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Ratio Testing for Changes in the Long Memory Indexes in Presence of Breaks in Mean

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Abstract

In this paper, we consider the problem of detecting for breaks in the long memory indexes in the presence of breaks in mean. The limiting distribution is derived under the null hypothesis and the alternative hypothesis, the ratio tests also diverge to infinity as the sample size grows. These results show that the rejection rate seriously depends on the magnitude of change points. Finally, Monte Carlo study presents that our test has reasonably good size and power properties.

Key words: Break in mean; Long memory series; Ratio tests; Asymptotic properties

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INTRODUCTION

The problem of testing for structural breaks has an important issue in time series analysis since the change points are often interpreted as serious risks in econometrics and neglecting breaks can make radically misleading decisions. Most of the breaks can occur in the mean, variance or quantiles of a time series. For instance, Bai (1994) adopted least square estimation method to detect a shift in linear processes. Kokoszka and Leipus (1998) studied the change point in the mean of dependent observations. Perron (2005) introduced methodological issues related to estimation, testing and computation in the context of structural changes in the linear models in detail. Lebarbier (2005) considered and Detected multiple change-points in the mean of Gaussian process by model selection. Jin (2009) employed subsampling tests for the mean change point with heavy-tailed innovations. Zhao, Xia and Tian (2010) adopted ratio test to detect variance change point in linear process with long memory, in comparison with the existing CUSUM of squares (SCUSUM) test , the ratio test does not need to estimate the long memory parameter and it can be used more conveniently. Bai (2010) used the least squares method and the quasi maximum likelihood (QML) method to estimate breaks in means and in variances for panel data and found QML method was more efficient than the least squares even if there is no change in the variances. Qi (2014) structured Bootstrap monitoring for mean changes of nonparametric regression models by wavelets and indicated that their procedure have good power and short detection delay in the monitoring of structural change of nonparametric regression models. Li (2015) discussed variance change points detection in panel data models and proposed a CUSUM based statistic to test if there is a variance change point in panel data models. Recently, Khaleghi and Ryabko (2016) relied on nonparametric regression methods for testing and estimating breaks in highly dependent time series.

On the other hand, many scholars already have studied the innovations which are long memory series for a long time and one of the focus is on estimating parameters and detecting change points. Beran (1996) researched and detected a change point in the long memory parameter. Wang and Wang (2006) studied Changes of variance problem for linear processes with long memory and investigated the asymptotic properties of the test statistics. Wang (2009) utilized the GPH estimation of spatial long memory parameter to investigate a stationary long memory random fields. Shao (2011) proposed a simple testing procedure to test for a change point in the mean of a possibly long range dependent time series and estimated memory indexes with Local Whittle method, the test can be used to discriminate between a stationary long memory and short range dependent time series with a change point in mean. Hou and Perron (2014) Modified local Whittle estimator for long memory processes in the presence of low frequency (and other) contaminations. Recently, Gustavo (2015) adopted A Two-Stage Approach to analyse long memory series subject to structural change, which showed TSF methodology results in accurate and more robust forecasts when applied to long memory series with a break in the mean, these researches are in the case of constant indexes of long memory to analyze and study. In fact, it is possible to use models with long memory innovations including change points in both index and mean in a variety of practical problems.

In this paper, the goal of the article is to detect change points with ratio statistics to show the existence of change points in the long memory indexes in presence of breaks in mean. Therefore, the primary contributions of this paper include three aspects. First, we derive the asymptotic distribution of the proposed ratio tests diverge to infinity with the rate of T^{1-2d_0} under the null hypothesis. Second, under the alternative hypothesis, the ratio tests also diverge to infinity as the sample size grows. Third, the Monte Carlo study shows that our test has reasonably good size and power properties.

The remainder of the paper is structured as follows. Section 1 introduces some models, assumptions and test statistics. Section 2 contains the main results. Monte Carlo simulations are collected in Section 3. Section 4 draws a conclusion. Finally, all proofs are given in the appendix.

1. MODEL, ASSUMPTION AND STATISTIC

In the last two decades, we have witnessed a rapid development for statistical inference of long range dependent time series; see Beran (1994), Robinson (2003) among others for book-length treatments of this topic. Let

$$(1-L)^d z_t=\varepsilon_t,\ t\in Z,$$

where L is the backward shift operator and $\{\varepsilon_t\}$ is a mean zero covariance stationary dependent process. We say that the process $\{z_t\}$ possesses long memory if $d\in(0,0.5)$ and short memory if $d\in(-0.5,0)$.

In order to study a stochastic process $\{y_t\}$ existing change points in indexes, we consider the following linear regression model given by:

$$y_t=\alpha+z_t,\ t=1,2,\cdots,T,$$

where α is an arbitrary constant , and z_t is a stationary long memory series with index $d\in(0,0.5)$.

The null hypothesis can be described as

$$H_0: y_t=\begin{cases}\alpha_1+z_t^{d_0} & t=1,2,\cdots,[T\lambda]\\ \alpha_2+z_t^{d_0} & t=[T\lambda]+1,\cdots,T\end{cases}.$$

The alternative hypothesis is

$$H_1: y_t=\begin{cases}\alpha_{1[t\le T\lambda]}+z_t^{d_0} & t=1,2,\cdots,[T\tau^*]\\ \alpha_{2[t>T\lambda]}+z_t^{d_1} & t=[T\tau^*]+1,\cdots,T\end{cases}.$$

where λ, $\tau^*$are unknown and $[T\lambda]$,$[T\tau^*]$ are the integer part of $T\lambda$ and $T\tau^*$, $d_0\neq d_1$. For the purpose of asymptotic analysis, we make the following assumption.

Assumption 1. There exists a $d\in(0,0.5)$, such that as $T\to\infty$,

$$T^{-(\frac{1}{2}+d)}\sum_{t=1}^{[Tr]}\{z_t-Ez_t\}\Rightarrow C_d B_d(r),\ r\in[0,1].$$

where the symbol $\Rightarrow$ signifies weak convergence of the associated probability measures, C_d is a positive and $B_d(\cdot)$ is the fractional Brownian motion. Marinucci and Robinson (1999) has given as follows:

$$B_d(\tau)\equiv\frac{1}{\Gamma(1+d)}\{\int_0^\tau(\tau-s)^d W(s)+\int_{-\infty}^0[(\tau-s)^d-(-s)^d]\mathrm{d}W(s)\}$$

where $\Gamma(\cdot)$ is the Gamma function and $W(s)$ is a standard Brownian motion. The assumption has been extensively studied in the literature; see, Davidson, Jame, De, and Robert (2000), Mandelbrot and Vanness (1968).

Before expressing the test statistics, let $\hat{z}_t$, $t=1,2,\cdots,T$ be the residuals from the regression of y_t on a constant. Then, let S_t be the following partial sum process:

$$S_t=\sum_{j=1}^{t}\hat{z}_j \quad \text{for } t=1,2,\cdots,T.$$

Next, we can give some definitions about partial sum process respectively before and after break:

$$S_{1,t}(\tau)=\sum_{j=1}^{t}\hat{z}_j \quad \text{for } t=1,2,\cdots,[T\tau],$$

$$S_{2,t}(\tau)=\sum_{j=[T\tau]+1}^{t}\hat{z}_j \quad \text{for } t=[T\tau]+1,\cdots,T.$$

The ratio test is defined as follows:

$$\Xi_T(\tau)=\max\frac{[(1-\tau)T]^{-2}\sum_{[T\tau]+1}^{T}S_{2,t}(\tau)^2}{[T\tau]^{-2}\sum_{1}^{[T\tau]}S_{1,t}(\tau)^2}.$$

2. MAIN RESULTS

Theorem 2.1. Suppose that Assumption 1 is true for z_t under null hypothesis, then

(a) If $\tau<\lambda$, then $\Xi_T(\tau)=O_P(T^{1-2d_0})$.

