CSCanada
Management Science and Engineering
Vol. 9, No. 1, 2015, pp. 19-24
DOI: 10.3968/6117

ISSN 1913-0341 [Print]
ISSN 1913-035X [Online]
www.cscanada.net
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Analysis of Duopoly Output Game With Different Decision-Making Rules

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Supported by the National Natural Science Foundation of China (71173248); the National Social Science Foundation of China (20142571); the Science and Technology Research Project of Henan Province Education Department (14A630045).
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Received 12 December 2014; accepted 15 February 2015
Published online 26 March 2015

Abstract

The main objective of this paper is to study the effects of output adjustment speed and weight on the dynamics and aggregate profits of a duopoly game model with heterogeneous players. In duopoly game process, one player chooses the bounded rationality strategy and the other adopts the adaptive expectation method. Also the linear inverse demand function and nonlinear cost function are used. The paper analyzes the stability of fixed points, and studies the dynamics of the duopoly model. Then an adaptive controller is constructed to maximize profits by controlling output chaos. Theoretical analysis and numerical simulations show that the duopoly game model has two equilibrium points: one is unstable and the other is locally stable. High output adjustment speed can cause chaotic variation of the outputs, which will decrease the profit of the firm with bounded rationality. The weight variation has little effect on inducing output chaos. The firm with bounded rationality has strong motives to suppress output chaos to maximize its profit. Numerical experiments to verify the effectiveness of the designed controller in this paper. From the profit point of view, the adaptive expectation method is better than the bounded rationality strategy.

Key words: Duopoly game; Bounded rationality; Adaptive expectation; Chaos control; Numerical simulation

Lu, Y. L. (2015). Analysis of Duopoly Output Game With Different Decision-Making Rules. *Management Science and Engineering*, *9*(1), 19-24. Available from: URL: http://www.cscanada.net/index.php/mse/article/view/6117
DOI: http://dx.doi.org/10.3968/6117

INTRODUCTION

The duopoly is a market structure where has only two firms in the market producing the same or homogeneous products. The history of duopoly game study can date back to Cournot who first introduced the formal theory of oligopoly in 1838. However, in the past many years the theoretical research of duopoly game model was not very deeply. One reason is that most of the duopoly model have nonlinearity which has induced the difficulties in theoretical analysis, and the other is that people have not discovered the chaotic phenomena at that time. Since Ott et al. (1990) introduced the OGY method for controlling chaos the chaotic dynamics analysis of duopoly game model has attracted strong interests of many researchers. On the basis of different cost functions, different inverse demand functions and different expectation for rivals' output, in recent years the dynamics of various duopoly game models have been studied extensively. For instance, Yassen et al. (2003), Yao and Xu (2006), Du et al. (2009) investigated the dynamical properties of duopoly game model with bounded rational method and quadratic cost functions. Elsadany et al. (2010) considered the cooperation situation of a duopoly game with bounded rationality. Chen et al. (2007) studied the problems of stability and control of a duopoly game model with hyperbolic inverse demand function and linear cost function and naive expectation for rival's output. Elsadany et al. (2010) discussed the dynamic behavior of a delayed duopoly game with bounded rationality. Kamalinejad et al. (2010) adopted the linear regression expectations method to study a Cournot game problem. Naimzada and Ricchiuti et al. (2011) analyzed the game dynamics under incomplete knowledge of the demand function. Du et al. (2010) proposed a limiter method to maximize player's profits in bounded rational game. Cánovas & Paredes (2010) gave a general result of Cournot game model under linear feedback control. Agiza et al. (2004) and Zhang et al. (2007) investigated the dynamical characteristics of duopoly game model for the situation that each player uses different expectations of its rival's output. In general, the game model will become more complex and have more abundant dynamical characters when the extra nonlinear term is introduced. To the best of our knowledge, so far the dynamics of duopoly game with quadratic cost functions, in which one player is bounded rationality and the other thinks with adaptive expectation, has not been studied. Motivated by the above analysis, the main objective of this paper is to study the dynamics of a duopoly game model with different decision-making rules and to improve the profits by controlling chaos.

