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Copula-Based Dependence Analysis of U.S. Stock Index and Futures Time Series in Financial Crisis

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Abstract
Copula modeling has become an increasingly popular tool in finance to model assets returns dependency as it can overcome the limitations of correlation when extreme losses occurred. In this study, we discussed the choice of an appropriate copula function aimed at adequately capturing the dependence between the return time series of S&P 500 stock index and futures in U.S. financial crisis. By comparing with the Gaussian, Student's t, Gumbel, Clayton and Frank copula, we concluded that Gumbel copula function can provide a better fit to the empirical data, and therefore well extract the dependence structure between S&P 500 stock index and futures in financial crisis.
Key words: Stock index; Dependence; Copula; Financial crisis

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INTRODUCTION
Modeling the dependence structure of assets returns time series has become an active topic of research in the finance field in recent years. A popular used dependence measure is correlation, which indicates the strength and direction of a linear relationship between two random variables. The best known correlation measure is the Pearson correlation coefficient. It is a reasonable measure when the random variables are normally distributed. But research shows that the multivariate normal distribution is inadequate because it underestimates both tail thickness of the marginal distributions and their dependence structure. Especially in financial crisis, data are approximated with more skewed distributions because of occasional, extreme losses. In 2008 financial crisis, U.S. stock markets have suffered their worst volatile trading days in memory, and various stock indices have fallen dramatically. The Dow Jones and S&P 500 are on course to record their worst yearly returns since the Great Depression. Meanwhile, U.S. stock index futures fluctuated a day after the Dow Jones snapped a seven-day losing streak. There is a number of empirical evidence that the dependence between many important asset returns is non-normal in crisis. Pearson correlation coefficient is not an appropriate dependence measure for very fat-tailed risks when extreme losses occurred. This inadequacy of correlation requires an appropriate dependence measure. Copula method may be the right tool for the job, which is applied to research on non-normal dependence of financial time series.

The primary motivation for this paper is as follows. Copula models for financial time series are used to extract the dependence between stock index futures and its underlying asset when crisis breaks out. In this study, we empirically examined the return time series of S&P 500 stock index and futures in 2008 financial crisis. It is concluded that, stock index and futures are not normally distributed in financial crisis and Gumbel copula function can provide a better fit to the empirical data.

Since Longin and Solnik (2001) have shown that the correlation between market returns is higher in case of extreme events, the number of papers on copula theory in finance and economics has grown enormously. Schmidt (2002) discusses the tail dependence property for some well-known examples of elliptical distributions. Cherubini et al. (2004) focus primarily on applications of copulas in mathematical finance and derivatives pricing. Rodriguez (2007) models dependence with switching-parameter copulas to study financial contagion. Bouye and Salmon (2009) introduce an approach to nonlinear regression model based on the copula function that defines the dependency structure between the variables. Guegan and Zhang (2010) propose a dynamic copula for measuring dependence in multivariate financial data.

1. COPULA MODELS

As is known to us, non-normality at the univariate level is associated with leptokurtosis phenomena, and the fat-tail problem. The use of copula functions enables us to model these features. A copula function links *n* univariate marginal distributions to a full multivariate distribution resulting in a joint distribution function of n standard uniform random variables. Consider a vector random variable, $X = [X_1, X_2, \ldots, X_n]$, with joint distribution F and marginal distributions $F_1, F_2, \ldots, F_n$. Sklar's theorem provides the mapping from the individual distribution functions to the joint distribution function:

$$F(x) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)), \quad \forall x \in \mathbb{R}^n. \qquad (1)$$

From any multivariate distribution F, we can extract the marginal distributions Fi, and the copula C, which captures the dependency structure among $X_1, X_2, \ldots, X_n$. And, more useful for time series model, given any set of marginal distributions ($F_1, F_2, \ldots, F_n$) and any copula C, equation (1) can be used to obtain a joint distribution with the given marginal distributions.

1.1 Elliptical Copulas

Copulas can be distinguished in the Elliptical and Archimedean family. Elliptical copulas are the ones with elliptical distributions and therefore symmetry in the tails. Two frequently used copulas in this family are the Gaussian and the student's t copula.

(1) Gaussian copula

Assume there are two random variables X and Y, the Gaussian copula is defined by

$$C_{Gaussian}(u, v; \rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} dx\, dy \qquad (2)$$

where Φρ is the bivariate standardized Gaussian cumulative distribution function (cdf) and the letter Φ represents the univariate standardized Gaussian cdf.

