^+(t,x)$ satisfy the PDE formula as follows:

