Strong Convergence and Stability of Jungck-Multistep-SP Iteration for Generalized Contractive-Like Inequality Operators

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Abstract

We introduce the Jungck-multistep-SP iteration and prove some convergence as well as stabiilty results for a pair of weakly compatible generalized contractive-like inequality operators defined on a Banach space. As corollaries, the results show that the Jungck-SP and Jungck-Mann iterations can also be used to approximate the common fixed points of such operators. The results are improvements, generalizations and extensions of the work of Chugh and Kumar (2011). Consequently, several results in literature are generalized.

Key words: Jungck-multistep-SP iteration; Banach space; Stability; Convergence

INTRODUCTION

Most physical systems whose equations are of the form $f(x) = y$, can be formulated by transforming the equation into a fixed point equation $x = Tx$ and then apply an approximate fixed point theorem to get information on the existence and uniqueness of fixedpoint, that is, the solution of the original equation. The Picard, Mann, Ishikawa, Noor and multistep iterations have been commonly used to approximate the fixed points of several classes of single quasi-contractive operators. For example see Berinde (2004), Chatterjea (1974), Kannan (1969) and Zamfirescu (1972).

Let X be a Banach space, K, a nonempty convex subset of X and $T : K \rightarrow K$ a self map of K.

Definition 1.1. Let $x_0 \in K$. The Picard iteration scheme $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = Tx_n, \ n \geq 0 \qquad (1.1)$$

Definition 1.2. For any given $x_0 \in K$, the Mann iteration scheme (Mann, 1953) $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n \qquad (1.2)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Definition 1.3. Let $x_0 \in K$. The Ishikawa iteration scheme (Ishikawa, 1974) $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n \qquad (1.3)$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Observe that if $\beta_n = 0$ for each n, then the Ishikawa iteration process (1.3) reduces to the Mann iteration scheme (1.2).

Definition 1.4. Let $x_0 \in K$. The Noor iteration (or three-step) scheme (Noor, 2000) $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tz_n \qquad (1.4)$$
$$z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Observe that if $\gamma_n = 0$ for each n, then the Noor iteration process (1.4) reduces to the Ishikawa iteration scheme (1.3).

Definition 1.5. Let $x_0 \in K$. The multistep iteration scheme (Rhoades & Soltuz, 2004) $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n^1$$
$$y_n^i = (1 - \beta_n^i)x_n + \beta_n^i Ty_n^{i+1}, \ i = 1, 2, ..., p-2, \qquad (1.5)$$
$$y_n^{p-1} = (1 - \beta_n^{p-1})x_n + \beta_n^{p-1} Tx_n, \ p \geq 2$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}, i = 1, 2, \ldots p-1$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Observe that that the multistep iteration is a generalization of the Noor, Ishikawa and the Mann iterations. In fact, if $p = 1$ in (1.5), we have the Mann iteration (1.2); if $p = 2$ in (1.5), we have the Ishikawa iteration (1.3) and if $p = 3$, we have the Noor iterations (1.4).

One of the most general quasi contractive operators which has been studied by several authors is the Zamfirescu operators.

Suppose X is a Banach space. The map $T : X \to X$ is called a Zamfirescu operator if

$$\|Tx - Ty\| \leq h \max\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2},$$

$$\frac{\|x - Ty\| + d\|y - Tx\|}{2}\} \tag{1.6}$$

where $0 \leq h < 1$ (Zamfirescu, 1972).

It is known that the operators satisfying (1.6) are generalizations of Kannan maps (Kannan, 1969) and Chatterjea maps (Chatterjea, 1972). Zamfirescu (1972) proved that the Zamfirescu operator has a unique fixed point which can be approximated by Picard iteration (1.1). Berinde (2004) showed that Ishikawa iteration can be used to approximate the fixed point of a Zamfirescu operator when X is a Banach space while it was shown by Olareru (2006) that if X is generalised to a complete metrizable locally convex space (which includes Banach spaces), the Mann iteration can be used to approximate the fixed point of a Zamfirescu operator. Several researchers have studied the convergence rate of these iterations with respect to the Zamfirescu operators. For example, it has been shown that the Picard iteration (1.1) converges faster than the Mann iteration (1.2) when dealing with the Zamfirescu operators. For example, see Popescu (2007). It is still a subject of research as to conditions under which the Mann iteration will converge faster than the Ishikawa or vice-versa when dealing with the Zamfirescu operators.

