CSCanada
International Business and Management
Vol. 6, No. 1, 2013, pp. 51-54
DOI:10.3968/j.ibm.1923842820130601.1115

ISSN 1923-841X [Print]
ISSN 1923-8428 [Online]
www.cscanada.net
www.cscanada.org

Wealth Optimization Models with Stochastic Volatility and Continuous Dividends

YANG Yunfeng[a],*; JIN Hao[a]

[a] School of Science XI'AN University of Science and Technology, XI'AN, 710054, China.
*Corresponding author.

Supported by NSFC(71103143), SSTF(2009KRM99), NSFS (2011JQ1016), NPFC(20110491672, 2012T50809), SSFE(12JK0858), XUST(201240).

Received 1 January 2013; accepted 14 February 2013

Abstract

This paper study the problem of wealth optimization.It is established that the behavior model of the stock pricing process is jump-diffusion driven by a count process and stochastic volatility. Supposing that risk assets pay continuous dividend regarded as the function of time. It is proved that the existence of an optimal portfolio and unique equivalent martingale measure by stochastic analysis methods. The unique equivalent martingale measure ,the optimal wealth process, the value function and the optimal portfolio are deduced.

Key words: Jump-Diffusion process; Stochastic volatility; Dividends; Incomplete financial market; Wealth optimization

YANG Yunfeng, JIN Hao (2013). Wealth Optimization Models with Stochastic Volatility and Continuous Dividends. *International Business and Management, 6*(1), 51-54. Available from: http://www.cscanada.net/index.php/ibm/article/view/j.ibm.1923842820130601.1115 DOI: http://dx.doi.org/10.3968/j.ibm.1923842820130601.1115

INTRODUCTION

The wealth optimization problem and the portfolioselection theory are always the kernel problems on financial mathematics. The domestic and foreign scholars have done a great dral of researches on the wealth optimization problem and obtained many results which is instructive to financial practice. When markets are complete, the existence of optimal strategies can be found Merton (1), Jeanblanc and Pontier (2), Follmer and Leukert (3), Pham (4), Nakano (5) discussed continuous and jump-diffusion modes.

In this paper, We define the wealth optimization problem:

$$V(t,x,y)=\sup_{\pi\in A}E[U(X^{x,\pi}(T))\,|\,X^{x,\pi}(t)=x,Y(t)=y]$$

where □□□(*t*)is the wealth process and □ is the set of a dmissible portfolios. When the wealth is equal to x at the time *t*. we consider an economic agent whose behavior facing the risk is determined by a utility function (6).Utility function is non decreasing, strictly concave, obviously □(□)admits an inverse *I*(□).He invests his wealth in the two assets and wants to maximize the expected utility of wealth at time □Our work extends those studies and analyses the wealth optimization problem when markets is incomplete and driven by discontinuous prices.We consider that price of underlying asset price obeys jump-diffusion process, jump process generalized conforms to the actual situation of stock price movement. This paper discusses jump-diffusion asset price model being driven by a count proces that more general than Poisson process. Supposing that risk assets pay continuous dividend regarded as the function of time. It is proved that the existence of an optimal portfolio and unique equivalent martingale measure by stochastic analysis methods. The unique equivalent martingale measure ,the optimal wealth process, the value function and the optimal portfolio are deduced.

1. ASSUMPTIONS AND MODEL

Let $(\Omega,F,P,(F_t)_{0\le t\le T})$ be a probability space .The market is built with a bond *B*(*t*) and a risky asset □(*t*). We suppose that *B*(*t*) and □(*t*) satisfy differential equation

d*B*(*t*) □*B*(*t*)*r*(*t*)d*t* *B*(0) □ 1 (1)

d□(t) □ □(*t*)((□(□(*t*)) - □(□(*t*))d*t* □ □(□(*t*))*d*W$_1$(*t*) □ □(□(*t*))d*M*(*t*)) (2)