(b) If $\tau\geq\lambda$, then $\Xi_T(\tau)=O_P(T^{2d_0-1})$.

Then

$$\Xi_T(\tau)=\max(\Xi_T(\tau),\Xi_T^{-1}(\tau))=O_P(T^{1-2d_0}).$$

Remark 2.1. The result shows that the limiting distribution depends strongly on the long memory index d_0 and sample size T.

Theorem 2.2. Suppose that Assumption 1 is true for z_t under alternative hypothesis, then

(a) If $\tau^*<\lambda$, $\tau\le\tau^*$, then $\Xi_T(\tau)=O_P(T^{1-2d_0})$;
$\tau^*<\tau\le\lambda$, $\Xi_T(\tau)=\infty$; $\lambda\le\tau$, $\Xi_T(\tau)=O_P(1)$

(b) If $\tau^*\geq\lambda$, $\le\lambda$, then $\Xi_T(\tau)=O_P(T^{1-2d_0})$;

$\lambda < \tau, \leq \tau^*$, $\Xi_T(\tau)=O_P(1)$; $\tau^* \leq \tau$, $\Xi_T(\tau)=O_P(1)$.

Then

$$\Xi_T(\tau)=\max(\Xi_T(\tau), \Xi_T^{-1}(\tau))=\infty.$$

Remark 2.2. These results show that statistics diverge to infinity as the sample size grows under the alternative hypothesis in the case of $\tau^* < \lambda$ or $\tau^* \geq \lambda$.

3. MONTE CARLO STUDY

In this section we use Monte Carlo study to evaluate the test in Section 2 and Section 3. All simulation are based on 1,000 replication. We report empirical rejection frequencies of the tests with T=500, 800, 1000 for tests run at α=0.95.

We consider the data generating processes, henceforth DGP's, which satisfy:

$$y_t = \alpha + z_t,\ t=1,2,\cdots,T,$$

where the innovations z_t is a stationary long memory series with indexes d. Subsequently, we consider the same model above allowing a change in mean α:

$$H_0 : y_t = \begin{cases} \alpha_1 + z_t^{d_0} & t = 1,2,\cdots,[T\lambda] \\ \alpha_2 + z_t^{d_0} & t = [T\lambda]+1,\cdots,T \end{cases}.$$

In addition, we also consider the model allowing changes in mean α and index d:

$$H_1 : y_t = \begin{cases} \alpha_{1[t\leq T\lambda]} + z_t^{d_0} & t = 1,2,\cdots,[T\tau^*] \\ \alpha_{2[t>T\lambda]} + z_t^{d_1} & t = [T\tau^*]+1,\cdots,T \end{cases}.$$

where $\Delta=\alpha_2-\alpha_1$ and d_0, $d_1 \in \{0, 0.1, 0.2, 0.3, 0.4\}$, the specific numerical simulations are expressed as follows:

Table 1
Empirical Size of Critical Value *P*=21.503

T		500			800			1000		
		λ								
	Δ	0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
d_0= 0	0	0.062	0.058	0.069	0.052	0.052	0.055	0.047	0.054	0.050
	0.2	0.184	0.293	0.279	0.220	0.406	0.357	0.296	0.501	0.451
	0.5	0.797	0.944	0.918	0.918	0.994	0.986	0.965	0.999	0.998
	1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
d_0= 0.1	0	0.058	0.082	0.082	0.060	0.062	0.058	0.059	0.059	0.044
	0.2	0.181	0.333	0.273	0.217	0.409	0.349	0.225	0.486	0.471
	0.5	0.785	0.953	0.921	0.923	0.994	0.985	0.962	0.976	0.998
	1	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
d_0= 0.2	0	0.037	0.045	0.030	0.040	0.042	0.053	0.041	0.036	0.040
	0.2	0.136	0.119	0.140	0.181	0.223	0.240	0.102	0.301	0.316
	0.5	0.408	0.774	0.757	0.739	0.953	0.951	0.844	0.985	0.978
	1	0.980	1.000	0.998	0.998	1.000	1.000	0.999	1.000	1.000
d_0= 0.3	0	0.056	0.039	0.036	0.057	0.049	0.037	0.045	0.044	0.033
	0.2	0.124	0.121	0.137	0.079	0.221	0.212	0.123	0.301	0.293
	0.5	0.405	0.756	0.764	0.711	0.946	0.939	0.837	0.984	0.973
	1	0.976	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000
d_0= 0.4	0	0.060	0.090	0.054	0.052	0.430	0.047	0.050	0.049	0.053
	0.2	0.110	0.214	0.180	0.128	0.140	0.230	0.312	0.324	0.204
	0.5	0.501	0.617	0.630	0.705	0.718	0.948	0.912	0.825	0.952
	1	0.908	1.000	0.949	0.950	0.978	0.980	0.979	1.000	1.000

We now discuss the main conclusions that can be drawn from Table 1. It shows the rejection rate at 95% under the null hypothesis and illustrates that ratio tests have a good size. For a given value of d_0, the size increases as Δ grows; In addition, for a given value of λ and Δ, the size reduces with increasing of d_0, e.g. T=500, λ=0.3 and Δ=0.2, if d=0, the size is 0.184, if d=0.1, the size is 0.181, if d=0.2, the size is 0.136.

Table 2
Empirical Power of the Ratio Test (*α*=95%)