Jungck was the first to introduce an iteration scheme, which is now called Jungck iteration scheme (Jungck, 1976) to approximate the common fixed points of what is now called Jungck contraction maps. Singh *et al.* (2005) introduced the Jungck-Mann iteration procedure and discussed it's stability for a pair of contractive maps. Olatinwo and Imoru (2008), Olatinwo (2008) built on that work to introduce the Jungck-Ishikawa and Jungck-Noor iteration schemes and used their convergences to approximate the coincidence points (not common fixed points) of some pairs of generalized contractive-like operators with the assumption that one of each of the pairs of maps is injective. However, a coincidence point for a pair of quasicontractive maps need not be a common fixed point. In 2010, Olaleru & Akewe (2010) introduced the Jungck-multistep iteration and show that its convergence can be used to approximate the common fixed points of

those pairs of contractive-like operators without assuming the injectivity of any of the operators. Hence the iterative sequence considered in Olaleru and Akewe (2010) is a generalization of the those used in Olatinwo and Imoru (2008) and Olatinwo (2008). The fact that the injectivity of any of the maps is not assumed in Olaleru and Akewe (2010) and the common fixed points of those maps are approximated and not just the coincidence points make the corollary of the results in Olaleru & Akewe (2010) an improvement of the results of Olatinwo (2008), Olatinwo and Imoru (2008). Consequently, a lot of results dealing with convergence of Picard, Mann, Ishikawa and multistep iterations for single quasicontractive operators on Banach spaces were generalized. Several stability results are proved in literature, some of the authors whose stability results are of paramount importance in fixed point iterative processes are: Bhagwati & Ritu (2011); Chugh & Kumar (2011); Olatinwo (2008); Osilike (1995); Singh *et al.* (2005).

PRELIMINARIES

Let X be a Banach space, Y be an arbitrary set and $S, T : Y \to X$ such that $T(Y) \subseteq S(Y)$.

Then we have the following definitions.

Definition 2.1 (Jungck, 1976). For any $x_0 \in Y$, there exists a sequence $\{x_n\}_{n=0}^{\infty} \in Y$ such that $Sx_{n+1} = Tx_n$. The Jungck iteration is defined as the sequence $\{Sx_n\}_{n=1}^{\infty}$ such that

$$Sx_{n+1} = Tx_n, \ n \geq 0 \tag{2.1}$$

This procedure becomes Picard iteration when $Y = X$ and $S = I_d$ where I_d is the identity map on X.

Similarly, the Jungck contraction maps are the maps S, T satisfying

$$d(Tx, Ty) \leq k \, d(Sx, Sy), \ \ 0 \leq k < 1 \text{ for all } x, y \in Y \tag{2.2}$$

If $Y = X$ and $S = I_d$, then maps satisfying (2.2) become the well known contraction maps.

Definition 2.2 (Singh *et al.*, 2005). For any given $u_0 \in Y$, the Jungck-Mann iteration scheme $\{Su_n\}_{n=1}^{\infty}$ is defined by

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n \tag{2.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ are real sequneces in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Definition 2.3 (Olatinwo & Imoru, 2008). Let $x_0 \in Y$. The Jungck-Ishikawa iteration scheme $\{Sx_n\}_{n=1}^{\infty}$ is defined by

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n$$
$$Sy_n = (1 - \beta_n)Sx_n + \beta_n Tx_n \tag{2.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Definition 2.4 (Olatinwo, 2008). Let $x_0 \in Y$. The Jungck-Noor iteration (or three-step) scheme $\{Sx_n\}_{n=1}^{\infty}$ is defined by

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n$$

$$Sy_n = (1 - \beta_n)Sx_n + \beta_n Tz_n \qquad (2.5)$$
$$Sz_n = (1 - \gamma_n)Sx_n + \gamma_n Tx_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences

in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Definition 2.5 (Olaleru & Akewe, 2010). Let $x_0 \in Y$. The Jungck-multistep iteration scheme $\{Sx_n\}_{n=1}^{\infty}$ is defined by

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n^1$$
$$Sy_n^i = (1 - \beta_n^i)Sx_n + \beta_n^i Ty_n^{i+1}, \ i = 1, 2, ..., k–2, \qquad (2.6)$$
$$Sy_n^{k-1} = (1 - \beta_n^{k-1})Sx_n + \beta_n^{k-1}Tx_n, \ p \geq 2$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}$, $i = 1, 2, ...k - 1$ are real sequences

in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Observe that that the Jungck-multistep iteration is a generalization of the Jungck-Noor, Jungck-Ishikawa and the Jungck-Mann iterations. In fact, if $k = 2$ and $\beta_n^1 = 0$ in (2.6), we have the Jungck-Mann iteration (2.3); if $k = 2$ in (2.6), we have the Jungck-Ishikawa iteration (2.4) and if $k = 3$, we have the Jungck-Noor iterations (2.5).