$\mathrm{d}Y(t) = \mathrm{b}(Y(t))\mathrm{d}t + a(Y(t))(\rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2(t))$ (3)

where ris□free interest rate □(t)□volatility □(Y(t)) □ □□ continuous dividends □(Y(t))□Standard Wiener process $\{W_1(t), 0 \le t \le T\}$ and $\{W_2(t), 0 \le t \le T\}$ are independent□

$M(t) = N(t) - \int_0^t \lambda(s)ds, T \ge t \ge 0$ is the compensated martingale of non e□plosive counting process $\{N(t), 0 \le t \le T\}$ with intensity parameter □(t)□

Lemma 1 □or all P^* in P □there e□ist predictable processes □(t)□□(t)□□(t)□ satisfy

□(Y(t)) □□(Y(t)) □□(t) + □(Y(t))□(t) + □(t)□(Y(t))□(t) = □(□)

□(Y(t)) + $a(Y(t))\rho$□(t) + $a(Y(t))\sqrt{1-\rho^2}$□(t) = □ (5)

Proof□Since □ in □□then $\frac{dP^*}{dP}|\mathcal{F}_t = L(t)$ is the □ martingale□applying martingale representation theorem□ there e□ists□(t)□□(t)□□(t) such that

d□(t) = □(t□)((□)dW_1(□)□W_2(□) + □(□)d□(□))

that is □(t) = □(□W□)$_t$ □(□W_2)$_t$ □□(1 +□)M□

$$= \exp\{\int_0^t \theta_1(s)dW_1(s) - \frac{1}{2}\int_0^t \theta_1^2(s)ds + \int_0^t \theta_2(s)dW_2(s) - \frac{1}{2}\int_0^t \theta_2^2(s)ds\} \exp\{\int_0^t \ln(1+\theta_3(s))dN(s) - \int_0^t \lambda(s)\theta_3(s)ds\}]$$

applying □irsanov theorem□$W_1^*(t) = W_1(s) - \int_0^t \theta_1(s)ds$ □ $W_2^*(t) = W_2(s) - \int_0^t \theta_2(s)ds$ are standard Wiener process under the martingale measure□□□ $M^*(t) = N(t) - \int_0^t \lambda(s)(1+\theta_3(s))ds$ is □ martingale□So $\tilde{S}(t) = \frac{S(t)}{B(t)}$ and $Y(t)$ satisfy

$$\mathrm{d}\tilde{S}(t) = \tilde{S}(t-)(\mu(Y(t)) - \tau(Y(t)) - r(t))\mathrm{d}t + \sigma(Y(t))\mathrm{d}W_1^*(t) + \sigma(Y(t))\theta_1(t)dt + \varphi(Y(t))\mathrm{d}M^*(t) + \varphi(Y(t))\lambda(t)\theta_3(t)dt) = \tilde{S}(t-)(\mu(Y(t)) - \tau(Y(t)) - r(t) + \sigma(Y(t))\theta_1(t) + \lambda(t)\varphi(Y(t))\theta_3(t))\mathrm{d}t + \sigma(Y(t))\mathrm{d}W_1^*(t) + \varphi(Y(t))\mathrm{d}M^*(t))$$

$$\mathrm{d}Y(t) = b(Y(t))\mathrm{d}t + a(Y(t))\rho \mathrm{d}W_1^*(t) + a(Y(t))\rho\theta_1(t)dt + a(Y(t))\sqrt{1-\rho^2}\,\mathrm{d}W_2^*(t) + a(Y(t))\sqrt{1-\rho^2}\,\theta_2(t)dt)$$
$$= (b(Y(t)) + a(Y(t))\rho\theta_1(t) + a(Y(t))\sqrt{1-\rho^2}\,\theta_2(t))\mathrm{d}t + a(Y(t))\rho \mathrm{d}W_1^*(t) + a(Y(t))\sqrt{1-\rho^2}\,\mathrm{d}W_2^*(t))$$

Since $\tilde{S}(t)$ and $Y(t)$ are □ martingale□ then

□(Y(t)) □□(Y(t)) □□(t) + □(Y(t))□(t) + □(t)□(Y(t))□(t) = □

$b(Y(t)) + a(Y(t))\rho\theta_1(t) + a(Y(t))\sqrt{1-\rho^2}\,\theta_2(t) = 0$

We define the investor wealth process □□(□) is standara self□financing way□the investor wealth process □□(□) satisfy