The remainder of this paper is organized as follows. Section 2 gives the duopoly game model and analyzes the stability of boundary equilibrium and Nash equilibrium. Also, the effects of output adjustment speed and weight on the dynamics of the duopoly game model with different players are studied by numerical simulations. In Section 3, we present an adaptive controller to improve the profits by suppressing outputs chaos. Finally, some conclusions are drawn in Section 4.

1. DUOPOLY MODEL AND ANALYSIS

1.1 Modelling Duopoly Output Game

Suppose that there are only two firms in the market producing homogeneous products which can be perfectly substituted. There is a duopoly competition between the two firms. For the convenience of narration, in this paper one firm is labeled by k=1 and the other is labeled by k=2. Recently Agiza and Elsadany (2002) have studied the dynamics of duopoly model by assuming the inverse demand function is linear and decreasing:

$$p=f(Q)=a-bQ,$$

where $Q=q_1+q_2$ is the total output of the products and $a,b>0$. Here a denotes the maximal price of the product in the market and b represents the effect of unit product quantity on price. The cost function of the two firms has the following form (Du et al., 2010):

$$C_k(q_k)=c_k+d_kq_k+e_kq^2_k,\ k=1,2,$$

where c_k represents the positive fixed cost and $d_kq_k+e_kq^2_k$ denotes the variable cost. From the economic knowledge, we know that the cost function $C_k(q_k)$ climbs with the increase of the product output q_k. Hence we can assume: $C'_k(q_k)>0$ and $C''_k(q_k)>0, k=1,2$. Then we can obtain $d_k>0$, $e_k>0$. Also the marginal profit of the k th firm is less than the maximal price of the product to make the two-players' game on the rails. This means $d_k>2e_kq_k<a$, k=1, 2. Finally, we assume that the decision-making of the two firms takes place only in the discrete-time periods t=0, 1, 2,⋯. Based on the above assumptions, the profit function of the two firms at the period t is:

$$\begin{aligned} L_k(q_1(t),q_2(t)) &= pq_k - C_k \\ &= q_k(a-bQ)-(c_k+d_kq_k+e_kq_k^2) \\ &\quad k=1,2. \end{aligned} \tag{1}$$

The marginal profit function of the two firms at the period t is:

$$\frac{\partial L_k(q_1(t),q_2(t))}{\partial q_k} = a-bQ-bq_k-d_k-2e_kq_k \quad k=1,2. \tag{2}$$

From $\partial L_k/\partial q_k=0$, we can find the optimal output $q^*_k(t)$ which can maximize the profit of the k th firm at the period t. The expression of $q^*_k(t)$ is of the following form:

$$q_k^*(t)=\frac{a-bq_j(t)-d_k}{2(b+e_k)}\quad k,j=1,2,\ k\neq j. \tag{3}$$

The firms (players) are assumed having homogeneous expectations rules for computing their expected outputs in many literatures (Yassen et al., 2003; Yao & Xu, 2006; Du et al., 2009); Chen & Chen, 2007; Elsadany, 2010; Du et al., 2010; Agiza et al., 2002). However, in fact, different firms may have different making-rules, so we consider a duopoly game with two heterogeneous firms in this paper. The first firm (labeled by k=1) is a bounded rational player and the second firm (labeled by k=2) chooses the adaptive expectations. That is to say, the first firm adjusts its output at next period based on the local estimate of its marginal profit. For instance, the first firm will increase (decreases) its output at the period t=1 if the firm's marginal profit of the period t is positive (negative). So the output adjustment model of the first firm has the following form:

$$q_1(t+1)=q_1(t)+\beta q_1(t)\frac{\partial L_1}{\partial q_1}, \tag{4}$$

where β is a positive parameter and denotes the relative output adjustment speed of the first firm. The second firm thinks with adaptive expectations and computes its output of the period t=1 with weighs between the last period's output $q_2(t)$ and the estimated optimal output $q^*_2(t)$ at the period t. Hence the dynamic equation of the second firm has the form:

