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Constrained Probabilistic Continuous Review Inventory System with Mixture Shortage and Stochastic Lead Time Demand

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Abstract

This paper derives the probabilistic continuous review backorders and lost sales inventory system when the order cost is a function of the order quantity. Our objective is to minimize the expected annual total cost under a restriction on the expected annual holding cost when the lead time demand follows some continuous distributions by using the Lagrangian method. Some published special cases are deduced and an illustrative numerical example with some graphs is added.

Key words: Probabilistic model; Mixture shortage inventory system; Lost sales; Varying order cost; Holding cost; Safety stock; Lead time demand; Continuous distributions.

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INTRODUCTION

The two basic questions that any inventory control system has to answer are when and how much to order. Over the years, hundreds of papers and books have been published presenting models for doing this under a wide variety of conditions and assumptions. Most of authors have shown that if demand that cannot be filled from stock is backordered or using the lost sales model.

Several $\langle Q,r\rangle$ inventory models with mixture of backorders and lost sales were proposed by Posner and Yansouni[1], Montgomery et al.[2], Rosenberg[3], Kyung[4]. Almost all the previous research works used γ as a fraction of unsatisfied demand that will be backordered (the remaining fraction $(1-\gamma)$ completely lost) to model partial backorders. Since it is optimal to allow some stockouts if all customers will wait $(\gamma=1)$ and it is optimal to either allow no stockouts or lose all sales if customers have no patience $(\gamma=0)$.

Also, Rabinowitz et al.[5] modeled a $\langle Q,r\rangle$ inventory system using a control variable, which limits the maximum number of backorders allowed to accumulate during a cycle. Also, Zipkin[6] shows that if demands occurring during a stockout period are lost sales rather than backorders, the optimal policy is to have either no stockouts or all stockouts.

Recently, Fergany and El-wakeel[7,8] introduced probabilistic lostsales models with normal distribution and other continuous distribution. Also, El-Wakeel[9] derived a probabilistic inventory back-orders model with uniform distribution.

In this paper, we assume that both backorder and lost sales costs are independent of the duration of the stockout and γ is the backorder fraction. Also, we deduced the model with varying order cost when the demand is a random variable, the lead-time is constant and the distribution of the lead time demand is known under the holding cost constraint. The situation will be considered in which a single item is stocked to meet a probabilistic demand. When the number of units on hand and on order reaches the reorder point r, action is initiated to procure a replenishment quantity Q.

1. ASSUMPTIONS AND NOTATIONS

The following assumptions are adopted for developing our model: The system is a continuous review which means that the demands are recorded as they occur and the stock level is known at all times. An order quantity of size Q per cycle is placed every time the stock level reaches a certain reorder point r (Q and r are two decision variables). Thus, the assumptions are:

1– The cycle is defined as the time between two successive arrivals of orders and assume that the system repeats itself in the sense that the inventory position varies between r and $r+Q$ during each cycle.

2– The average number of cycles per year can be written as $n=\overline{D}/Q$ and then the inventory cycle is $N=Q/\overline{D}$.

3– There is never more than a single order outstanding.

The following notations are adopted for developing our model:

$\overline{D}=$ The average rate of annual demand,
$Q=$ A decision variable representing the order quantity per cycle,
$r=$ A decision variable representing the reorder point,
$N=$ The inventory cycle,
$n=$ The average number of cycles per year,
$L=$ The lead time between the placement of an order and its receipt,
$\gamma=$ A fraction of unsatisfied demand thatwillbackordered,
$x=$ The continuous random variable represents the demand during L (lead time demand),
$f(x)=$ The probability density function of the lead time demand and $F(x)$ its distribution function,
$r-x=$ The randomvariable represents the net inventory when the procurement quantity arrives if the lead time demand $x\leq r$,
$E(r-x)=ss=$ Safety stock $=$ The expected net inventory

$$=\int_0^r (r-x)f(x)\mathrm{d}x=r-E(x)+\int_r^\infty (x-r)f(x)\mathrm{d}x$$

$\overline{H}=$ The average on hand inventory $=$ (Max. on hand $+$ Min.on hand)$/2=(ss+Q+ss)/2$