$d_0 \to d_1$	T		500			800			1000		
	Δ	λ	τ								
			0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
0.4→0.3	0.2	0.3	0.381	0.451	0.490	0.398	0.393	0.501	0.540	0.602	0.550
		0.5	0.400	0.412	0.485	0.440	0.531	0.522	0.639	0.614	0.708
		0.7	0.394	0.430	0.391	0.493	0.519	0.601	0.624	0.652	0.625
	0.5	0.3	0.823	0.800	0.856	0.899	0.930	0.952	0.992	0.960	0.991
		0.5	0.910	0.871	0.887	0.966	0.948	0.928	0.998	0.985	0.996
		0.7	0.870	0.850	0.849	0.941	0.928	0.940	1.000	0.989	1.000
0.4→0.1	0.2	0.3	0.448	0.466	0.485	0.474	0.483	0.502	0.624	0.639	0.645
		0.5	0.470	0.502	0.511	0.482	0.624	0.549	0.651	0.660	0.718
		0.7	0.454	0.471	0.498	0.525	0.544	0.620	0.733	0.710	0.730
	0.5	0.3	0.860	0.856	0.880	1.000	0.970	0.998	1.000	0.997	1.000
		0.5	0.882	0.880	0.910	0.980	0.981	1.000	0.990	1.000	0.998
		0.7	0.923	0.908	0.875	0.988	0.979	0.981	1.000	1.000	1.000
0.3→0.2	0.2	0.3	0.331	0.404	0.419	0.418	0.422	0.429	0.592	0.597	0.611
		0.5	0.408	0.420	0.415	0.433	0.432	0.425	0.637	0.626	0.568
		0.7	0.394	0.413	0.391	0.434	0.437	0.431	0.564	0.549	0.589
	0.5	0.3	0.763	0.720	0.739	0.870	0.885	0.869	0.981	0.973	0.992
		0.5	0.731	0.780	0.727	0.893	0.893	0.904	0.984	0.989	0.968
		0.7	0.767	0.759	0.769	0.905	0.892	0.919	0.967	0.959	0.966
0.3→0.1	0.2	0.3	0.461	0.504	0.509	0.512	0.507	0.513	0.609	0.597	0.624
		0.5	0.482	0.469	0.495	0.518	0.520	0.532	0.650	0.631	0.628
		0.7	0.494	0.501	0.510	0.514	0.530	0.529	0.641	0.637	0.636
	0.5	0.3	0.838	0.841	0.844	0.867	0.885	0.893	0.979	0.981	0.982
		0.5	0.799	0.849	0.785	0.934	0.935	0.927	0.991	0.979	0.971
		0.7	0.838	0.840	0.839	0.925	0.928	0.939	0.970	0.979	0.968
0.2→0	0.2	0.3	0.402	0.415	0.422	0.460	0.454	0.501	0.524	0.554	0.571
		0.5	0.409	0.428	0.417	0.504	0.487	0.492	0.561	0.600	0.578
		0.7	0.414	0.424	0.420	0.467	0.488	0.478	0.545	0.601	0.525
	0.5	0.3	0.810	0.825	0.817	0.931	0.938	0.940	0.990	0.987	0.979
		0.5	0.857	0.841	0.850	0.954	0.927	0.938	0.998	1.000	0.998
		0.7	0.835	0.840	0.855	0.924	0.953	0.944	0.990	0.994	1.000

Table 3
Empirical Power of the Ratio Test (*α*=95%)

$d_0 \to d_1$	T		500			800			1000		
	Δ	λ	τ								
			0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
0→0.2	0.2	0.3	0.413	0.417	0.421	0.471	0.465	0.487	0.527	0.534	0.568
		0.5	0.409	0.423	0.425	0.500	0.494	0.486	0.579	0.612	0.614
		0.7	0.418	0.430	0.431	0.471	0.493	0.484	0.564	0.600	0.570
	0.5	0.3	0.824	0.827	0.822	0.928	0.931	0.926	0.986	0.983	0.974
		0.5	0.836	0.839	0.844	0.950	0.932	0.946	0.987	0.977	0.990
		0.7	0.860	0.856	0.855	0.926	0.941	0.937	0.991	0.978	0.997

To be continued

Continued

$d_0 \to d_1$	T		500			800			1000		
	Δ	λ	τ								
0.1→0.3	0.2	0.3	0.463	0.500	0.496	0.509	0.504	0.511	0.600	0.588	0.596
		0.5	0.478	0.455	0.475	0.514	0.521	0.531	0.644	0.640	0.624
		0.7	0.483	0.511	0.504	0.509	0.526	0.528	0.637	0.633	0.626
	0.5	0.3	0.826	0.829	0.835	0.869	0.879	0.891	0.977	0.979	0.983
		0.5	0.800	0.844	0.785	0.937	0.940	0.929	0.988	0.979	0.972
		0.7	0.824	0.833	0.840	0.933	0.927	0.931	0.972	0.965	0.955
0.1→0.4	0.2	0.3	0.441	0.465	0.454	0.564	0.530	0.577	0.633	0.636	0.637
		0.5	0.481	0.500	0.484	0.593	0.624	0.632	0.647	0.671	0.709
		0.7	0.444	0.456	0.502	0.525	0.548	0.633	0.729	0.718	0.733
	0.5	0.3	0.856	0.863	0.858	1.000	0.987	1.000	1.000	1.000	1.000
		0.5	0.874	0.867	0.907	0.993	0.977	0.994	0.991	0.987	1.000
		0.7	0.911	0.924	0.869	0.990	1.000	0.984	0.999	1.000	1.000
0.2→0.3	0.2	0.3	0.326	0.400	0.407	0.406	0.415	0.432	0.573	0.600	0.612
		0.5	0.405	0.415	0.413	0.431	0.428	0.424	0.634	0.598	0.621
		0.7	0.388	0.404	0.389	0.440	0.426	0.430	0.521	0.533	0.567
	0.5	0.3	0.699	0.702	0.718	0.866	0.875	0.865	0.979	0.968	0.988
		0.5	0.724	0.738	0.729	0.891	0.884	0.895	0.983	0.977	0.957
		0.7	0.755	0.761	0.753	0.900	0.913	0.920	0.965	0.947	0.960
0.3→0.4	0.2	0.3	0.378	0.464	0.488	0.387	0.386	0.500	0.541	0.600	0.535
		0.5	0.412	0.409	0.456	0.454	0.543	0.533	0.624	0.594	0.666
		0.7	0.400	0.411	0.389	0.490	0.508	0.544	0.704	0.674	0.633
	0.5	0.3	0.813	0.808	0.844	0.889	0.901	0.933	0.987	0.955	0.964
		0.5	0.912	0.869	0.864	0.947	0.954	0.943	0.987	0.992	0.975
		0.7	0.857	0.855	0.860	0.933	0.934	0.950	0.997	1.000	0.997

Tables 2-3 indicate the rejection rate are more greater with the larger distance from d_0 to d_1 and mean changes Δ under alternative hypothesis. If we set a value of d_0, the power increases with declining of d_1 in Table 2. Similarly, for a given value of d_0, the power increases as d_1 grows in Table 3. Meanwhile, it might be not intuitive that the power of breaks of equal distance. In addition, we found that mean change points have serious impact on power, if we set values of d_0 and d_1, the power increases as Δ grows, e.g. T=1000, λ=0.3, τ=0.7 and Δ=0.2, the power is 0.568 when d is from 0 to 0.2; Δ=0.5, but the power is 0.974 in Table 3. Ultimately, we observe that the effect of mean changes is greater than index points, mean change points have a decisive role in our study. On the whole, ratio tests depend on means and memory indexes, it is able to reject the null hypothesis and accept the alternative hypothesis to prove the existence of change points.

CONCLUSION

In this paper, change points are considered in the long memory indexes and means detected by the ratio test in the regression model. The asymptotic distribution of our tests diverge to infinity under the null hypothesis and is divergent as the sample size increases under the alternative hypothesis. Moreover, the Monte Carlo studies have been conducted to investigate the performance of our test procedures and show the existence of change points in memory indexes may be unambiguous. Overall, the simulation results reveal the reject rate heavily depends on the magnitude of change points.