$$q_2(t+1)=w\cdot q_2(t)+(1-w)\cdot q_2^*(t), \tag{5}$$

where $w\in(0,1)$ represents the weight. From Equations (2), (3), (4) and (5), we can get the dynamic output game model of the two firms in the following form:

$$\begin{cases} q_1(t+1)=q_1(t)+\beta q_1(t)[a-2bq_1-bq_2 \\ \qquad -d_1-2e_1q_1] \\ q_2(t+1)=w\cdot q_2(t)+(1-w)\dfrac{a-bq_1-d_2}{2(b+e_2)} \end{cases} \tag{6}$$

1.2 Fixed Points and Its Stability Analysis

Based on the economical significance, we know that the outputs of the two firms are nonnegative. So we only discuss the nonnegative equilibriums in this paper. Suppose $q_1(t+1)=q_1(t)$, $q_2(t+1)=q_2(t)$ and solve Equation (6), we can get two fixed points:

$$E_0=\left(0,\frac{a-d_2}{2(b+e_2)}\right) \text{ and } E_1=(q^*_1,q^*_2),$$

here

$$q_1^*=\frac{ab+bd_2+2ae_2-2bd_1-2d_1e_2}{3b^2+4b(e_1+e_2)+4e_1e_2},$$

$$q_2^*=\frac{ab+bd_1+2ae_1-2bd_2-2d_2e_1}{3b^2+4b(e_1+e_2)+4e_1e_2}.$$

The fixed point E_0 is called boundary equilibrium because the output of the first firm is zero. This means that the first firm is put out the market. The fixed point E_1 is known as Nash equilibrium which should meet the below conditions to ensure the nonnegative outputs:

$$ab+bd_2+2ae_2-2bd_1-2d_1e_2>0$$

and

$$ab+bd_1+2ae_1-2bd_2-2d_2e_1>0. \quad (7)$$

In order to study the stability of equilibrium points E_0 and E_1, we consider the Jacobian matrix of Equation (6) which has the following form:

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $a_{11}=1+\beta[a-4bq_1-4e_1q_1-bq_2-d_1]$, $a_{12}=-\beta bq_1$, $a_{21}=[b(w-1)]/(2b+2e_2)$, $a_{22}=w$. Hence we can investigate the stability of equilibrium points via analyzing the eigenvalues of the Jacobian matrix at the corresponding fixed points.

Proposition 1 The boundary equilibrium E_0 is a saddle point of system (6).

Proof. The Jacobian matrix at the boundary equilibrium E_0 has the following form:

$$A\big|_{E_0}=\begin{pmatrix} 1+\beta[a-\dfrac{ab-bd_2}{2(b+e_2)}-d_1] & 0 \\ \dfrac{b(w-1)}{2(b+e_2)} & w \end{pmatrix},$$

which has two eigenvalues: $\lambda_1=w$ and

$$\lambda_2=1+\beta[a-\frac{ab-bd_2}{2(b+e_2)}-d_1]$$
$$=1+\beta\cdot\frac{ab+bd_2+2ae_2-2bd_1-2d_1e_2}{2(b+e_2)}.$$

From Equation (7), $w\in(0,1)$, and β, b, e_2 are positive parameters, we can obtain $|\lambda_1|<1$ and $|\lambda_2|>1$. Therefore, E_0 is a saddle equilibrium point of Equation (6). This completes the proof.

Proposition 2 The unique Nash equilibrium E_1 is locally stable.