$X^{x,\pi}(s) = m_1(s)B(s) + m_2(s)S(s) \qquad 0 \le t \le s \le T$

□et m$_2$(□)□(□) = □(□)□ then□□(□) satisfy

d□□(□) = r(□)□□(□)□ □ □(□)((□(Y(□)) □ □Y(t)) □ r())□ + □((□(□))dW_1(□) + □(Y(□))d□(□))

and $\tilde{X}^{x,\pi}(s) = \frac{X^{x,\pi}(s)B(t)}{B(s)}$ satisfy

$$d\tilde{X}^{x,\pi}(s) = \tilde{\pi}(s)(\mu(Y(s)) - \tau(Y(t)) - r(s))ds + \tilde{\pi}(s)\sigma((Y(s))\mathrm{d}W_1(s) + \tilde{\pi}(s)\varphi(Y(s))\mathrm{d}M(s)$$
$$= \tilde{\pi}(s)(\mu(Y(s)) - \tau(Y(t)) - r(s) + \sigma(Y(s))\theta_1(s) + \lambda(s)\varphi(Y(s))\theta_3(s))ds + \tilde{\pi}(s)\sigma((Y(s))\mathrm{d}W_1^*(s) + \tilde{\pi}(s)\varphi(Y(s))\mathrm{d}M^*(s)$$

by **Lemma 1,** we have

$$d\tilde{X}^{x,\pi}(s) = \tilde{\pi}(s)\sigma((Y(s))\mathrm{d}W_1^*(s) + \tilde{\pi}(s)\varphi(Y(s))\mathrm{d}M^*(s) \quad (□)$$

so $\tilde{X}^{x,\pi}(s)$ □$0 \le t \le s \le T$ is a □ martingale□

Lemma 2 □unction

$$F(z) = z + \lambda\varphi - \frac{\lambda\varphi}{V_x'}V_x'(t, x + \frac{\varphi}{\sigma^2}(r - \mu + \tau - \rho\sigma a\frac{V_{xy}''}{V_x'} - z)\frac{V_x'}{V_{xx}''}, y)$$

e□ists uni□ue zero point$z_0 \in R$ □

□et

$$\theta_3 = \frac{1}{V_x'}V_x'(t, x + \frac{\varphi}{\sigma^2}(r - \mu + \tau - \rho\sigma a\frac{V_{xy}''}{V_x'} - z)\frac{V_x'}{V_{xx}''}, y) - 1 \ □$$

we have □(□) e□ists uni□ue zero poin z□=□□□ □so e□ists uni□ue zero point□ such that

$$\theta_3 = \frac{1}{V_x'}V_x'(t, x + \frac{\varphi}{\sigma^2}(r - \mu + \tau - \rho\sigma a\frac{V_{xy}''}{V_x'} - \lambda\varphi\theta_3)\frac{V_x'}{V_{xx}''}, y) - 1 \quad (□)$$

□ogether with (□)□(5) and (□we can define a uni□ue e□uivalent martingale measure□□

2. MAIN RESULTS

Proposition 1 We assume that utility function satisfies a polynomial growth condition□then the optimal trategy π^* is given by

$$\pi^*(t) = \frac{r(t) - \mu(y) + \tau(y) - \lambda(t)\varphi(y)\theta_3(t)}{\sigma^2(y)}\frac{V_x'(t,x,y)}{V_{xx}''(t,x,y)} - \frac{\rho a(y)}{\sigma(y)}\frac{V_{xy}''(t,x,y)}{V_{xx}''(t,x,y)}$$