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APPENDIX

Proof of theorem 2.1. First, discussing the first case of $\tau < \lambda$, for the denominator

$$T^{-2d_0}[T\tau]^{-2}\sum_{1}^{T\tau} S_{1,\ t}(\tau)^2 \Rightarrow \tau^{-2}C_{d_0}{}^2\int_0^\tau \psi_2(s,\tau)^2\,\mathrm{d}s\ .$$

Then, for the nominator, if $T\tau < t \le T\lambda$, we have

$$\bar{y} = \frac{1}{T-T\tau}\sum_{j=[T\tau]+1}^{T} y_j$$

$$= \frac{1}{T-T\tau}(\sum_{j=[T\tau]+1}^{[T\lambda]} y_j + \sum_{j=[T\lambda]+1}^{T} y_j)$$

$$= \frac{\lambda-\tau}{1-\tau}\alpha_1 + \frac{1-\lambda}{1-\tau}\alpha_2 = \tilde{\alpha}\ .$$

$$\sum_{t=[T\tau]+1}^{[T\lambda]}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2 = \sum_{t=[T\tau]+1}^{[T\lambda]}[\sum_{j=[T\tau]+1}^{t}(\alpha_1 + z_j - \bar{y})]^2$$

$$= T(\lambda-\tau)[(t-T\tau)(\alpha_1-\tilde{\alpha})]^2$$

$$= T(\lambda-\tau)[\frac{(t-T\tau)(1-\lambda)}{1-\tau}(\alpha_1-\alpha_2)]^2\ .$$

$$T^{-1}\cdot[(1-\tau)T]^{-2}\sum_{t=[T\tau]+1}^{[T\lambda]}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2=T^{-1}\cdot[(1-\tau)T]^{-2}T(\lambda-\tau)[\frac{(t-T\tau)(1-\lambda)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$=T^{-1}\cdot T(1-\tau)^{-2}(\lambda-\tau)[T^{-1}\cdot\frac{(t-T\tau)(1-\lambda)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow(\lambda-\tau)[\frac{\tau(\lambda-1)}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2 .$$

If $T\lambda < t \leq T$, we have

$$\sum_{t=[T\lambda]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2=\sum_{t=[T\lambda]+1}^{T}[\sum_{j=[T\tau]+1}^{[T\lambda]}(\alpha_1+z_j-\bar{y})+\sum_{j=[T\lambda]+1}^{t}(\alpha_2+z_j-\bar{y})]^2$$

$$=\sum_{t=[T\lambda]+1}^{T}[T(\lambda-\tau)\alpha_1+(t-T\lambda)\alpha_2-(t-T\tau)\tilde{\alpha}]^2$$

$$=T(1-\lambda)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2 .$$

$$T^{-1}\cdot[(1-\tau)T]^{-2}\sum_{t=[T\lambda]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2=T^{-1}\cdot[(1-\tau)T]^{-2}T(1-\lambda)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$=T^{-1}\cdot(1-\tau)^{-2}T(1-\lambda)[T^{-1}\cdot\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow(1-\lambda)[\frac{\lambda-\tau}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2 .$$

According to the above analysis, we can obtain

$$\Xi_T(\tau)=\frac{O_P(T)+O_P(T)}{O_P(T^{2d_0})}=\frac{2O_P(T)}{O_P(T^{2d_0})} .$$

Then, discussing the second case of $\lambda \leq \tau$, for the nominator

$$T^{-2d_0}[(1-\tau)T]^{-2}\sum_{T\tau+1}^{T}S_{2,\ t}(\tau)^2\Rightarrow(1-\tau)^{-2}C_{d_0}{}^2\int_{\tau}^{1}\psi_1(s,\tau)^2\,\mathrm{d}s .$$

For the denominator, if $1<t \leq T\lambda$, we have

$$\bar{y}=\frac{1}{T\tau}\sum_{j=1}^{[T\tau]}y_j$$

$$=\frac{1}{T\tau}(\sum_{j=1}^{[T\lambda]}y_j+\sum_{j=[T\lambda]+1}^{[T\tau]}y_j)$$

$$=\frac{\lambda}{\tau}\alpha_1+\frac{\tau-\lambda}{\tau}\alpha_2=\tilde{\alpha}_1 .$$

$$\sum_{t=1}^{[T\lambda]}(\sum_{j=1}^{t}\hat{z}_j)^2=\sum_{t=1}^{[T\lambda]}[\sum_{j=1}^{t}(\alpha_1+z_j-\bar{y})]^2$$

$$=T\lambda[t(\alpha_1-\tilde{\alpha}_1)]^2$$

$$=T\lambda[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2 .$$

$$T\cdot[T\tau]^{-2}\sum_{t=1}^{[T\lambda]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T\cdot[T\tau]^{-2}T\lambda[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$= T\cdot T^{-2}T\lambda[\frac{t(\tau-\lambda)}{\tau^2}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow \lambda[\frac{t(\tau-\lambda)}{\tau^2}(\alpha_1-\alpha_2)]^2 .$$

If $T\lambda < t \le T\tau$, we have

$$\sum_{t=[T\lambda]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2 = \sum_{t=[T\lambda]+1}^{[T\tau]}[\sum_{j=1}^{[T\lambda]}(\alpha_1+z_j-\bar{y})+\sum_{j=[T\lambda]+1}^{t}(\alpha_2+z_j-\bar{y})]^2$$

$$= \sum_{t=[T\lambda]+1}^{[T\tau]}[T\lambda\alpha_1+(t-T\lambda)\alpha_2-t\tilde{\alpha}_1]^2$$

$$= T(\tau-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2 .$$

$$T^{-1}\cdot[T\tau]^{-2}\sum_{t=[T\lambda]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T^{-1}\cdot[T\tau]^{-2}T(\tau-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$= T^{-1}\cdot\tau^{-2}T(\tau-\lambda)[T^{-1}\cdot\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow (\tau-\lambda)[\frac{\lambda}{\tau}(\alpha_1-\alpha_2)]^2 .$$

According to the above analysis, we can obtain

$$\Xi_T(\tau) = \frac{O_P(T^{2d_0})}{O_P(T^{-1})+O_P(T)} .$$

This proves the theorem.

Proof of theorem 2.2. First, discussing the first case of $\tau^* < \lambda$, and if $\tau \le \tau^*$, for the denominator

$$T^{-2d_0}[T\tau]^{-2}\sum_{1}^{T\tau}S_{1,\ t}(\tau)^2 \Rightarrow \tau^{-2}C_{d_0}{}^2\int_0^\tau \psi_2(s,\tau)^2\mathrm{d}s .$$

For the nominator, if $T_\tau < t \le T\tau^*$, we have

$$\bar{y} = \frac{1}{T-T\tau}\sum_{j=[T\tau]+1}^{T}y_j$$

$$= \frac{1}{T-T\tau}(\sum_{j=[T\tau]+1}^{[T\tau^*]}y_j+\sum_{j=[T\tau^*]+1}^{[T\lambda]}y_j+\sum_{j=[T\lambda]+1}^{T}y_j)$$

$$= \frac{1}{T-T\tau}[\sum_{j=[T\tau]+1}^{[T\tau^*]}(\alpha_1+z_{1j})+\sum_{j=[T\tau^*]+1}^{[T\lambda]}(\alpha_1+z_{2j})+\sum_{j=[T\lambda]+1}^{T}(\alpha_2+z_{2j})]$$

$$= \frac{1}{T-T\tau}[(T\tau^*-T\tau)\alpha_1+(T\lambda-T\tau^*)\alpha_1+(T-T\lambda)\alpha_2]$$

$$= \frac{\lambda-\tau}{1-\tau}\alpha_1+\frac{1-\lambda}{1-\tau}\alpha_2 = \tilde{\alpha} .$$

$$\sum_{t=[T\tau]+1}^{[T\tau^*]} (\sum_{j=[T\tau]+1}^{t} \hat{z}_j)^2 = \sum_{t=[T\tau]+1}^{[T\tau^*]} [\sum_{j=[T\tau]+1}^{t} (\alpha_1 + z_{1j} - \bar{y})]^2$$