At the Nash equilibrium point E_1, the Jacobian matrix is $A\big|_{E_1}=\begin{pmatrix} * & ** \\ \dfrac{b(w-1)}{2(b+e_2)} & w \end{pmatrix}$,

where

$$*=[(-2ab^2-2abe_1-4abe_2-4ae_1e_2-2b^2d_2$$
$$+4b^2d_1+4bd_1e_2-2e_1bd_2+4e_1bd_1$$
$$+4e_1d_1e_2)\beta+(3b^2+4be_1+4be_2$$
$$+4e_1e_2)]/[3b^2+4be_1+4be_2+4e_1e_2],$$

$$**=\frac{(2bd_1+2d_1e_2-2ae_2-ab-bd_2)}{3b^2+4b(e_1+e_2)+4e_1e_2}\cdot\beta b.$$

The characteristic equation of the Jacobian matrix $A|_{E_1}$ is of the form

$$\lambda^2-Tr\cdot\lambda+Det=0,$$

where "Tr" is the trace and "Det" is the determinant of the Jacobian matrix $A|_{E_1}$. Based on the stability theory of discrete systems, we know that the sufficient and necessary conditions for the stability of Nash equilibrium E_1 is that the eigenvalues of the Jacobian matrix $A|_{E_1}$ are inside the unit circle on the complex plane. This is true if and only if the following Jury's stability criteria (see Ref. Agiza et al., 2002; Jury & Blanchard, 1961 and therein) are satisfied:

① $Det<1$,
② $1-Tr+Det>0$,
③ $1+Tr+Det>0$.

From the above three criteria, we can define a local stability region on the parameters' space to ensure that the Nash equilibrium E_1 is stable. This indicates that unique Nash equilibrium E_1 is only locally stable. For instance, let $a=10$, $b=1$, $d_1=1$, $e_1=1$, $d_2=1$, $e_2=1.1$ (Du et al., 2010), and assume β and w be variable, so the Jury's stability criteria become:

④ $1+\frac{9}{20}\beta-w+\frac{27}{4}w\beta>0$,
⑤ $\frac{27}{4}\beta-\frac{27}{4}w\beta>0$,
⑥ $2-\frac{153}{20}\beta+2w-\frac{27}{4}w\beta>0$.

The three inequalities define a region in (w, β) plane. We call it the local stability region of the Nash equilibrium which has been depicted in Figure 1. That is to say, for the value of (w, β) inside the stability region in Figure 1 and the above other given parameters, the Nash equilibrium is stable. However, once the increase of β brings the point (w, β) out the stability region, more complex phenomena of outputs evolution will occur such as bifurcation and chaos.

1.3 Numerical Simulation and Analysis

The main purpose of this section is to study the dynamics of the duopoly game model (6) with the variation of

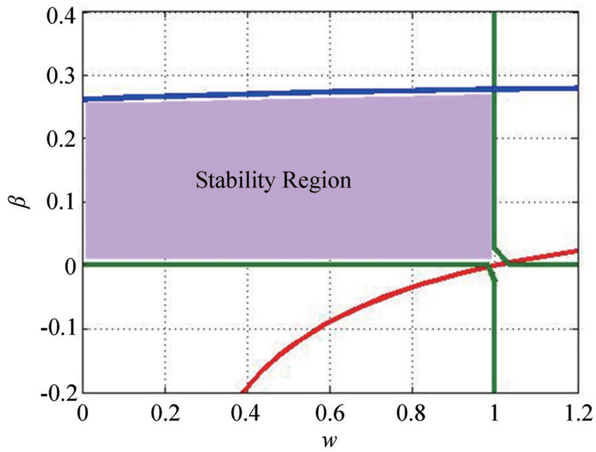

Figure 1
Local Stability Region of the Nash Equilibrium

parameters w and β. To this goal, some various numerical evidences are presented to show the effects of parameter variation on the dynamics and aggregate profits in this section, such as bifurcation diagrams, maximal Lyapunov exponents, strange attractor, and so on. The numerical experiments are carried out by setting (Du et al., 2010): a=10, b=1, c_1=1.1, d_1=1, e_1=1, c_2=1, d_2=1, e_2=1.1. Figures. 2-4 show the bifurcation diagram, maximal Lyapunov exponent curve and aggregate profit curves of the two firms with respect to the output adjustment speed β when w=0.5. It can be seen clearly from Figure 2 that the outputs of the two firms appear period-doubling bifurcation and chaos phenomenon as the output adjustment speed β increases. Figure 3 indicates that the maximal Lyapunov exponent is positive when β>0.351. Here the point (w,β)(0.5,0.351) is outside the stability region (see Figure 1). The positive maximal Lyapunov exponent means the existence of chaos. Hence, Figure 3 is concordant with Figure 2 for identifying chaos arising.