Observe that if $X = Y$ and $S = I_d$, then the Jungck-multistep (2.6), Jungck-Noor (2.5), Jungck-Ishikawa (2.4) and the Jungck-Mann (2.3) iterations respectively become the multistep (1.5), Noor (1.4), Ishikawa (1.3) and the Mann (1.2) iterative procedures.

Definition 2.6 (Chugh & Kumar, 2011). Let $x_0 \in Y$. The Jungck-SP iteration scheme $\{Sx_n\}_{n=1}^{\infty}$ is defined by

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n$$
$$Sy_n = (1 - \beta_n)Sz_n + \beta_n Tz_n \qquad (2.7)$$
$$Sz_n = (1 - \gamma_n)Sx_n + \gamma_n Tx_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences

in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We now consider the following conditions. X is a Banach space and Y a nonempty set such that $T(Y) \subseteq S(X)$ and $S, T : Y \to X$. For $x, y \in Y$ and $h \in (0,1)$:

$$\| Tx - Ty \| \leq h \ max\{\| Sx - Sy \|, \tfrac{\|Sx-Tx\|+\|Sy-Ty\|}{2}, \tfrac{\|Sx-Ty\|+\|Sy-Tx\|}{2}\} \ (2.8)$$

$$\| Tx - Ty \| \leq h \ max\{\| Sx - Sy \|, \tfrac{\|Sx-Tx\|+\|Sy-Ty\|}{2}, \| Sx - Ty \| \| Sy$$
$$\| Sx - Ty \|, \| Sy - Tx \|\}, \qquad (2.9)$$

$$\| Tx - Ty \| \leq \delta \| Sx - Sy \| + L \| Sx - Tx \|,$$
$$L > 0, \ 0 < \delta < 1 \qquad (2.10)$$

$$\| Tx - Ty \| \leq \tfrac{\delta\|Sx-Sy\|+\phi(\|Sx-Tx\|)}{1+M\|Sx-Tx\|}, \ 0 \leq \delta < 1, M \geq 0 \qquad (2.11)$$

$$\| Tx - Ty \| \leq \delta \| Sx - Sy \| + \phi(\| Sx - Tx \|), \ 0 \leq \delta < 1 \ (2.12)$$

where $\phi : \Re_+ \to \Re_+$ is a monotone increasing sequence with $\phi(0) = 0$.

Remark 2.7. Observe that if $X = Y$ and $S = I_d$, (2.8) is the same as the Zamfirescu operator (1.6) already studied by several authors; (2.9) becomes the operator studied by Rhoades (1976); while (2.10) becomes the operator introduced by (Osilike, 1995). Operators satisfying (2.11)

and (2.12) were introduced by Olatinwo (2008).

A comparison of the four maps show the following.

Proposition 2.8 (Olaleru and Akewe, 2010). $(2.8) \Rightarrow (2.9) \Rightarrow (2.10) \Rightarrow (2.11) \Rightarrow (2.12)$ *but the converses are not true*. For details of Proof see Olaleru and Akewe (2010).

Bosede (2010) proved some convergence results for the Jungck-Ishikawa and Jungck-Mann iteration processes by using the following more general contractive condition than the Zamfirescu operator

$$\| Tx - Ty \| \leq e^{L\|Sx-Tx\|}[\delta \| Sx - Sy \| + 2\delta \| Sx - Tx \|],$$
$$0 \leq \delta < 1 \qquad (2.13)$$

for all $x, y \in Y$ where $L \geq 0$.

Motivated by the work of Bosede (2010), Chugh and Kumar (2011), introduced the following contractive-like inequality operators and proved strong convergence and stability results for the Jungck-SP iterative scheme (2.7).

$$\| Tx - Ty \| \leq e^{L\|Sx-Tx\|}[\delta \| Sx - Sy \| + 2\delta \| Sx - Tx \|],$$
$$0 \leq \delta < 1 \qquad (2.14)$$

for all $x, y \in Y$ where $L \geq 0$ and $\phi : \Re_+ \to \Re_+$ is a monotone increasing sequence with $\phi(0) = 0$.

However, inspired by the work of Chugh & Kumar (2011), we introduce the following Jungck-multistep-SP and use it to approximate the common fixed point using the contractive condition (2.14).