Proof. The associated H-J-B equation takes the following form

$$V_t^{'} + \sup_{\pi}\{[xr + \pi(\mu(y) - \tau(y) - \lambda\varphi(y) - r)]V_x^{'} +$$

$$\pi\rho\sigma(y)a(y)V_{xy}^{"} + \frac{1}{2}\pi^2\sigma^2(y)V_{xx}^{"}$$

$$+\frac{1}{2}a^2(y)V_{yy}^{"} + b(y)V_y^{'} + \lambda[V(t, x + \pi\varphi(y), y) - V]\} = 0$$

$$V(T, x, y) = U(x)$$

□e have

$$(\mu(y) - \tau(y) - \lambda\varphi(y) - r)V_x^{'} + \pi^*\sigma^2(y)V_{xx}^{"} + \rho\sigma(y)a(y)V_{xy}^{"}$$

$$+\lambda\varphi(y)V_x^{'}(t, x + \pi^*\varphi(y), y) = 0$$

□et $r - \mu - \rho\sigma a\frac{V_{xy}^{"}}{V_x^{'}} - \pi^*\sigma^2\frac{V_{xx}^{"}}{V_x^{'}} = z$, this equation equivalently

$$z + \lambda\varphi - \frac{\lambda\varphi}{V_x^{'}}V_x^{'}(t, x + \frac{\varphi}{\sigma^2}(r - \mu + \tau - \rho\sigma a\frac{V_{xy}^{"}}{V_x^{'}} - z)\frac{V_x^{'}}{V_{xx}^{"}}, y) = 0$$

Thus,we prove that

$$\pi^*(t) = \frac{r(t) - \mu(y) + \tau(y) - \lambda(t)\varphi(y)\theta_3(t)}{\sigma^2(y)}\frac{V_x^{'}(t,x,y)}{V_{xx}^{"}(t,x,y)}$$

$$-\frac{\rho a(y)}{\sigma(y)}\frac{V_{xy}^{"}(t,x,y)}{V_{xx}^{"}(t,x,y)}.$$

□et $Z^{(t,z)}(s) =$

$$z\exp\{-\int_t^s r(u)du\}\frac{\varepsilon(\theta_1 W_1)_s\,\varepsilon(\theta_2 W_2)_s\,\varepsilon[(1+\theta_3)M]_s}{\varepsilon(\theta_1 W_1)_t\,\varepsilon(\theta_2 W_2)_t\,\varepsilon[(1+\theta_3)M]_t}$$

$0 \le t \le s \le T$, o□viously

$Z^{(t,z)}(s) = zZ^{(t,1)}(s) = z\frac{B(t)L(s)}{B(s)L(t)}$ is a P martingale.

□et $\mathrm{X}(t,z,y) = □^*[□^{-\int_t^T r(u)du}□(Z^{(t,z)}(T))]$, we have

$$\mathrm{X}(t,z,y) = □^*\{□^*[□^{-\int_t^T r(u)du}□(Z^{(t,z)}(T))|\mathsf{F}_t]\}$$

$$= □\{\frac{L(T)}{L(t)}□[□^{-\int_t^T r(u)du}L(T)□(Z^{(t,z)}(T))|\mathsf{F}_t]\}$$

$$= \frac{1}{z}□\{L(T)□[Z^{(t,z)}(T)□(Z^{(t,z)}(T))|\mathsf{F}_t]\}$$

$$= \frac{1}{z}□[Z^{(t,z)}(T)□(Z^{(t,z)}(T))|\mathsf{F}_t]$$

Proposition 2 The optimal wealth process $□^{x,\pi^*}(s)$ satisfies

$$□^{x,\pi^*}(s) = \frac{1}{Z^{(t,1)}(s)}□[□(\mathrm{X}^{-1}(t,x,y)Z^{(t,1)}(T))Z^{(t,1)}(T)|\mathsf{F}_s](0 \le t \le s \le T).$$