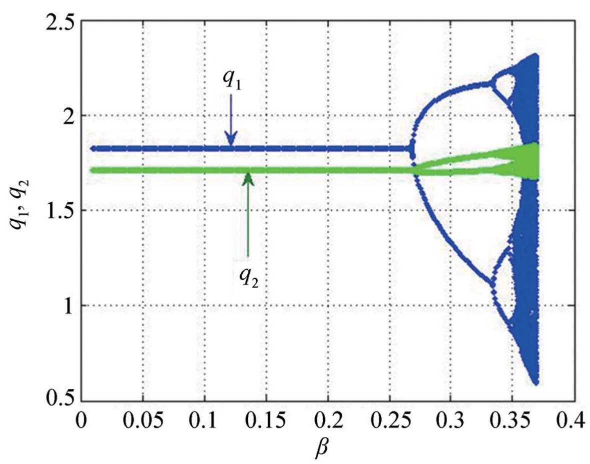

Figure 2
Bifurcation Diagram for *w*=0.5

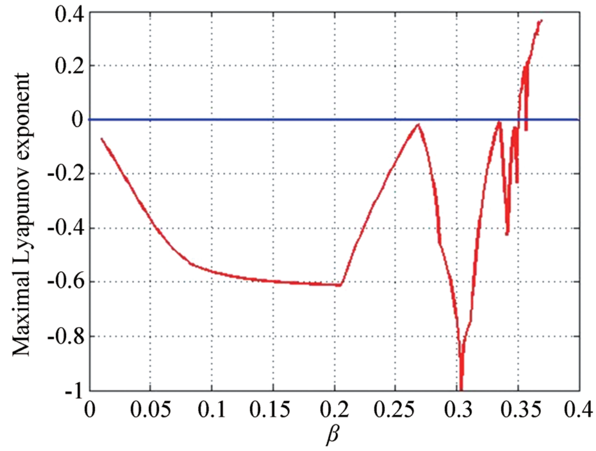

Figure 3
The Maximal Lyapunov Exponent Curve for *w*=0.5

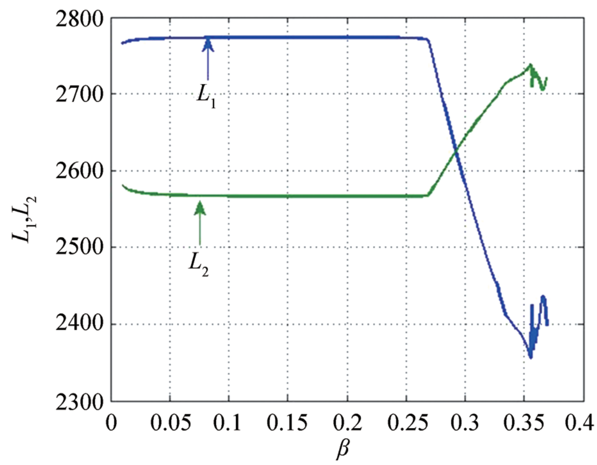

Figure 4
The Profit Curves vs. *β* for *w*=0.5
L_1: Profit of the First Firm, L_2: Profit of the Second Firm