$$= \sum_{t=[T\tau]+1}^{[T\tau^*]} [(t - T\tau)(\alpha_1 - \tilde{\alpha})]^2$$

$$= \sum_{t=[T\tau]+1}^{[T\tau^*]} [\frac{(t - T\tau)(1-\lambda)}{1-\tau}(\alpha_1 - \alpha_2)]^2$$

$$= T(\tau^* - \tau)[\frac{(t - T\tau)(1-\lambda)}{1-\tau}(\alpha_1 - \alpha_2)]^2 .$$

$$T^{-1} \cdot [(1-\tau)T]^{-2} \sum_{t=[T\tau]+1}^{[T\tau^*]} (\sum_{j=[T\tau]+1}^{t} \hat{z}_j)^2 = T^{-1} \cdot [(1-\tau)T]^{-2} T(\tau^* - \tau)[\frac{(t - T\tau)(1-\lambda)}{1-\tau}(\alpha_1 - \alpha_2)]^2$$

$$= T^{-1} \cdot (1-\tau)^{-2} T(\tau^* - \tau)[T^{-1} \cdot \frac{(t - T\tau)(1-\lambda)}{1-\tau}(\alpha_1 - \alpha_2)]^2$$

$$\Rightarrow (\tau^* - \tau)[\frac{\tau(\lambda - 1)}{(1-\tau)^2}(\alpha_1 - \alpha_2)]^2 .$$

If $T\tau^* < t \leq T\lambda$, we have

$$\sum_{t=[T\tau^*]+1}^{[T\lambda]} (\sum_{j=[T\tau]+1}^{t} \hat{z}_j)^2 = \sum_{t=[T\tau^*]+1}^{[T\lambda]} [\sum_{j=[T\tau]+1}^{[T\tau^*]} (y_j - \bar{y}) + \sum_{j=[T\tau^*]+1}^{t} (y_j - \bar{y})]^2$$

$$= \sum_{t=[T\tau^*]+1}^{[T\lambda]} [\sum_{j=[T\tau]+1}^{[T\tau^*]} (\alpha_1 + z_{1j}) + \sum_{j=[T\tau^*]+1}^{t} (\alpha_1 + z_{2j}) - \sum_{j=[T\tau]+1}^{t} \bar{y}]^2$$

$$= \sum_{t=[T\tau^*]+1}^{[T\lambda]} [(T\tau^* - T\tau)\alpha_1 + (t - T\tau^*)\alpha_1 - (t - T\tau)\tilde{\alpha}]^2$$

$$= \sum_{t=[T\tau^*]+1}^{[T\lambda]} [(t - T\tau)(\alpha_1 - \tilde{\alpha})]^2$$

$$= T(\lambda - \tau^*)[\frac{(t - T\tau)(1-\lambda)}{1-\tau}(\alpha_1 - \alpha_2)]^2 .$$

$$T^{-1} \cdot [(1-\tau)T]^{-2} \sum_{t=[T\tau^*]+1}^{[T\lambda]} (\sum_{j=[T\tau]+1}^{t} \hat{z}_j)^2 = T^{-1} \cdot [(1-\tau)T]^{-2} T(\lambda - \tau^*)[\frac{(t - T\tau)(1-\lambda)}{1-\tau}(\alpha_1 - \alpha_2)]^2$$

$$= T^{-1} \cdot (1-\tau)^{-2} T(\lambda - \tau^*)[T^{-1} \cdot \frac{(t - T\tau)(1-\lambda)}{1-\tau}(\alpha_1 - \alpha_2)]^2$$

$$\Rightarrow (\lambda - \tau^*)[\frac{\tau(\lambda - 1)}{(1-\tau)^2}(\alpha_1 - \alpha_2)]^2 .$$

If $T\lambda < t \leq T$, we have

$$\sum_{t=[T\lambda]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2=\sum_{t=[T\lambda]+1}^{T}[\sum_{j=[T\tau]+1}^{[T\tau^*]}(y_j-\bar{y})+\sum_{j=[T\tau^*]+1}^{[T\lambda]}(y_j-\bar{y})+\sum_{j=[T\lambda]+1}^{t}(y_j-\bar{y})]^2$$

$$=\sum_{t=[T\lambda]+1}^{T}[\sum_{j=[T\tau]+1}^{[T\tau^*]}(\alpha_1+z_{1j})+\sum_{j=[T\tau^*]+1}^{[T\lambda]}(\alpha_1+z_{2j})+\sum_{j=[T\lambda]+1}^{t}(\alpha_2+z_{2j})-\sum_{j=[T\tau]+1}^{t}\bar{y}]^2$$

$$=\sum_{t=[T\lambda]+1}^{T}[(T\lambda-T\tau)\alpha_1+(t-T\lambda)\alpha_2-(t-T\tau)\tilde{\alpha}]^2$$

$$=T(1-\lambda)[\frac{(\lambda-\tau)(T-t)}{1-\tau}(\alpha_1-\alpha_2)]^2 .$$

$$T^{-1}\cdot[(1-\tau)T]^{-2}\sum_{t=[T\lambda]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2=T^{-1}\cdot[(1-\tau)T]^{-2}T(1-\lambda)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$=T^{-1}\cdot(1-\tau)^{-2}T(1-\lambda)[T^{-1}\cdot\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow(1-\lambda)[\frac{\lambda-\tau}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2 .$$

According to the above analysis, we can obtain

$$\Xi_T(\tau)=\frac{O_P(T)+O_P(T)+O_P(T)}{O_P(T^{2d_0})}=\frac{3O_P(T)}{O_P(T^{2d_0})} .$$

If $\tau^*<\tau\le\lambda$, for the denominator, we have

$$\sum_{t=1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2=\sum_{t=1}^{[T\tau^*]}(\sum_{j=1}^{t}\hat{z}_j)^2+\sum_{t=[T\tau^*]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2$$

$$=\sum_{t=1}^{[T\tau^*]}[\sum_{j=1}^{t}(y_j-\bar{y})]^2+\sum_{t=[T\tau^*]+1}^{[T\tau]}[\sum_{j=1}^{[T\tau^*]}(y_j-\bar{y})+\sum_{j=[T\tau^*]+1}^{t}(y_j-\bar{y})]^2$$

$$=\sum_{t=1}^{[T\tau^*]}[\sum_{j=1}^{t}(\alpha_1+z_{1j}-\bar{y})]^2+\sum_{t=[T\tau^*]+1}^{[T\tau]}[\sum_{j=1}^{[T\tau^*]}(\alpha_1+z_{1j}-\bar{y})+\sum_{j=[T\tau^*]+1}^{t}(\alpha_1+z_{2j}-\bar{y})]^2$$

$$=\sum_{t=1}^{[T\tau^*]}(t\alpha_1-t\bar{y})^2+\sum_{t=[T\tau^*]+1}^{[T\tau]}(t\alpha_1-t\bar{y})^2$$

$$=\sum_{t=1}^{[T\tau^*]}(t\alpha_1-t\alpha_1)^2+\sum_{t=[T\tau^*]+1}^{[T\tau]}(t\alpha_1-t\alpha_1)^2=0 .$$