$$=\frac{Q}{2}+r-E(x)+\int_r^\infty (x-r)f(x)\mathrm{d}x$$

$R(r)=$ The reliability function

$$=1-F(r)=\int_r^\infty f(x)\mathrm{d}x$$

$\overline{S}(r)=$ The expected value of shortages per cycle

$$=\int_r^\infty (x-r)f(x)\mathrm{d}x$$

$c_b=$ The backorders cost per cycle,
$c_o=$ The order cost per cycle,
$C_o(Q)=C_oQ^\beta=$ The varying order cost per cycle, $0<\beta<1$, where *beta* is a constant real number selected to provide the best fit of estimated expected cost function.
$c_h=$ The holding cost per year,
$c_l=$ The lost sales cost per cycle,
$K=$ The limitation on the expected annual holding cost.

2. THE MATHEMATICAL MODEL

It is possible to develop the expected annual total cost where it consisted of three components: the expected varying order cost, the expected holding cost and the expected shortage cost as follows:

$$\begin{aligned}E(\text{Total Cost})=&E(\text{Order Cost})+E(\text{Holding Cost})\\&+E(\text{Shortage Cost})\end{aligned}$$

I.e.,

$$E(\mathrm{TC})=E(\mathrm{OC})+E(\mathrm{HC})+E(\mathrm{BC})+E(\mathrm{LC}) \tag{2.1}$$

where

$$E(\mathrm{OC})=C_o(Q)\cdot n=c_oQ^\beta\frac{\overline{D}}{Q}=c_o\overline{D}Q^{\beta-1} \tag{2.2}$$

$$\begin{aligned}E(\mathrm{HC})&=c_h\overline{H}=c_h\left(\frac{Q}{2}+r-E(x)+(1-\gamma)\overline{S}(r)\right)\\&=c_h\left(\frac{Q}{2}+r-E(x)+(1-\gamma)\int_r^\infty (x-r)f(x)\mathrm{d}x\right)\end{aligned} \tag{2.3}$$

$$E(\mathrm{BC})=c_b\cdot n\cdot\gamma\cdot\overline{S}(r)=\frac{c_b\gamma\overline{D}}{Q}\int_r^\infty (x-r)f(x)\mathrm{d}x \tag{2.4}$$

and

$$E(\mathrm{LC})=c_l\cdot n\cdot(1-\gamma)\cdot\overline{S}(r)=\frac{c_b(1-\gamma)\overline{D}}{Q}\int_r^\infty (x-r)f(x)\mathrm{d}x \tag{2.5}$$

Therefore

$$\begin{aligned}E\left[\mathrm{TC}(Q,r)\right]=&c_o\overline{D}Q^{\beta-1}+c_h\left(\frac{Q}{2}+r-E(x)\right)\\&+\frac{c_b\gamma\overline{D}}{Q}\overline{S}(r)+\left(\frac{c_l\overline{D}}{Q}+c_h\right)(1-\gamma)\overline{S}(r)\end{aligned} \tag{2.6}$$

Our objective is to minimize the expected annual total cost $E\left[\mathrm{TC}(Q,r)\right]$ under the expected holding cost constraint:

$$c_h\left(\frac{Q}{2}+r-E(x)+(1-\gamma)\overline{S}(r)\right)\leq K \tag{2.7}$$

To find the optimal values Q^* and r^* which minimize equation (2.6) under the constraint (2.7), we will use the Lagrange multiplier technique as follows:

$$\begin{aligned}L(Q,r,\lambda)=&c_o\overline{D}Q^{\beta-1}+c_h\left(\frac{Q}{2}+r-E(x)\right)\\&+\frac{c_b\gamma\overline{D}}{Q}\overline{S}(r)+\left(\frac{c_l\overline{D}}{Q}+c_h\right)(1-\gamma)\overline{S}(r)\\&+\lambda\left[c_h\left(\frac{Q}{2}+r-E(x)+(1-\gamma)\overline{S}(r)\right)-K\right]\end{aligned} \tag{2.8}$$

To find the optimal values Q^* and r^* can be found by setting each of the corresponding first partial derivatives of equation (2.8) equal to zero, then we obtain:

$$AQ^2-BQ^\beta 2[\gamma(M-G)+G]\overline{S}(r)=0 \tag{2.9}$$

and

$$R(r)=\frac{A\cdot Q}{\gamma M+(1-\gamma)(G+A\cdot Q)} \tag{2.10}$$

where$A=(a+\lambda)c_h$, $B=2(1-\beta)c_o\overline{D}$, G=$c_l\overline{D}$ and $M=c_b\overline{D}$. Clearly there is no closed form solution of equations (2.9) and (2.10).