It is easy to see from Figure 4 that the curve of the first firm's aggregate profit in 500 times iteration decrease when the outputs appear bifurcation or chaos for high output adjustment speed of β. Therefore, the Nash equilibrium profit is optimal for the first firm. But the opposite is true for the second firm. In order to illustrate the sensitive dependences on initial conditions, the curves of outputs evolution of q_1 and q_2 are depicted in Figure 5 for two different initial conditions (1.2, 1.3) and (1.21, 1.30), respectively. Here the q_1-coordinates of initial conditions differ by 0.01, and the other coordinates kept equal. Figure 5 show that the time series of the Equation (6) is sensitive dependence to initial conditions, i.e., complex dynamics behaviors occur in this duopoly game model. The strange attractor with specific structure in Figure 6 further confirms the existence of chaos. Let β=0.34, which is inside the bifurcation region and close to chaotic region (see Figure 2), Figure 7 shows the maximal Lyapunov exponent curve vs. weight w with the above given parameters. It must be noted that point of (w,0.34) is outside the local stability region (see Figure 1), where $w \in (0,1)$. Still and all, one can see from Figure 7 that the maximal Lyapunov exponent is only positive as w>0.98. The above numerical experiments indicate that high output adjustment speed can cause chaotic variation of outputs evolution which will decrease the profit of the first firm with bounded rationality, and the maximal Lyapunov exponent is not sensitive to the weight w.

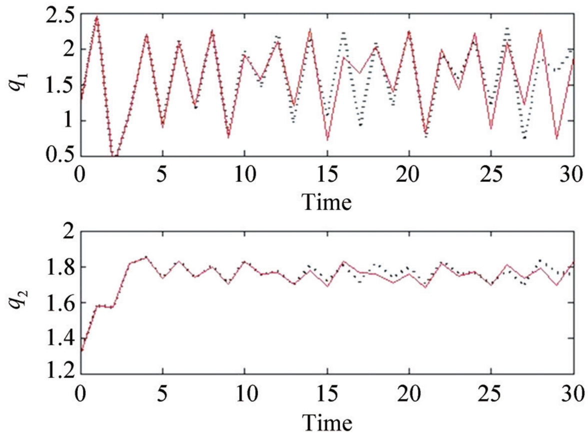

Figure 5
Time History of Outputs q_1 and q_2 With Two Different Initial Values (1.2, 1.3) and (1.21, 1.30) for β=0.36, and w=0.5

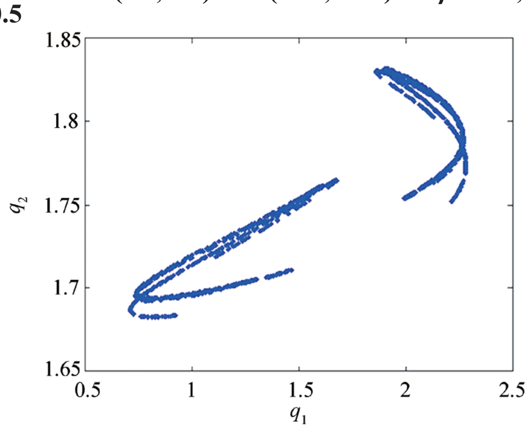

Figure 6
Strange Attractor for β=0.36, w=0.5

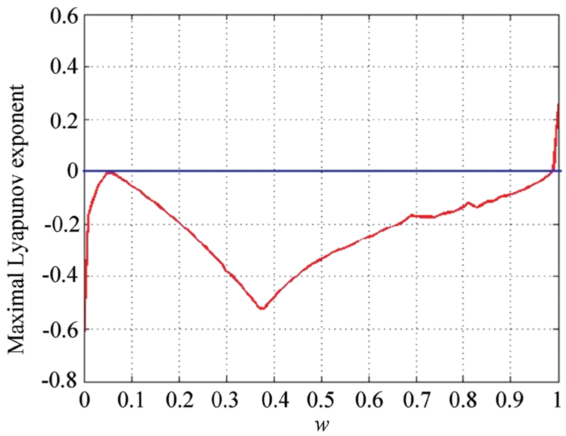

Figure 7
The Maximal Lyapunov Exponent Curve for β=0.34

2. THE STRATEGY OF CHAOS CONTROL

We know from the above analysis that the high output adjustment speed can cause the bifurcation or chaos phenomenon of output evolution which will decrease the aggregate profit of the first firm with bounded rationality. So the first firm has strong motivation to suppress the bifurcation or chaos of output evolution. When the output of the first firm shows violent fluctuations, the first firm can suppress output fluctuations to enhance its performance via adaptive speed adjustment method. Assume that the output un-stability is aroused by high adjustment speed of the first firm. The adaptive controller can be designed as below:

$$\beta(t)=\beta(t\text{-}1)+u(t),$$
$$u(t)=-\min[\varepsilon,\eta\cdot(q_1(t)\text{-}q_1(t\text{-}1))]. \qquad (8)$$