(2) Student's t copula

$$C_t(u, v; \rho, \upsilon) = t_{\rho,\upsilon}(t_\upsilon^{-1}(u), t_\upsilon^{-1}(v)) = \int_{-\infty}^{t_\upsilon^{-1}(u)} \int_{-\infty}^{t_\upsilon^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{x^2 - 2\rho xy + y^2}{\upsilon(1-\rho^2)}\right)^{-\frac{\upsilon+2}{2}} dxdy \qquad (3)$$

1.2 Archimedean Copulas

In comparison to Elliptical copulas, Archimedean copulas are constructed using a generator *øα*(*t*), indexed by the parameter α. By choosing the generator, one obtains a family of Archimedean copulas. The formulas of Gumbel, Clayton and Frank copula for the bivariate cases are given as follows.

$$C_{\text{Gumbel}}(u, v; \alpha) = \exp\left\{-[(-\ln u)^\alpha + (-\ln v)^\alpha]^{1/\alpha}\right\} \qquad (5)$$

$$C_{\text{Clayton}}(u, v; \alpha) = \max\left[(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}, 0\right] \qquad (6)$$

$$C_{\text{Frank}}(u, v; \alpha) = -\frac{1}{\alpha}\ln\left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right) \qquad (7)$$

2. DATA AND EMPIRICAL RESULTS

2.1 Data

We investigate the dependence between the S&P 500 stock index and futures time series data in U.S. financial crisis. On September 15th, 2008, bankruptcy of the investment bank "Lehman Brothers" in U.S. marked the beginning of global crisis. It resulted in a number of bank failures and sharp reductions in the value of stock worldwide. Then, a great many of the world's stock exchanges experienced the worst declines in their history, with drops of around 10% in most indices. For this reason, the empirical data covers the period from September 15th, 2008 to July 31st, 2009 when the stock and futures markets suffered a dramatic fluctuation in the crisis. The test data is the natural logarithm return of the closing price. All the estimation process is carried out in Matlab 7.7.0. Some descriptive statistics are presented in Table 1. As previously found in other studies, returns exhibit excess kurtosis and skewness. It is also illustrated in Figure 1. Distributions of return series are founded to be non-normality.

Table 1
Descriptive Statistics of S&P 500 Stock Index and Futures Returns Series

Series	Mean	Std. Dev.	Skewness	Kurtosis	Jarque-Bera	Probability
S&P 500	-0.0011	0.0300	-0.0483	4.6509	140.8539	0.0000
S&P 500 futures	-0.0011	0.0303	0.1808	5.7877	295.1440	0.0000

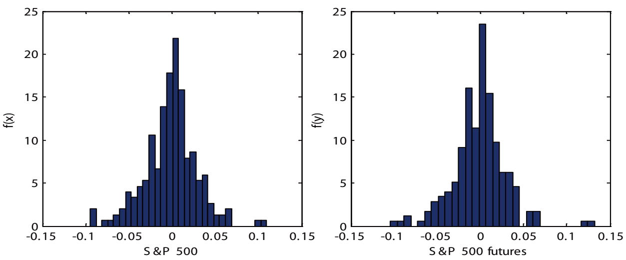

Figure 1
Frequency Histograms of S&P 500 and S&P 500 Futures Returns Series

2.2 Copula Choice for S&P 500 Stock Index and Futures

The parameters and formulas of Elliptical and Archimedean copulas were estimated as follows.

2.2.1 Elliptical Copulas

The correlation coefficient ρ of Gaussian copula function is 0.9843 and the function is

$$\hat{C}_{Gaussian}(u,v)=\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2p\sqrt{1-0.9843^2}}e^{-\frac{x^2-2\times0.9843xy+y^2}{2(1-0.9843^2)}}dxdy \tag{7}$$

The correlation coefficient ρ of Student's t copula is 0.9871 and the degree of freedom is 3. Then the function is

$$\hat{C}_t(u,v)=\int_{-\infty}^{t_3^{-1}(u)}\int_{-\infty}^{t_3^{-1}(v)}\frac{1}{2p\sqrt{1-0.9871^2}}(1+\frac{x^2-2\times0.9871xy+y^2}{3\times(1-0.9871^2)})^{-\frac{3+2}{2}}dxdy \tag{8}$$

The density functions $c(u,v)$ of Gaussian and Student's t copula are plotted in Figure 2. It is illustrated that both these copulas have symmetric tails and strong tail dependence exists between S&P 500 stock index and futures returns series. However, they have different characteristics in terms of tail dependence. The density function of Student's t copula has a little stronger tail than Gaussian copula.

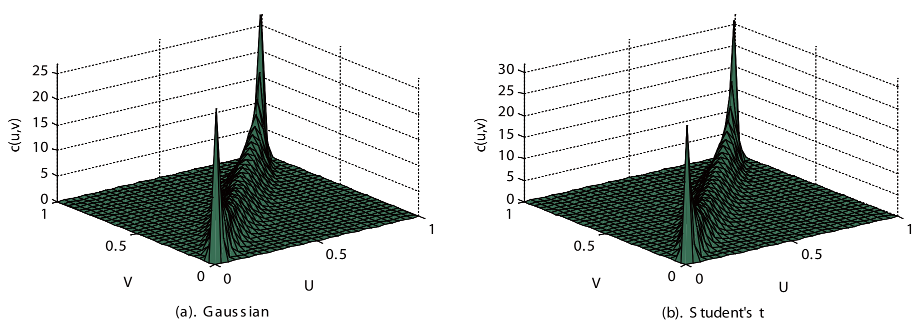

Figure 2
The Density Function C (U,V) of Gaussian and Student's T Copula

2.2.2 Archimedean Copulas

The parameters α of Archimedean copulas are estimated and the formulas of Gumbel copula, Clayton copula and Frank copula are followed in Table 2.