3. LEAD-TIME DEMAND

3.1 Lead-time Demand Follows Uniform Distribution

Assume that the lead-time demand follows the uniform distribution as follows:

$$f(x)=\frac{1}{b};0\leq x\leq b \quad \text{with} \quad E(x)=\frac{b}{2} \quad \text{and} \quad R(r)=1-\frac{r}{b}$$

and

$$\overline{S}=\frac{r^2}{2b}-r+\frac{b}{2} \tag{3.1}$$

By substituting from equation (3.4) into equations (2.9) and (2.10) and then solving them simultaneously, we can obtain the optimal order quantity from the following equation:

$$\begin{aligned}&A^3(1-\gamma)^2Q^{*4-\beta}+2A^2(1-\gamma)\left[\gamma M+(1-\gamma)G\right]Q^{*3-\beta}\\&\quad -A^2B(1-\gamma)^2Q^{*2}+A\left[\gamma M+(1-\gamma)G\right]Q^{*2-\beta}\\&\quad -2AB(1-\gamma)\left[\gamma M+(1-\gamma)G-Ab\right]Q^*\\&\quad -B\left[\gamma M+(1-\gamma)G-Ab\right]^2=0\end{aligned} \tag{3.2}$$

Also, the optimal reorder level is given by:

$$R^*=\frac{\lfloor\gamma(M-AQ^*)+(1-\gamma)G\rfloor b}{\gamma M+(1-\gamma)(G+AQ^*)} \tag{3.3}$$

Thus we can obtain the optimal values Q^* and r^* by solving equations (3.2) and (3.3) respectively for different values of β and vary λ until the smallest positive value is found such that the constraint holds. Hence the minimum expected annual total cost is:

$$\begin{aligned}E\left[TC(Q,r)\right]&=c_o\overline{D}Q^{*\beta-1}+c_h\left(\frac{Q^*}{2}+\frac{r^{*2}}{2b}\right)\\&\quad+\gamma\left(\frac{c_b\overline{D}}{Q^*}-c_h\right)+\frac{c_l(1-\gamma)\overline{D}}{Q^*}\left(\frac{r^{*2}}{2b}-r^*+\frac{b}{2}\right)\end{aligned} \tag{3.4}$$

3.2 Lead-time Demand Follows Exponential Distribution

Lead-time demand follows Exponential distribution:

$$f(x)=\alpha e^{\alpha};x\geq 0,\quad \alpha\geq 0,\theta>0$$

then,

$$\overline{S}(r)=\frac{1}{\alpha}e^{-\alpha r},r\geq 0 \tag{3.5}$$

Similarly, substituting from equation (3.5) into equations (2.9) and (2.10) and solving them simultaneously, we get:

$$\begin{aligned}&\alpha A^2(1-\gamma)Q^{*3-\beta}+\alpha A\left[\gamma M+(1-\gamma)G\right]Q^{*2-\beta}\\&\quad-\alpha AB(1-\gamma)Q^*+2A\left[\gamma M+(1-\gamma)G\right]Q^{*1-\beta}\\&\quad-\alpha B\left[\gamma M+(1-\gamma)G\right]=0\end{aligned} \tag{3.6}$$

and

$$r^*=\frac{1}{\alpha}\ln\left[(1-\gamma)+\frac{G+\gamma(M-G)}{AQ^*}\right] \tag{3.7}$$