Here ε and η are small positive real, which ensure the $\beta(t)$ changing slowly. So the Equations (6) and (8) form a controlled duopoly game model. The parameters are taken as a=10, b=1, c_1=1.1, d_1=1, e_1=1, c_2=1, d_2=1, e_2=1.1,

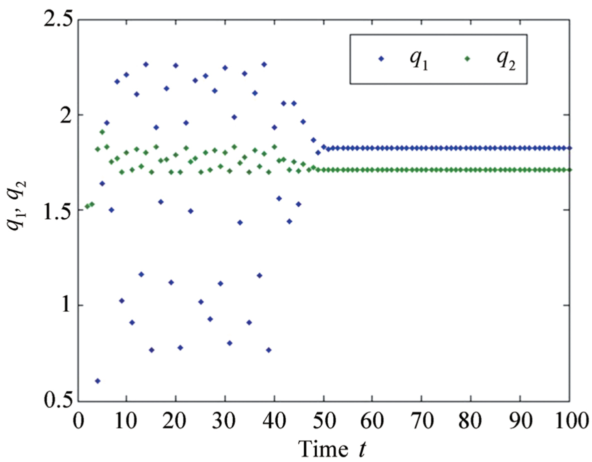

Figure 8
The Output Evolution of the Two Firms Under Control (the Controller Is Activated at t>40)

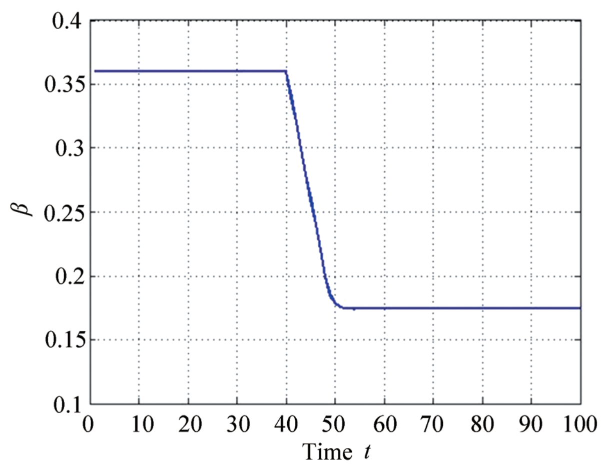

Figure 9
The Evolution Curve of Output Adjustment Speed

β=0.36, w=0.5 to ensure the chaotic behavior of the output variation. The numerical experiment results are shown in Figures 8-9. It can be seen clearly from Figure 8 that the outputs of the two firms can be controlled to Nash equilibrium point via about 10 times iteration when the controller is activated at t>40. Compared with the Figure 2, we can see from Figure 9 that the output adjustment speed $\beta(t)$ is reposefully dropped to the region of 1-period via adaptive method.

CONCLUSION

In this paper we studied a duopoly game model with heterogeneous players, linear inverse demand function and nonlinear cost function. The first player is bounded rationality and the second player uses the adaptive expectations. The theoretical analysis and numerical experiments show that the second player can at least gain equilibrium profit, while the first player's profit will decrease when the outputs appear bifurcation and/or chaos phenomenon. Obtaining the equilibrium profit is the optimal strategy of the first firm via suppressing the bifurcation or chaos of outputs evolution. The adaptive adjustment speed method proposed in this paper is effective to stable the outputs of the two players to Nash equilibrium. From the profit point of view, the adaptive expectation method has an advantage over the bounded rational strategy.

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