Table 2
Estimated Results of Archimedean Copulas

Archimedean copulas	Estimated parameter α	Formulas of copulas
Gumbel copula	9.1269	$\hat{C}_{Gumbel}(u,v)=\exp\left\{-[(-\ln u)^{9.1269}+(-\ln v)^{9.1269}]^{1/9.1269}\right\}$
Clayton copula	9.2345	$\hat{C}_{Clayton}(u,v)=\max\left[(u^{-9.2345}+v^{-9.2345}-1)^{-1/9.2345},0\right]$
Frank copula	34.8682	$\hat{C}_{Frank}(u,v)=-\frac{1}{34.8682}\ln(1+\frac{(e^{-34.8682u}-1)(e^{-34.8682v}-1)}{e^{-34.8682}-1})$

The density functions $c(u,v)$ of Gumbel, Clayton and Frank copulas are plotted in Figure 3. As shown that, they have different characteristics in terms of tail dependence. The Gumbel copula has asymmetric tails and the upper tail is stronger. The Clayton copula also has asymmetric tails, but differently, the lower tail is stronger than upper. The lower left tails are best described with Clayton copulas while the upper right tails are best described with Gumbel copula. Different with the former two copulas, the density functions of Frank copula are symmetry in the tails.

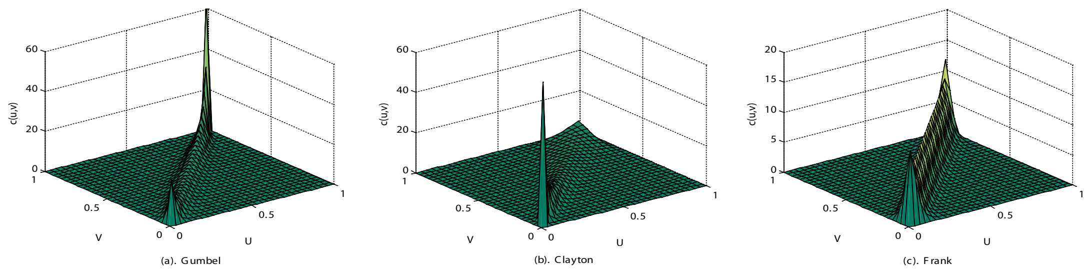

Figure 3
The Density Function c(u, v) of Gumbel, Clayton and Frank Copula

2.2.3 Comparison and Evaluation

In order to choose an appropriate copula model to describe the dependence structure of data, we introduce empirical copula to evaluate performances of the Elliptical and Archimedean family. When analyzing data with an unknown underlying distribution, one can transform the empirical data distribution into an empirical copula by warping such that the marginal distributions become uniform. Mathematically the empirical copula frequency function is calculated by

$$\hat{C}_{Empirical}(u,v)=\frac{1}{n}\sum_{i=1}^{n} I_{[F_n(x_i)\le u]} I_{[G_n(y_i)\le v]},\quad I_{[F_n(x_i)\le u]}=\begin{cases}1 & F_n(x_i)\le u\\ 0 & else\end{cases} \tag{9}$$

The Euclidean distance between each copula function and empirical copula are computed as

$$d=\sqrt{\sum_{i=1}^{n}\left|\hat{C}_{Empirical}(u_i,v_i)-C(u_i,v_i)\right|^2} \tag{8}$$

Results of $d_{Gaussian}$, d_t, d_{Gumbel}, $d_{Clayton}$ and d_{Frank} are 0.0954, 0.0927, 0.0883, 0.2939 and 0.1304. By comparing distances, we found that the distance between Gumbel and empirical copula is the smallest. It is suggested that the Gumbel copula can provide a better fit to the empirical data and therefore well extract the dependence structure between S&P 500 stock index and futures in financial crisis.

CONCLUSION

In this study, we discussed the choice of an appropriate copula function aimed at adequately capturing the dependence between the return time series of S&P 500 stock index and futures in U.S.. In 2008 financial crisis, data are approximated with more skewed distributions because of extreme losses. The linear correlation is inadequate for non-normal distributions. In this case, the Elliptical and Archimedean family of copulas are employed to extract the dependence structure.

To choose an appropriate copula to describe the dependence structure, we introduce empirical copula to evaluate performances of copulas. By comparing the distances between each copula function and empirical copula, we concluded that the dependence between the return series of S&P 500 stock index and futures in 2008 financial crisis can be well captured by the Gumbel copula function.

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