Thus we can obtain the optimal values Q^* and r^* by solving equations (3.6) and (3.7) respectively for different values of β and vary λ until the smallest positive value is found such that the constraint holds. Hence the minimum expected annual total cost is:

$$\begin{aligned}E\left[TC(Q,r)\right]&=c_o\overline{D}Q^{*\beta-1}\\&\quad+c_h\left\{\frac{Q^*}{2}+r^*-\frac{1}{\alpha}\left[1-(1-\gamma)e^{-\alpha r^*}\right]\right\}\\&\quad+\left[c_b\gamma+c_l(1-\gamma)\right]\left(\frac{\overline{D}}{Q^*\alpha}e^{-\alpha r^*}\right)\end{aligned} \tag{3.8}$$

3.3 Lead-time Demand Follows Laplace Distribution

Similarly, consider the lead-time demand follows Laplace distribution; we can obtain the exact solution as follows:

$$f(x)=\frac{1}{2\theta}e^{\frac{-|x-\mu|}{\theta}};\quad -\infty<x<\infty,\quad -\infty<\mu<\infty,\quad \theta>0$$

then,

$$\overline{S}(r)=\frac{\theta}{2}e^{\frac{-(r-\mu)}{\theta}},R\geq 0,\theta>0 \tag{3.9}$$

Substituting from equation (3.9) into equations (2.9) and (2.10) and then solving them simultaneously, we get:

$$\begin{aligned}&A^2(1-\gamma)Q^{*3-\beta}+A\left[\gamma(M-G)+G\right]Q^{*2-\beta}-AB(1-\gamma)Q^*\\&\quad+2\theta A\left[\gamma(M-G)+G\right]Q^{*1-\beta}-B\left[\gamma(M-G)+G\right]=0\end{aligned} \tag{3.10}$$

and

$$r^*=\mu+\theta\ln\left(\frac{1-\gamma}{2}+\frac{\gamma M+(1-\gamma)G}{2AQ^*}\right) \tag{3.11}$$

Thus we can obtain the optimal values Q^* and r^* by solving equations (3.10) and (3.11) respectively for different values of β and vary λ until the smallest positive value is found such that the constraint holds. Hence the minimum expected annual total cost is:

$$\begin{aligned}E\left[TC(Q,r)\right]&=c_o\overline{D}Q^{*\beta-1}\\&\quad+c_h\left(\frac{Q^*}{2}+r^*-\mu+\frac{(1-\gamma)\theta}{2}e^{-\frac{r-\mu}{\theta}}\right)\\&\quad+\frac{\left[\gamma c_b+(1-\gamma)c_l\right]\overline{D}\theta}{2Q^*}e^{-\frac{r-\mu}{\theta}}\end{aligned} \tag{3.12}$$

4. SPECIAL CASES

Case 1: Let $\gamma=0$, $\beta=0$ and $K\to\infty\Rightarrow C_o(Q)=c_o$ and $\lambda=0$. Thus equations (2.9) and (2.10) become

$$Q^*=\sqrt{\frac{2\overline{D}\left(c_o+c_l\overline{S}(r)\right)}{c_h}} \quad \text{and} \quad R\left(r^*\right)=\frac{c_hQ^*}{c_hQ^*+c_l\overline{D}}$$

This is unconstrained lost sales continuous review inventory model with constant units of cost, which are the same results as in Hadley[10].

Table 1
The Optimal Solutions and the Min $E(\mathrm{TC})$ for Each Distribution at $\gamma = 0.7$

β	Uniform Distribution			Exponential Distribution			Laplace Distribution		
	Q^*	r^*	min$E$(TC)	Q^*	r^*	min$E$(TC)	Q^*	r^*	min$E$(TC)
0.1	1455	247.5	17625.9	964	492.2	25823	1542	203.7	17352.2
0.2	1459	245.4	27380.1	1074	436.7	38335	1566	191.7	26666.2
0.3	1466	241.7	47545.2	1187	379.3	61851	1589	180.0	45915.7
0.4	1479	235.4	89176.9	1301	321.2	107120	1612	168.6	85808.1
0.5	1499	225.0	175159	1413	263.8	196248	1632	157.9	168849.2
0.6	1529	209.3	352777	1517	209.4	375487	1652	148.0	341972
0.7	1569	187.9	720397	1606	161.4	742397	1667	139.6	704408.5
0.8	1610	165.6	1484510	1670	125.9	1504010	1679	133.5	1464200
0.9	1616	162.0	3081440	1680	120.7	3100160	1680	132.6	3060573.3