Thus, it is related to the index, we have

$$\sum_{t=1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2=\sum_{t=1}^{[T\tau^*]}(\sum_{j=1}^{t}z_{1j})^2+\sum_{t=[T\tau^*]+1}^{[T\tau]}(\sum_{j=1}^{t}z_{1j}+\sum_{j=[T\tau^*]+1}^{t}z_{2j})^2 .$$

$$T^{-2d_0}[T\tau]^{-2}\sum_{t=1}^{[T\tau^*]}(\sum_{j=1}^{t} z_{1j})^2 = T^{-1}\tau^{-2}\sum_{t=1}^{[T\tau^*]}(T^{-\frac{1}{2}-d_0}\sum_{j=1}^{t} z_{1j})^2$$

$$= \tau^{-2}\cdot T^{-1}\sum_{t=1}^{[T\tau^*]}(T^{-\frac{1}{2}-d_0}\sum_{j=1}^{t} z_{1j})^2$$

$$\Rightarrow \tau^{-2}\int_0^{\tau^*}(C_{d_0}B_{d_0}(s))^2\,\mathrm{d}s\,.$$

The same can be

$$T^{-2d_1}[T\tau]^{-2}\sum_{t=[T\tau^*]+1}^{[T\tau]}(\sum_{j=1}^{[T\tau^*]} z_{1j} + \sum_{j=[T\tau^*]+1}^{t} z_{2j})^2 \Rightarrow \tau^{-2}\int_{\tau^*}^{\tau}[(C_{d_0}B_{d_0}(\tau^*)) + (C_{d_1}B_{d_1}(s))]^2\,\mathrm{d}s\,.$$

For the nominator, we have

$$\bar{y} = \frac{1}{T-T\tau}\sum_{j=[T\tau]+1}^{T} y_j$$

$$= \frac{1}{T-T\tau}(\sum_{j=[T\tau]+1}^{[T\lambda]} y_j + \sum_{j=[T\lambda]+1}^{T} y_j)$$

$$= \frac{\lambda-\tau}{1-\tau}\alpha_1 + \frac{1-\lambda}{1-\tau}\alpha_2 = \tilde{\alpha}\,.$$

$$\sum_{t=[T\tau]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2 = \sum_{t=[T\tau]+1}^{[T\lambda]}[\sum_{j=[T\tau]+1}^{t}(y_j-\bar{y})]^2 + \sum_{t=[T\lambda]+1}^{T}[\sum_{j=[T\tau]+1}^{t}(y_j-\bar{y})]^2$$

$$= \sum_{t=[T\tau]+1}^{[T\lambda]}[\sum_{j=[T\tau]+1}^{t}(y_j-\bar{y})]^2 + \sum_{t=[T\lambda]+1}^{T}[\sum_{j=[T\tau]+1}^{[T\lambda]}(y_j-\bar{y}) + \sum_{j=[T\lambda]+1}^{t}(y_j-\bar{y})]^2$$

$$= \sum_{t=[T\tau]+1}^{[T\lambda]}[(t-T\tau)(\alpha_1-\tilde{\alpha})]^2 + \sum_{t=[T\lambda]+1}^{T}[(T\lambda-T\tau)\alpha_1 + (t-T\lambda)\alpha_2 - (t-T\tau)\tilde{\alpha}]^2$$

$$= T(\lambda-\tau)[\frac{(t-T\tau)(1-\lambda)}{1-\tau}(\alpha_1-\alpha_2)]^2 + T(1-\lambda)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2.$$

$$T^{-1}\cdot[(1-\tau)T]^{-2}\sum_{t=[T\tau]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2 \Rightarrow (\lambda-\tau)[\frac{\tau(\lambda-1)}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2 + (1-\lambda)[\frac{\lambda-\tau}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2.$$

$$\Xi_T(\tau) = \frac{O_P(T)}{O_P(T^{2d_0}) + O_P(T^{2d_1})}$$

If $\lambda \le \tau$, for the nominator

$$T^{-2d_1}[(1-\tau)T]^{-2}\sum_{T\tau+1}^{T} S_{2,\ t}(\tau)^2 \Rightarrow (1-\tau)^{-2}C_{d_1}{}^2\int_{\tau}^{1}\psi_1(s,\tau)^2\,\mathrm{d}s\,.$$

For the denominator, if $1 < t \le T\tau^*$, we have

$$\bar{y} = \frac{1}{T\tau}\sum_{j=1}^{[T\tau]} y_j$$

$$= \frac{1}{T\tau}(\sum_{j=1}^{[T\tau^*]} y_j + \sum_{j=[T\tau^*]+1}^{[T\lambda]} y_j + \sum_{j=[T\lambda]+1}^{[T\tau]} y_j)$$

$$= \frac{1}{T\tau}[T\tau^*\alpha_1 + T(\lambda-\tau^*)\alpha_1 + T(\tau-\lambda)\alpha_2]$$

$$= \frac{\lambda}{\tau}\alpha_1 + \frac{\tau-\lambda}{\tau}\alpha_2 = \tilde{\alpha}_1 .$$

$$\sum_{t=1}^{[T\tau^*]}(\sum_{j=1}^{t}\hat{z}_j)^2 = \sum_{t=1}^{[T\tau^*]}(\sum_{j=1}^{t}(\alpha_1 + z_{1j} - \bar{y}))^2$$

$$= T\tau^*(t\alpha_1 - t\tilde{\alpha}_1)^2$$

$$= T\tau^*[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2 .$$

$$T\cdot[T\tau]^{-2}\sum_{t=1}^{[T\tau^*]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T\cdot[T\tau]^{-2}T\tau^*[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow \tau^*[\frac{t(\tau-\lambda)}{\tau^2}(\alpha_1-\alpha_2)]^2 .$$

If $T\tau < t \le T\lambda$, we have

$$\sum_{t=[T\tau^*]+1}^{[T\lambda]}(\sum_{j=1}^{t}\hat{z}_j)^2 = \sum_{t=[T\tau^*]+1}^{[T\lambda]}[\sum_{j=1}^{[T\tau^*]}(\alpha_1 + z_{1j} - \bar{y}) + \sum_{j=[T\tau^*]+1}^{t}(\alpha_1 + z_{2j} - \bar{y})]^2$$

$$= T(\lambda-\tau^*)(t\alpha_1 - t\tilde{\alpha}_1)^2$$

$$= T(\lambda-\tau^*)[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2.$$

$$T\cdot[T\tau]^{-2}\sum_{t=[T\tau^*]+1}^{[T\lambda]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T\cdot[T\tau]^{-2}T(\lambda-\tau^*)[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow (\lambda-\tau^*)[\frac{t(\tau-\lambda)}{\tau^2}(\alpha_1-\alpha_2)]^2 .$$

If $T\lambda < t \le T\tau$, we have

$$\sum_{t=[T\lambda]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2 = \sum_{t=[T\lambda]+1}^{[T\tau]}[\sum_{j=1}^{[T\tau^*]}(y_j - \bar{y}) + \sum_{j=[T\tau^*]+1}^{[T\lambda]}(y_j - \bar{y}) + \sum_{j=[T\lambda]+1}^{t}(y_j - \bar{y})]^2$$

$$= \sum_{t=[T\lambda]+1}^{[T\tau]}[\sum_{j=1}^{[T\tau^*]}(\alpha_1 + z_{1j}) + \sum_{j=[T\tau^*]+1}^{[T\lambda]}(\alpha_1 + z_{2j}) + \sum_{j=[T\lambda]+1}^{t}(\alpha_2 + z_{2j}) - t\bar{y}]^2$$

$$= \sum_{t=[T\lambda]+1}^{[T\tau]}[T\lambda\alpha_1 + (t-T\lambda)\alpha_2 - t\tilde{\alpha}_1]^2$$

$$= T(\tau-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$T^{-1}\cdot[T\tau]^{-2}\sum_{t=[T\lambda]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T^{-1}\cdot[T\tau]^{-2}T(\tau-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow (\tau-\lambda)[\frac{\lambda}{\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$\Xi_T(\tau) = \frac{O_P(T^{2d_1})}{O_P(T^{-1})+O_P(T^{-1})+O_P(T)} = \frac{O_P(T^{2d_1})}{2O_P(T^{-1})+O_P(T)}$$