Definition 2.9. Let $x_o \in Y$. The Jungck-multistep-SP iterative process $\{Sx_n\}_{n=1}^{\infty}$ is defined by

$$Sx_{n+1} = (1 - \alpha_n)Sy_n^1 + \alpha_n Ty_n^1$$
$$Sy_n^i = (1 - \beta_n^i)Sy_n^{i+1} + \beta_n^i Ty_n^{i+1}, \ i = 1, 2, ..., k–2, \qquad (2.15)$$
$$Sy_n^{k-1} = (1 - \beta_n^{k-1})Sx_n + \beta_n^{k-1}Tx_n, \ p \geq 2$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}$, $i = 1, 2, ..., k–1$, are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Observe that (2.15) gives (2.7) if $k = 3$.

We need the following definition.

Definition 2.10 (Abbas and Jungck, 2008). A point $x \in X$ is called a coincident point of a pair of self maps S, T if there exists a point w (called a point of coincidence) in X such that $w = Sx = Tx$. Self-maps S and T are said to be weakly compatible if they commute at their coincidence points, that is, if $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

Chugh and Kumar (2011) proved that the Jungck-SP converges to the coincidence point of S,T defined by (2.14) when S is an injective operator. It was shown in Olatinwo (2008) that the Jungck-Ishikawa iteration converges to the coincidence point of S,T defined by (2.12) when S is an injective operator while the same convergence result was proved for Jungck-Noor when S,T are defined by (2.11) (Olatinwo, 2008). (We note that the maps satisfying (2.9) and of course (2.10)-(2.14) need not have a coincidence point (Olaleru & Akewe, 2010)). We rather prove the convergence of Jungck-multistep-SP iteration (2.15) to the *unique common fixed point* of S,T defined by

(2.14), without assuming that S is injective, provided the coincidence point exist for S,T.

Lemma 2.11 (Berinde, 2004): Let δ be a real number satisfying $0 \le \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $lim_n\varepsilon_n = 0$ then for any sequence of positive numbers $\{u_n\}_{n=0}^{y\infty}$ satisfying $u_{n+1} \le \delta u_n+\varepsilon_n$, $n = 0,1,2,...$, we have $lim_n u_n = 0$.

MAIN RESULTS

Theorem 3.1. Let X be a Banach space and $S,T\colon Y\rightarrow X$ for an arbitrary set Y such that (2.14) holds and $T(Y)\subseteq S(Y)$. Assume S and T have a coincidence point z such that $Tz = Sz = p$. For any $x_o \in Y$, the Jungck-multistep-SP iterative process (2.15) $\{Sx_n\}_{n=1}^{\infty}$ converges strongly to p.

Further, if $Y = X$ and S,T commute at p (i.e. S and T are weakly compatible), then p is the unique common fixed point of S, T.

Proof. In view of (2.14) and (2.15) coupled with the fact that $Tz = Sz = p$, we have

$$\| Sx_{n+1} - p \|=\| (1-\alpha_n)Sy_n^1 + \alpha_n Ty_n^1 -(1-\alpha_n+\alpha_n)p \|$$
$$\le (1-\alpha_n)\| Sy_n^1 - p \|+\alpha_n\| Ty_n^1 - p \|$$
$$\le (1-\alpha_n)\| Sy_n^1 - p \|+\alpha_n\| Tz - Ty_n^1 \|$$
$$\le (1-\alpha_n)\| Sy_n^1 - p \|+\alpha_n e^{L\|Sz-Tz\|}[\delta\| Sz - Sy_n^1 \| \quad(3.1)$$
$$+\phi(\| Sz - Tz \|)]$$
$$= (1-\alpha_n)\| Sy_n^1 - p \|+\delta\alpha_n\| p - Sy_n^1 \|$$
$$= [1-\alpha_n(1-\delta)]\| Sy_n^1 - p \|$$

An application of (2.15) and (2.14) give

$$\| Sy_n^1 - p \|\le (1-\beta_n^1)\| Sy_n^2 - p \|+\beta_n^1\| Tz - Ty_n^2 \|$$
$$\le (1-\beta_n^1)\| Sy_n^2 - p \|+\beta_n^1 e^{L\|Sz-Tz\|}[\delta\| Sz - Sy_n^2 \|$$
$$+\phi(\| Sz - Tz \|)] \quad(3.2)$$
$$= (1-\beta_n^1)\| Sy_n^2 - p \|+\delta\beta_n^1\| p - Sy_n^2 \|$$
$$= [1-\beta_n^1(1-\delta)]\| Sy_n^2 - p \|$$