Case 2: Let $\gamma = 1$, $\beta = 0$ and $K \to \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$. Thus equations (2.9) and (2.10) become

$$Q^* = \sqrt{\frac{2\overline{D}\left(c_o + c_b\overline{S}(r)\right)}{c_h}} \quad \text{and} \quad R(r^*) = \frac{c_h Q^*}{c_b \overline{D}}$$

This is unconstrained backorders continuous review inventory model with constant unit costs, which are the same results as in Hadley[10].

Note: When $\gamma = 1$, $\beta = 0$ and $K \to \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$.

Equations (3.2) and (3.3) give unconstrained simple model with constant units of cost and the lead-time demand follows the Uniform distribution, which are the same results as in Fabrycky, W., et al[11].

Equations (3.6) and (3.7) will be the form of unconstrained continuous review model with constant units of cost and the lead-time demand follows the Exponential distribution, which are agree with results of Hillier[12].

Equations (3.10) and (3.11) give unconstrained continuous review model with constant units of cost and the lead-time demand follows the Laplace distribution, which are the same results as in Nahmias[13].

5. AN ILLUSTRATIVE EXAMPLE

The cosmetics department of a large department store has recently introduced a constrained $\langle Q, r\rangle$ system with varying order cost and mixed shortages to control many items in the department. A particular type of expensive perfume has an annual demand rate equals 1600 units. The cost of placing an order amounts to $4000 and the inventory holding cost is $10. This particular perfume is not easy to obtain elsewhere, and hence demands occurring when the store is out of stock are partially backordered. The management estimates that 70% of unsatisfied demand will be backordered with backorder cost equals $600 and the remaining demand will be lost with cost $2000. There is a restriction that the average holding cost is either less than or equal $8500 per year and the procurement lead-time is constant. Determine Q^* and r^* when the lead time demand has the following distribution:

(i) Uniform distribution with $f(x) = 1/250, 0 \le x \le 250$ units.
(ii) Exponential distribution with $\alpha = 0.008$ units.
(iii) Laplace distribution with $\mu = 125$ and $\theta = 20$ units.

From the above example, we have the following parameters values: $\overline{D} = 1600$, $c_o = \$400$, $c_h = \$10$, $c_b = \$600$, $\gamma = 0.7$, $c_l = \$2000$ and $K = \$8500$. By solving the previous deduced equations for each distribution at different values of β, we obtain Table 1.

From the data given in Table 1, we can draw the optimal values of and against for all distributions as shown in the following Figures (1) and (2).

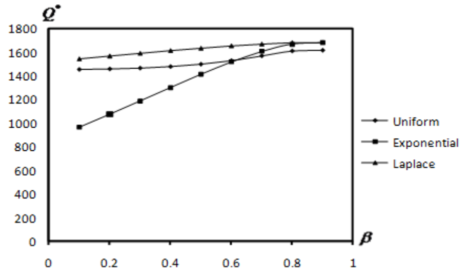

Figure 1
The Optimal Values of Q^* Against β

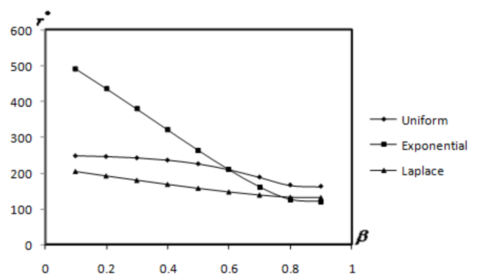

Figure 2
The Optimal Values of r^* Against β

CONCLUSION

This paper deducing our probabilistic (Q, r) model with mixed shortage when lead-time demand follows Uniform, Exponential and Laplace distributions. For such distributions, we can evaluate the solution of Q^* and r^* for each value of β and λ which yields our expected holding cost constraint and then obtain the minimum expected total cost. From the previous example, we can deduce that the least min$E$(TC) obtained when the lead-time demand follows Laplace distribution and its optimal min$E$(TC) will be at $\beta = 0.1$.

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