Then, discussing the second case of $\lambda \le \tau^*$, if $\tau \le \lambda$, for the denominator

$$T^{-2d_0}[T\tau]^{-2}\sum_{1}^{T\tau}S_{1,\ t}(\tau)^2 \Rightarrow \tau^{-2}C_{d_0}{}^2\int_0^\tau \psi_2(s,\tau)^2\,\mathrm{d}s \,.$$

For the nominator, if $T\tau < t \le T\lambda$, we have

$$\sum_{t=[T\tau]+1}^{[T\lambda]}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2 = \sum_{t=[T\tau]+1}^{[T\lambda]}[\sum_{j=[T\tau]+1}^{t}(\alpha_1+z_{1j}-\bar{y})]^2$$

$$= \sum_{t=[T\tau]+1}^{[T\lambda]}[(t-T\tau)\alpha_1-(t-T\tau)\tilde{\alpha}]^2$$

$$= T(\lambda-\tau)[\frac{(t-T\tau)(1-\lambda)}{1-\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$T^{-1}\cdot[(1-\tau)T]^{-2}\sum_{t=[T\tau]+1}^{[T\lambda]}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2 = T^{-1}\cdot[(1-\tau)T]^{-2}T(\lambda-\tau)[\frac{(t-T\tau)(1-\lambda)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow (\lambda-\tau)[\frac{\tau(\lambda-1)}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2 \quad .$$

If $T\lambda < t \le T\tau^*$, we have

$$\sum_{t=[T\lambda]+1}^{[T\tau^*]}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2 = \sum_{t=[T\lambda]+1}^{[T\tau^*]}[\sum_{j=[T\tau]+1}^{[T\lambda]}(\alpha_1+z_{1j}-\bar{y})+\sum_{j=[T\lambda]+1}^{t}(\alpha_2+z_{1j}-\bar{y})]^2$$

$$= \sum_{t=[T\lambda]+1}^{[T\tau^*]}[T(\lambda-\tau)\alpha_1+(t-T\lambda)\alpha_2-(t-T\tau)\tilde{\alpha}]^2$$

$$= T(\tau^*-\lambda)[T(\lambda-\tau)\alpha_1+(t-T\lambda)\alpha_2-(t-T\tau)\tilde{\alpha}]^2$$

$$= T(\tau^*-\lambda)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$T^{-1}\cdot[(1-\tau)T]^{-2}\sum_{t=[T\lambda]+1}^{[T\tau^*]}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2 = T^{-1}\cdot[(1-\tau)T]^{-2}T(\tau^*-\lambda)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow (\tau^*-\lambda)[\frac{\lambda-\tau}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2 \quad .$$

If $T\tau^* < t \le T$, we have

$$\sum_{t=[T\tau^*]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2=\sum_{t=[T\tau^*]+1}^{T}[\sum_{j=[T\tau]+1}^{[T\lambda]}(y_j-\bar{y})+\sum_{j=[T\lambda]+1}^{[T\tau^*]}(y_j-\bar{y})+\sum_{j=[T\tau^*]+1}^{t}(y_j-\bar{y})]^2$$

$$=\sum_{t=[T\tau^*]+1}^{T}[\sum_{j=[T\tau]+1}^{[T\lambda]}(\alpha_1+z_{1j})+\sum_{j=[T\lambda]+1}^{[T\tau^*]}(\alpha_2+z_{1j})+\sum_{j=[T\tau^*]+1}^{t}(\alpha_2+z_{2j})-\sum_{j=[T\tau]+1}^{t}\bar{y}]^2$$

$$=\sum_{t=[T\tau^*]+1}^{T}[(T\lambda-T\tau)\alpha_1+(t-T\lambda)\alpha_2-(t-T\tau)\tilde{\alpha}]^2$$

$$=T(1-\tau^*)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$T^{-1}\cdot[(1-\tau)T]^{-2}\sum_{t=[T\tau^*]+1}^{T}(\sum_{j=[T\tau]+1}^{t}\hat{z}_j)^2=T^{-1}\cdot[(1-\tau)T]^{-2}T(1-\tau^*)[\frac{(T-t)(\lambda-\tau)}{1-\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow(1-\tau^*)[\frac{\lambda-\tau}{(1-\tau)^2}(\alpha_1-\alpha_2)]^2 \quad .$$

According to the above analysis, we can obtain

$$\Xi_T(\tau)=\frac{O_P(T)+O_P(T)+O_P(T)}{O_P(T^{2d_0})}=\frac{3O_P(T)}{O_P(T^{2d_0})} \quad .$$

If $\lambda<\tau\le\tau^*$, for the denominator, we have

$$\sum_{t=1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2=\sum_{t=1}^{[T\lambda]}(\sum_{j=1}^{t}\hat{z}_j)^2+\sum_{t=[T\lambda]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2$$

$$=\sum_{t=1}^{[T\lambda]}[\sum_{j=1}^{t}(y_j-\bar{y})]^2+\sum_{t=[T\tau^*]+1}^{[T\tau]}[\sum_{j=1}^{[T\lambda]}(y_j-\bar{y})+\sum_{j=[T\lambda]+1}^{t}(y_j-\bar{y})]^2$$

$$=\sum_{t=1}^{[T\lambda]}[\sum_{j=1}^{t}(\alpha_1+z_{1j}-\bar{y})]^2+\sum_{t=[T\lambda]+1}^{[T\tau]}[\sum_{j=1}^{[T\lambda]}(\alpha_1+z_{1j}-\bar{y})+\sum_{j=[T\lambda]+1}^{t}(\alpha_2+z_{1j}-\bar{y})]^2$$

$$=\sum_{t=1}^{[T\lambda]}(t\alpha_1-t\bar{y})^2+\sum_{t=[T\lambda]+1}^{[T\tau]}(T\lambda\alpha_1+(t-T\lambda)\alpha_2-t\bar{y})^2$$

$$=T\lambda(t\alpha_1-t\tilde{\alpha}_1)^2+T(\tau-\lambda)[T\lambda\alpha_1+(t-T\lambda)\alpha_2-t\tilde{\alpha}_1]^2$$

$$=T\lambda[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2+T(\tau-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$T\cdot[T\tau]^{-2}T\lambda[\frac{t(\tau-\lambda)}{\tau^2}(\alpha_1-\alpha_2)]^2\Rightarrow\lambda[\frac{t(\tau-\lambda)}{\tau^{2^2}}(\alpha_1-\alpha_2)]^2$$

$$T^{-1}\cdot[T\tau]^{-2}T(\tau-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2\Rightarrow(\tau-\lambda)[\frac{\lambda}{\tau}(\alpha_1-\alpha_2)]^2$$