An application of (2.15) and (2.14) also give

$$\| Sy_n^2 - p \|\le (1-\beta_n^2)\| Sy_n^3 - p \|+\beta_n^2\| Tz - Ty_n^3 \|$$
$$\le (1-\beta_n^2)\| Sy_n^3 - p \|+\beta_n^2 e^{L\|Sz-Tz\|}[\delta\| Sz - Sy_n^3 \|$$
$$+\phi(\| Sz - Tz \|)] \quad(3.3)$$
$$= (1-\beta_n^2)\| Sy_n^3 - p \|+\delta\beta_n^1\| p - Sy_n^3 \|$$
$$= [1-\beta_n^2(1-\delta)]\| Sy_n^3 - p \|$$

Similarly, an application of (2.15) and (2.14) give

$$\| Sy_n^3 - p \|\le (1-\beta_n^3)\| Sy_n^4 - p \|+\beta_n^3\| Tz - Ty_n^4 \|$$
$$\le (1-\beta_n^3)\| Sy_n^4 - p \|+\beta_n^3 e^{L\|Sz-Tz\|}[\delta\| Sz - Sy_n^4 \|$$
$$+\phi(\| Sz - Tz \|)]$$
$$= (1-\beta_n^3)\| Sy_n^4 - p \|+\delta\beta_n^1\| p - Sy_n^4 \| \quad(3.4)$$
$$= [1-\beta_n^3(1-\delta)]\| Sy_n^4 - p \|$$

Continuing the above process we have

$$\| Sx_{n+1} - p \|\le [1-\alpha_n(1-\delta)][1-\beta_n^1(1-\delta)]$$
$$[1-\beta_n^2(1-\delta)][1-\beta_n^3(1-\delta)] \quad(3.5)$$
$$...[1-\beta_n^{k-2}(1-\delta)]\| Sy_n^{k-1} - p \|$$

An application of (2.15) and (2.14) also give

$$\| Sy_n^{k-1} - p \|\le (1-\beta_n^{k-1})\| Sx_n - p \|+\beta_n^{k-1}\| Tz - Tx_n \|$$
$$\le (1-\beta_n^{k-1})\| Sx_n - p \|$$
$$+\beta_n^{k-1}e^{L\|Sz-Tz\|}[\delta\| Sz - Sx_n \|+\phi(\| Sz - Tz \|)] \quad(3.6)$$
$$= (1-\beta_n^{k-1})\| Sx_n - p \|+\delta\beta_n^{k-1}\| p - Sx_n \|$$
$$= [1-\beta_n^{k-1}(1-\delta)]\| Sx_n - p \|$$

Substituting (3.6) in (3.5), we have

$$\| Sx_{n+1} - p \|\le [1-\alpha_n(1-\delta)][1-\beta_n^1(1-\delta)]$$
$$[1-\beta_n^2(1-\delta)][1-\beta_n^3(1-\delta)]$$
$$...[1-\beta_n^{k-2}(1-\delta)][1-\beta_n^{k-1}(1-\delta)]\| Sx_n - p \|$$
$$\le [1-\alpha_n(1-\delta)]\| Sx_n - p \| \quad(3.7)$$
$$\le \prod_{j=0}^{n}[1-\alpha_j(1-\delta)]\| Sx_0 - p \|$$
$$\le e^{-(1-\delta)\sum_{j=0}^{\infty}\alpha_j}\| Sx_0 - p \|$$

Since $0 \le \delta < 1$, $\alpha_j \in [0,1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, so $e^{-(1-\delta)\sum_{j=0}^{\infty}\alpha_j} \to 0$ as $n \to \infty$

Thus $\lim_{n\to\infty}\| Sx_{n+1} - p \|= 0$

Therefore, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Next we show that p is unique. Suppose there exists another point of coincidence p^*. Then there is an $z^* \in X$ such that $Tz^* = Sz^* = p^*$. Hence, using (2.14) we have

$$\| z - z^* \|=\| Tz - Tz^* \|$$
$$\le e^{L\|Sz-Tz\|}[\delta\| Sz - Sz^* \|+\phi(\| Sz - Tz \|)]$$
$$= \delta\| z - z^* \|$$

Since $\delta < 1$, then $z = z^*$ and so p is unique.