Proof Since $\frac{□^{x,\pi^*}(s)B(t)}{B(s)}(0 \le t \le s \le T)$ is a P martingale, applying Bayes□s rule,

$$x = □^*[\frac{□^{x,\pi^*}(T)B(t)}{B(T)}|\mathsf{F}_t]□[\frac{L(T)}{L(t)}|\mathsf{F}_t]$$

$$= □[[\frac{□^{x,\pi^*}(T)B(t)}{B(T)}]\frac{L(T)}{L(t)}|\mathsf{F}_t]$$

$$= □[□^{x,\pi^*}(T)Z^{(t,1)}(T)|\mathsf{F}_t]$$

and

$$x = \mathrm{X}(t, \mathrm{X}^{-1}(t,x,y), y) = □[Z^{(t,1)}(T)□(\mathrm{X}^{-1}(t,x,y)Z^{(t,1)}(T))|\mathsf{F}_t]$$

we have

$$□^{x,\pi^*}(T) = □(\mathrm{X}^{-1}(t,x,y)Z^{(t,1)}(T))$$

so

$$\frac{□^{x,\pi^*}(s)B(t)}{B(s)} = □^*[\frac{□^{x,\pi^*}(T)B(t)}{B(T)}|\mathsf{F}_s]$$

$$= □[\frac{□^{x,\pi^*}(T)B(t)L(T)}{B(T)L(t)}|\mathsf{F}_s]/□[\frac{L(T)}{L(t)}|\mathsf{F}_s]$$

$$= \frac{L(t)}{L(s)}□[\frac{□^{x,\pi^*}(T)B(t)L(T)}{B(T)L(t)}|\mathsf{F}_s]$$

$$= \frac{L(t)}{L(s)}□[\frac{□(\mathrm{X}^{-1}(t,x,y)Z^{(t,1)}(T))B(t)L(T)}{B(T)L(t)}|\mathsf{F}_s]$$

then

$$□^{x,\pi^*}(s) = \frac{1}{Z^{(t,1)}(s)}□[□(\mathrm{X}^{-1}(t,x,y)Z^{(t,1)}(T))Z^{(t,1)}(T)|\mathsf{F}_s]$$

Proposition 3 □et

$□(t,z,y) = □[U(□(zZ^{(t,1)}(T)))]$, the value function and the optimal portfolio are given □y

$$V(t,x,y) = □(t, \mathrm{X}^{-1}(t,x,y), y)$$

$$\pi^*(s) = \frac{r - \mu + \tau - \lambda\varphi\theta_3}{\sigma^2}\frac{\mathrm{X}^{-1}(s, □^{x,\pi^*}(s), □(s))}{\mathrm{X}_x^{-1\,\prime}(s, □^{x,\pi^*}(s), □(s))}$$

$$-\frac{\rho a}{\sigma}\frac{\mathrm{X}_y^{-1\,\prime}(s, □^{x,\pi^*}(s), □(s))}{\mathrm{X}_x^{-1\,\prime}(s, □^{x,\pi^*}(s), □(s))}$$

Proof. □or $0 < \xi < z$,

$$z\mathrm{X}(t,z,y) - \xi\mathrm{X}(t,\xi,y) - \int_{\xi}^{z}\mathrm{X}(t,u,y)du$$

$$= □[Z^{(t,z)}(T)□(Z^{(t,z)}(T)) - Z^{(t,\xi)}(T)□(Z^{(t,\xi)}(T)) -$$

$$\int_{\xi}^{z}\frac{1}{u}Z^{(t,u)}(T)□(Z^{(t,u)}(T))du]$$

$$= □[Z^{(t,z)}(T)□(Z^{(t,z)}(T)) - Z^{(t,\xi)}(T)□(Z^{(t,\xi)}(T)) -$$

$$\int_{\xi Z^{(t,1)}(T)}^{zZ^{(t,1)}(T)} I(v)dv]$$
$$= E[U(I(zZ^{(t,1)}(T))) - U(I(\xi Z^{(t,1)}(T)))]$$
$$= G(t,z,y) - G(t,\xi,y)$$

Then we easily see that $G_z'(t,z,y) = z\mathrm{X}_z'(t,z,y)$.