For the nominator, we have

$$\bar{y} = \frac{1}{T - T\tau} \sum_{j=[T\tau]+1}^{T} y_j$$

$$= \frac{1}{T - T\tau} (\sum_{j=[T\tau]+1}^{[T\tau^*]} y_j + \sum_{j=[T\tau^*]+1}^{T} y_j)$$

$$= \alpha_2 \quad .$$

$$\sum_{t=[T\tau]+1}^{T} (\sum_{j=[T\tau]+1}^{t} \hat{z}_j)^2 = \sum_{t=[T\tau]+1}^{[T\tau^*]} [\sum_{j=[T\tau]+1}^{t} (y_j - \bar{y})]^2 + \sum_{t=[T\tau^*]+1}^{T} [\sum_{j=[T\tau]+1}^{t} (y_j - \bar{y})]^2$$

$$= \sum_{t=[T\tau]+1}^{[T\tau^*]} [\sum_{j=[T\tau]+1}^{t} (y_j - \bar{y})]^2 + \sum_{t=[T\tau^*]+1}^{T} [\sum_{j=[T\tau]+1}^{[T\tau^*]} (y_j - \bar{y}) + \sum_{j=[T\tau^*]+1}^{t} (y_j - \bar{y})]^2$$

$$= \sum_{t=[T\tau]+1}^{[T\tau^*]} [(t - T\tau)(\alpha_2 - \bar{y})]^2 + \sum_{t=[T\lambda]+1}^{T} [(t - T\tau)(\alpha_2 - \bar{y})]^2 = 0 \quad .$$

Thus, it is related to the index, we have

$$\sum_{t=[T\tau]+1}^{T} (\sum_{j=[T\tau]+1}^{t} \hat{z}_j)^2 = \sum_{t=[T\tau]+1}^{[T\tau^*]} (\sum_{j=[T\tau]+1}^{t} z_{1j})^2 + \sum_{t=[T\tau^*]+1}^{T} (\sum_{j=[T\tau]+1}^{[T\tau^*]} z_{1j} + \sum_{j=[T\tau^*]+1}^{t} z_{2j})^2$$

$$T^{-2d_0} [(1-\tau)T]^{-2} \sum_{t=[T\tau]+1}^{[T\tau^*]} (\sum_{j=[T\tau]+1}^{t} z_{1j})^2 = T^{-1} (1-\tau)^{-2} \sum_{t=[T\tau]+1}^{[T\tau^*]} (T^{-\frac{1}{2}-d_0} \sum_{j=[T\tau]+1}^{t} z_{1j})^2$$

$$\Rightarrow (1-\tau)^{-2} \int_{\tau}^{\tau^*} (C_{d_0} B_{d_0}(s))^2 \, \mathrm{d}s \, .$$

The same can be

$$T^{-2d_1} [(1-\tau)T]^{-2} \sum_{t=[T\tau^*]+1}^{T} (\sum_{j=[T\tau]+1}^{[T\tau^*]} z_{1j} + \sum_{j=[T\tau^*]+1}^{t} z_{2j})^2 \Rightarrow (1-\tau)^{-2} \int_{\tau^*}^{1} [(C_{d_0} B_{d_0}(\tau^*)) + (C_{d_1} B_{d_1}(s))]^2 \, \mathrm{d}s \, .$$

According to the above analysis, we can obtain

$$\Xi_T(\tau) = \frac{O_P(T^{2d_0}) + O_P(T^{2d_1})}{O_P(T) + O_P(T^{-1})} \, .$$

If $\tau^* \leq \tau$, for the nominator

$$T^{-2d_1} [(1-\tau)T]^{-2} \sum_{T\tau+1}^{T} S_{2,\ t}(\tau)^2 \Rightarrow (1-\tau)^{-2} C_{d_1}{}^2 \int_{\tau}^{1} \psi_1(s,\tau)^2 \, \mathrm{d}s \, .$$

For the denominator, if $1 < t \leq T\lambda$, we have

$$\sum_{t=1}^{[T\lambda]} (\sum_{j=1}^{t} \hat{z}_j)^2 = \sum_{t=1}^{[T\lambda]} [\sum_{j=1}^{t} (\alpha_1 + z_{1j} - \bar{y})]^2$$

$$= T\lambda (t\alpha_1 - t\tilde{\alpha}_1)^2$$

$$= T\lambda [\frac{t(\tau - \lambda)}{\tau} (\alpha_1 - \alpha_2)]^2 \quad .$$

$$T\cdot[T\tau]^{-2}\sum_{t=1}^{[T\lambda]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T\cdot[T\tau]^{-2}T\lambda[\frac{t(\tau-\lambda)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow \lambda[\frac{t(\tau-\lambda)}{\tau^2}(\alpha_1-\alpha_2)]^2 \quad .$$

If $T\lambda < t \le \tau^*$, we have

$$\sum_{t=[T\lambda]+1}^{[T\tau^*]}(\sum_{j=1}^{t}\hat{z}_j)^2 = \sum_{t=[T\lambda]+1}^{[T\tau^*]}[\sum_{j=1}^{[T\lambda]}(\alpha_1+z_{1j}-\bar{y})+\sum_{j=[T\lambda]+1}^{t}(\alpha_2+z_{1j}-\bar{y})]^2$$

$$=\sum_{t=[T\lambda]+1}^{[T\tau^*]}[T\lambda\alpha_1+(t-T\lambda)\alpha_2-t\bar{y}]^2$$

$$=T(\tau^*-\lambda)[T\lambda\alpha_1+(t-T\lambda)\alpha_2-t\tilde{\alpha}_1]^2$$

$$=T(\tau^*-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$T^{-1}\cdot[T\tau]^{-2}\sum_{t=[T\lambda]+1}^{[T\tau^*]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T^{-1}\cdot[T\tau]^{-2}T(\tau^*-\lambda)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow (\tau^*-\lambda)[\frac{\lambda}{\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

If $T\tau^* < t \le T\tau$, we have

$$\sum_{t=[T\tau^*]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2 = \sum_{t=[T\tau^*]+1}^{[T\tau]}[\sum_{j=1}^{[T\lambda]}(\alpha_1+z_{1j}-\bar{y})+\sum_{j=[T\lambda]+1}^{[T\tau^*]}(\alpha_2+z_{1j}-\bar{y})+\sum_{j=[T\tau^*]+1}^{t}(\alpha_2+z_{1j}-\bar{y})]^2$$

$$=\sum_{t=[T\tau^*]+1}^{[T\tau]}[T\lambda\alpha_1+(t-T\lambda)\alpha_2-t\tilde{\alpha}_1]^2$$

$$=T(\tau-\tau^*)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

$$T^{-1}\cdot[T\tau]^{-2}\sum_{t=[T\tau^*]+1}^{[T\tau]}(\sum_{j=1}^{t}\hat{z}_j)^2 = T^{-1}\cdot[T\tau]^{-2}T(\tau-\tau^*)[\frac{\lambda(T\tau-t)}{\tau}(\alpha_1-\alpha_2)]^2$$

$$\Rightarrow (\tau-\tau^*)[\frac{\lambda}{\tau}(\alpha_1-\alpha_2)]^2 \quad .$$

Finally, combining results above we can obtain

$$\Xi_T(\tau)=\frac{O_P(T^{2d_1})}{O_P(T^{-1})+O_P(T)+O_P(T)}=\frac{O_P(T^{2d_1})}{O_P(T^{-1})+2O_P(T)} \quad .$$

The proof is completed.