Since S,T are weakly compatible, then $TSz = STz$ and so $Tp = Sp$. Hence p is a coincidence point of S,T and since the coincidence point is unique, then $p = z$ and hence $Sp = Tp = p$ and therefore p is the unique common fixed point of S,T. This ends the proof.

Theorem 3.1 leads to the following Corollaries:

Corollary 3.2. Let X be a Banach space and $S,T\colon Y\rightarrow X$ for an arbitrary set Y such that (2.14) holds and $T(Y)\subseteq S(Y)$. Assume S and T have a coincidence point z such that $Tz = Sz = p$. For any $x_o \in Y$, the Jungck-SP iterative process (2.7) $\{Sx_n\}_{n=1}^{\infty}$ converges strongly to p.

Further, if $Y = X$ and S,T commute at p (i.e. S and T are weakly compatible), then p is the unique common fixed point of S,T.

Corollary 3.3. Let X be a Banach space and $S,T\colon Y\rightarrow X$ for an arbitrary set Y such that (2.14) holds and $T(Y)\subseteq S(Y)$. Assume S and T have a coincidence point z such that $Tz = Sz = p$. For any $x_o \in Y$, the Jungck-Mann iterative process (2.3) $\{Sx_n\}_{n=1}^{\infty}$ converges strongly to p.

Further, if $Y = X$ and S,T commute at p (i.e. S and T are weakly compatible), then p is the unique common fixed point of S,T.

Remark 3.4. Weaker versions of Theorem 3.1 are the convergence results in Chugh and Kumar (2011) where S is assumed injective and the convergence is not to the common fixed point but to the coincidence point of S,T. Furthermore, the Jungck-multistep-SP iteration used in Theorem 3.1 is more general than the Jungck-SP used in Chugh and Kumar (2011).

STABILITY OF JUNGCK-MULTISTEP-SP ITERATIONS IN A BANACH SPACE

In this section, some stability results for the Jungck-multistep-SP iterative processes defined by (2.15) are established for generalized contractive-like inequality operators defined by (2.14). The stabilities of Jungck-SP and Jungck-Mann iterative processes follow as corollaries. The theorem is stated thus:

Theorem 4.1. Let X be a Banach space and $S,T: Y \to X$ for an arbitrary set Y such that (2.14) holds and $T(Y) \subseteq S(Y)$. For any $x_o \in Y$ and $0 \le \delta < 1$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-multistep-SP iterative process defined by (2.15) converging to p (that is $Sp = Tp = p$) with $0 < \alpha < \alpha_n$, $0 < \beta^i < \beta^i_n$ for $i = 1,2,...,k-1$ and all n. Then the Jungck-multistep-SP iterative process defined by (2.15) is (S,T)-stable.

Proof. Let $\{Sy_n\}_{n=0}^{\infty} \subset Y$, $\{Su_n^i\}_{n=0}^{\infty}$, for $i=1,2,...k-1$ *be real sequences in Y.*

Let $\varepsilon_n = \| Sy_{n+1} - (1-\alpha_n)Su_n^1 - \alpha_n Tu_n^1 \|$, $n = 0,1,2,...,$

where
$Su_n^i = (1 - \beta^i_n)Su_n^{i+1} + \beta^i_n Tu_n^{i+1}$, $i=1,2,...k-2$,
$Su_n^{k-1} = (1 - \beta^{k-1}_n)Sy_n + \beta^{k-1}_n Ty_n$, $i=1,2,...k-2$ and let $lim_{n\to\infty}\varepsilon_n = 0$.

Then, we shall prove that $lim_{n\to\infty}Sy_n = p$ using the generalized contractive-like inequality operators satisfying condition (2.14).

That is,
$$\| Sy_{n+1} - p \| \le \| Sy_{n+1} - (1-\alpha_n)Su_n^1 - \alpha_n Tu_n^1 \|$$
$$+ \| (1-\alpha_n)Su_n^1 + \alpha_n Tu_n^1 - (1-\alpha_n+\alpha_n)p \|$$
$$\le \varepsilon_n + (1-\alpha_n)\| Su_n^1 - p \| + \alpha_n \| Tu_n^1 - p \|$$
$$= \varepsilon_n + (1-\alpha_n)\| Su_n^1 - p \| + \alpha_n \| Tz - Tu_n^1 \|$$
$$\le \varepsilon_n + (1-\alpha_n)\| Su_n^1 - p \|$$
$$+ \alpha_n e^{L\|Sz-Tz\|}[\delta \| Sz - Su_n^1 \| + \phi(\| Sz - Tz \|)]$$
$$= \varepsilon_n + (1-\alpha_n)\| Su_n^1 - p \| + \delta \| p - Su_n^1 \|$$
$$= [1-\alpha_n(1-\delta)]\| Su_n^1 - p \| + \varepsilon_n$$
(4.1)