$$V(t,x,y) = \sup_{\pi\in A} E[U(X^{x,\pi}(T)) \big| X^{x,\pi}(t) = x, Y(t) = y]$$
$$= E[U(X^{x,\pi^*}(T)) \big| X^{x,\pi^*}(t) = x, Y(t) = y]$$
$$= E[U(I(\mathrm{X}^{-1}(t,x,y)Z^{(t,1)}(T)))]$$
$$= G(t,\mathrm{X}^{-1}(t,x,y),y)$$

thus

$$V_x'(t,x,y) = G_z'(t,\mathrm{X}^{-1}(t,x,y),y)\mathrm{X}_x^{-1}(t,x,y) = \mathrm{X}^{-1}(t,x,y)$$
$$V_{xx}''(t,x,y) = \mathrm{X}^{-1}{}_x'(t,x,y),\ V_{xy}''(t,x,y) = \mathrm{X}^{-1}{}_y'(t,x,y)$$

applying **Proposition 1**, we have

$$\pi^*(s) = \frac{r-\mu+\tau-\lambda\varphi\theta_3}{\sigma^2}\frac{\mathrm{X}^{-1}(s,X^{x,\pi^*}(s),Y(s))}{\mathrm{X}^{-1}{}_x'(s,X^{x,\pi^*}(s),Y(s))} - \frac{\rho a}{\sigma}\frac{\mathrm{X}^{-1}{}_y'(s,X^{x,\pi^*}(s),Y(s))}{\mathrm{X}^{-1}{}_x'(s,X^{x,\pi^*}(s),Y(s))}$$

Proposition 4 (1) Assume that $U(x) = \ln x$, $0 < x < \infty$, then

$$\theta_3(s) = \frac{\varphi(r-\mu+\tau-\lambda\varphi\theta_3)}{\sigma^2 - \varphi(r-\mu+\tau-\lambda\varphi\theta_3)},$$

$$X^{x,\pi^*}(s) = \frac{x}{Z^{(t,1)}(s)},\ V(t,x,y) = \ln x - E[\ln Z^{(t,1)}(T)],$$

$$\pi^*(s) = \frac{\mu-r-\tau+\lambda\varphi\theta_3}{\sigma^2}X^{x,\pi^*}(s).$$

(2) Assume that $U(x) = \frac{x^{\gamma}}{\gamma}, 0 < x < \infty, 0 < \gamma < 1$, then for $s(0 \le t \le s \le T)$, $\theta_1(s)$, $\theta_2(s)$ and $\theta_3(s)$ satisfy (4), (5) and

$$(\gamma-1)\sigma^2(1+\theta_3)^{\frac{1}{\gamma-1}} = (\gamma-1)\sigma^2 + \varphi(r-\mu+\tau-\rho\sigma a(1-\gamma)\frac{\mathrm{X}_y'(s,1,Y(s))}{\mathrm{X}(s,1,Y(s))} - \lambda\varphi\theta_3)$$

the optimal wealth, the value function and the optimal portfolio are given by

$$X^{x,\pi^*}(s) = x\frac{\gamma(s)\,Z^{\frac{1}{\gamma-1}}(s)}{\gamma(t)\,Z^{\frac{1}{\gamma-1}}(t)}\exp\{\int_t^s \frac{-\gamma}{2(\gamma-1)^2}[\theta_1^2+\theta_2^2] - \lambda[(1+\theta_3)^{\frac{\gamma}{\gamma-1}} - \frac{\gamma}{\gamma-1}\theta_3 - 1]d\gamma\}$$

$$V(t,x,y) = \frac{x^{\gamma}}{\gamma}\mathrm{X}^{1-\gamma}(t,1,y)$$

$$\pi^*(s) = \frac{r-\mu+\tau-\lambda\varphi\theta_3}{\sigma^2}\frac{X^{x,\pi^*}(s)}{\gamma-1} + \frac{\rho a}{\sigma}X^{x,\pi^*}(s)\frac{\mathrm{X}_y'(s,1,Y(s))}{\mathrm{X}(s,1,Y(s))}$$

where

$$\mathrm{X}(t,1,y) = \exp\{\int_t^T \frac{-\gamma r}{\gamma-1} + \frac{\gamma}{2(\gamma-1)^2}[\theta_1^2+\theta_2^2] + \lambda[(1+\theta_3)^{\frac{\gamma}{\gamma-1}} - \frac{\gamma}{\gamma-1}\theta_3 - 1]d\gamma\}.$$

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