Using (2.15) and (2.14), we have the following estimates,

$$\| Su_n^1 - p \| \le (1-\beta^1_n)\| Su_n^2 - p \| + \beta^1_n \| Tu_n^2 - p \|$$
$$= (1-\beta^1_n)\| Su_n^2 - p \| + \beta^2_n \| Tz - Tu_n^2 \|$$
$$\le (1-\beta^1_n)\| Su_n^2 - p \|$$
$$+ \beta^2_n e^{L\|Sz-Tz\|}[\delta \| Sz - Su_n^2 \| + \phi(\| Sz - Tz \|)]$$
$$= (1-\beta^1_n)\| Su_n^2 - p \| + \delta \| p - Su_n^2 \|$$
$$= [1-\beta^1_n(1-\delta)\| Su_n^2 - p \|$$
(4.2)

An application of (2.15) and (2.14) also give
$$\| Su_n^2 - p \| \le (1-\beta^2_n)\| Su_n^3 - p \| + \beta^2_n \| Tz - Tu_n^3 \|$$
$$\le (1-\beta^2_n)\| Su_n^3 - p \|$$
$$+ \beta^2_n e^{L\|Sz-Tz\|}[\delta \| Sz - Su_n^3 \| + \phi(\| Sz - Tz \|)]$$
$$= (1-\beta^2_n)\| Su_n^3 - p \| + \delta\beta^2_n \| p - Su_n^3 \|$$
$$= [1-\beta^2_n(1-\delta)]\| Su_n^3 - p \|$$
(4.3)

Similarly, an application of (2.15) and (2.14) give
$$\| Su_n^3 - p \| \le (1-\beta^3_n)\| Su_n^4 - p \| + \beta^3_n \| Tz - Tu_n^4 \|$$
$$\le (1-\beta^3_n)\| Su_n^4 - p \|$$
$$+ \beta^3_n e^{L\|Sz-Tz\|}[\delta \| Sz - Su_n^4 \| + \phi(\| Sz - Tz \|)]$$
$$= (1-\beta^3_n)\| Su_n^4 - p \| + \delta\beta^3_n \| p - Su_n^4 \|$$
$$= [1-\beta^3_n(1-\delta)]\| Su_n^4 - p \|$$
(4.4)

An application of (2.15) and (2.14) also give
$$\| Su_n^{k-2} - p \| \le (1-\beta^{k-2}_n)\| Su_n^{k-1} - p \| + \beta^{k-2}_n \| Tz - Tu_n^{k-1} \|$$
$$\le (1-\beta^{k-2}_n)\| Su_n^{k-1} - p \|$$
$$+ \beta^{k-2}_n e^{L\|Sz-Tz\|}[\delta \| Sz - Su_n^{k-1} \| + \phi(\| Sz - Tz \|)]$$
$$= (1-\beta^{k-2}_n)\| Su_n^{k-1} - p \| + \delta\beta^{k-2}_n \| p - Su_n^{k-1} \|$$
$$= [1-\beta^{k-2}_n(1-\delta)]\| Su_n^{k-1} - p \|$$
(4.5)

Combining (4.1), (4.2), (4.3) (4.4) and (4.5), we have
$$\| Sy_{n+1} - p \| \le [1-\alpha_n(1-\delta)][1-\beta^1_n(1-\delta)]$$
$$[1-\beta^2_n(1-\delta)][1-\beta^3_n(1-\delta)]$$
$$...[1-\beta^{k-2}_n(1-\delta)]\| Su_n^{k-1} - p \| + \varepsilon_n$$
(4.6)

An application of (2.15) and (2.14) also give
$$\| Su_n^{k-1} - p \| \le (1-\beta^{k-1}_n)\| Sy_n - p \| + \beta^{k-1}_n \| Tz - Ty_n \|$$
$$\le (1-\beta^{k-1}_n)\| Sy_n - p \|$$
$$+ \beta^{k-1}_n e^{L\|Sz-Tz\|}[\delta \| Sz - Sy_n \| + \phi(\| Sz - Tz \|)]$$
$$= (1-\beta^{k-1}_n)\| Sy_n - p \| + \delta\beta^{k-1}_n \| p - Sy_n \|$$
$$= [1-\beta^{k-1}_n(1-\delta)]\| Sy_n - p \|$$
(4.7)

Substituting (4.7) in (4.6), we have
$$\| Sy_{n+1} - p \| \le [1-\alpha_n(1-\delta)][1-\beta^1_n(1-\delta)]$$
$$[1-\beta^2_n(1-\delta)][1-\beta^3_n(1-\delta)]$$
$$...[1-\beta^{k-2}_n(1-\delta)][1-\beta^{k-1}_n(1-\delta)]\| Sy_n - p \| + \varepsilon_n$$
(4.8)

Using $0 < \alpha \le \alpha_n$ and $\delta \in [0,1)$, we have
$[1-\alpha_n(1-\delta)][1-\beta^1_n(1-\delta)][1-\beta^2_n(1-\delta)][1-\beta^3_n(1-\delta)]...$
$.[1-\beta^{k-2}_n(1-\delta)][1-\beta^{k-1}_n(1-\delta)] < 1$.

Using Lemma (2.11), (4.8) yields $\lim_{n\to\infty} Sy_{n+1} = p$.
Conversely, let $lim_{n\to\infty}Sy_n = p$, we show that $lim_{n\to\infty}\varepsilon_n = 0$ as follows:

$$\varepsilon_n = \| Sy_{n+1} - (1-\alpha_n)Su_n^1 - \alpha_n Tu_n^1 \|$$

$$\leq \| Sy_{n+1} - p \| + \| (1-\alpha_n+\alpha_n)p - (1-\alpha_n)Su_n^1 - \alpha_n Tu_n^1 \|$$

$$\leq \| Sy_{n+1} - p \| + (1-\alpha_n) \| Su_n^1 - p \| + \alpha_n \| Tz - Tu_n^1 \| \quad (4.9)$$

$$\leq \| Sy_{n+1} - p \| + (1-\alpha_n) \| Su_n^1 - p \| + \delta\alpha_n \| p - Su_n^1 \|$$

$$= \| Sy_{n+1} - p \| + [1-\alpha_n(1-\delta)] \| Su_n^1 - p \|$$

However,

$$\| Su_n^1 - p \| \leq [1-\beta_n^1(1-\delta)][1-\beta_n^2(1-\delta)][1-\beta_n^3(1-\delta)]$$
$$...[1-\beta_n^{k-2}(1-\delta)][1-\beta_n^{k-1}(1-\delta)] \| Sy_n - p \| \quad (4.10)$$

Substituting (4.10) in (4.9), we have

$$\varepsilon_n \leq [1-\alpha_n(1-\delta)][1-\beta_n^1(1-\delta)][1-\beta_n^2(1-\delta)][1-\beta_n^3(1-\delta)]$$
$$...[1-\beta_n^{k-2}(1-\delta)][1-\beta_n^{k-1}(1-\delta)] \| Sy_n - p \| + \| Sy_{n+1} - p \|$$

$$(4.11)$$

Since $\lim_{n\to\infty} \| Sy_n - p \| = 0$ (by our assumption), it follows that $\lim_{n\to\infty}\varepsilon_n = 0$.

Therefore the Jungck-multistep-SP iterative scheme (2.15) is S,T-stable.

Theorem 4.1 yields the following corollaries:

Corollary 4.2. Let X be a Banach space and $S,T : Y \to X$ for an arbitrary set Y such that (2.14) holds and $T(Y) \subseteq S(Y)$. For any $x_o \in Y$ and $0 \leq \delta < 1$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-SP iterative process defined by (2.7) converging to p (that is $Sp = Tp = p$) with $0 < \alpha < \alpha_n$ and all n. Then the Jungck-SP iterative process defined by (2.7) is (S,T)-stable.

Corollary 4.3. Let X be a Banach space and $S,T : Y \to X$ for an arbitrary set Y such that (2.14) holds and $T(Y) \subseteq S(Y)$. For any $x_o \in Y$ and $0 \leq \delta < 1$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Mann iterative process defined by (2.3) converging to p (that is $Sp = Tp = p$) with $0 < \alpha < \alpha_n$ and all n. Then the Jungck-Mann iterative process defined by (2.3) is (S,T)-stable.

Remark 4.4. Weaker versions of Theorem 4.1 are the the stability results in Chugh and Kumar (2011) where S is assumed injective and the stability result is not to the common fixed point but to the coincidence point of S,T. Furthermore, the Jungck-multistep-SP iteration used in Theorem 4.1 is more general than the Jungck-SP used in Chugh and Kumar (2011).

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